Phase coexistence of gradient Gibbs measures

A gradient Gibbs measure is the projection to the gradient variables $\eta_b = \phi_y - \phi_x$ of the Gibbs measure of the form

$$P(\mathrm{d}\phi) = Z^{-1} \exp\Big\{-\beta \sum_{\langle x,y \rangle} V(\phi_y - \phi_x)\Big\} \mathrm{d}\phi,$$

where V is a potential, β is the inverse temperature and $\mathrm{d}\phi$ is the product Lebesgue measure. The simplest example is the (lattice) Gaussian free field $V(\eta) = \frac{1}{2}\kappa\eta^2$. A well known result of Funaki and Spohn asserts that, for any uniformly-convex V, the possible infinite-volume measures of this type are characterized by the *tilt*, which is a vector $u \in \mathbb{R}^d$ such that $E(\eta_b) = u \cdot b$ for any (oriented) edge b. I will discuss a simple example for which this result fails once V is sufficiently non-convex thus showing that the conditions of Funaki-Spohn's theory are generally optimal. The underlying mechanism is an order-disorder phase transition known, e.g., from the context of the q-state Potts model with sufficiently large q. Based on joint work with Roman Kotecký.

Phase transitions in (large-spin) quantum-spin systems

A well-known "principle" of solid-state physics asserts that once the temperature is positive, the quantum systems behave essentially classically. A similar "principle" is deemed to be true for the large $\mathcal{S}$ limit of quantum spin systems with spins of size $\mathcal{S}$. Mathematical results making precise statements to this extent are known but very few go beyond the information about the large $\mathcal{S}$ limit of the free energy (which, in its own right, is not enough to derive conclusive statements about phase transitions in the quantum models). I will show that, whenever one can prove a phase transition in the classical system using chessboard estimates, a similar transition occurs in the corresponding quantum spin $\mathcal{S}$-system, provided the temperature and the spin size obey $\beta \ll \sqrt{\mathcal{S}}$. This allows one to conclude the existence of temperature-driven phase transitions in systems that obey a certain large-entropy condition, and phase coexistence in systems where a symmetry breaking occurs via the mechanism of order-by-disorder. These have been inaccessible by methods available heretofore. The key estimate underlying the whole theory is an extension of Berezin-Lieb inequalities down to the level of matrix elements relative to the basis of coherent states. Based on joint work with Lincoln Chayes and Shannon Starr.

Scaling limit of simple random walk on supercritical percolation clusters

We consider a simple random walk on the (unique) infinite cluster $\mathscr{C}_\infty$ of bond percolation in $\mathbb{Z}^d$, where $d \geq 2$. At each unit of time, the walk picks one of its $2d$ neighbors in $\mathbb{Z}^d$ at random and attempts to move to it, but the move is suppressed if the corresponding bond is not present in $\mathscr{C}_\infty$. We will show that, under the "usual" scaling of space and time, in almost every realization of $\mathscr{C}_\infty$ the path distribution of this walk converges weakly to that of an isotropic d-dimensional Brownian motion. The proof is based on analysis of the "harmonic embedding" of $\mathscr{C}_\infty$ on which the corresponding walk is an L^2-martingale. Based on joint work with Noam Berger.

A perturbative proof of the Lee-Yang Circle Theorem

The Lee-Yang Circle Theorem [T.D. Lee and C.N. Yang, Phys. Rev. **87** (1952) 410–419] states that the complex zeros of a certain function of interest in statistical mechanics—namely, the partition function of the Ising model in a complex magnetic field—always lie on the unit circle in $\mathbb{C}$. This observation influenced dramatically the early understanding of phase transitions in statistical mechanics. I will present (the main ideas of) the original Lee & Yang's proof and then explain a new approach to this result based on perturbative expansions and contour methods. The perturbative approach is the basis of a general theory which allows one to control the above zeros in a large class of statistical-mechanical systems. The talk is based on a series of joint papers with C. Borgs, J.T. Chayes, L.J. Kleinwaks and R. Kotecký.

Graph distance in long-range percolation models

In 1967, using an ingenious sociological experiment, S. Milgram studied the length of acquaintance chains between “geometrically distant” individuals. The results led him to the famous conclusion that average two Americans are about six acquaintances (or “six handshakes”) away from each other. We will model the situation in terms of long-range percolation on $\mathbb{Z}^d$, where the nearest neighbor bonds represent the acquaintances due to geometric proximity—people living in the house next door—while long bonds are acquaintances established by other means—e.g., friends from college. The question is: What is the minimal number of bonds one needs to traverse to get from site x to site y.

Thus, in addition to the usual connections between nearest neighbors on $\mathbb{Z}^d$, any two sites $x, y \in \mathbb{Z}^d$ at Euclidean distance $|x-y|$ will be connected by an occupied bond independently with probability proportional to $|x-y|^{-s}$, where $s > 0$ is a parameter. Using $D(x,y)$ to denote the length of the shortest occupied path between x and y, the main question boils down to the asymptotic scaling of $D(x,y)$ as $|x-y| \to \infty$. I will discuss a variety of possible behaviors and mention known results and open problems. Then I will sketch the proof of the fact that, when $s \in (d, 2d)$, the distance $D(x,y)$ scales like $(\log|x-y|)^{\Delta}$, where Δ^{-1} is the binary logarithm of $2d/s$.

Mesoscopic droplet formation in the 2D Ising model

Droplet formation is a topic central to many parts of physical sciences—physical chemistry, physics of liquids and surfaces, solid-state physics—as well as applied mathematics and computer science. One reason is that the formation of a droplet serves as a bottleneck through which the system under consideration passes from one equilibrium (or steady) state to another. In mathematical physics, droplet formation is directly linked with the topic of Wulff construction whose task is to determine the macroscopic shape of a droplet on the basis of microscopic considerations. In the context of the Ising model and percolation, the latter has been the subject of intensive research in last 15 years.

I will discuss droplet formation in the two-dimensional Ising model in a finite $L \times L$ box in $\mathbb{Z}^2$, with plus boundary conditions and fixed magnetization (i.e., fixed total number of plus spins). The focus will lie on the critical scale for appearance/disappearance of a droplet which corresponds to having about $L^{4/3}$ extra minus spins above the "expected" value. Based on a simple large-deviation analysis, whose validity seems to extend beyond the model and dimension at hand, I will derive a universal scaling relation for the size of the droplet. The most striking conclusion is that there is a *minimal droplet size* beyond which the droplet prefers to melt into the background fluctuations. Based on joint work with Lincoln Chayes and Roman Kotecký.

Organized criticality in tree geometries

Self-organized criticality (SOC) is a state in which a dynamical system, typically referred to as a *sandpile*, evolves through a series of nearly-unstable stable states—a sequence of *avalanches*. While a lot has been said and even more written about the SOC, very little is known mathematically. I will discuss a new model of self-organized phenomena where the focus is on a single avalanche propagating through a fixed randomly-generated background configuration. (The latter can be thought of as a snapshot of a self-organized state; hence the word "organized" in the title.) The underlying lattice is a directed, rooted tree and the background configuration is sampled from an i.i.d. distribution μ.

The prime goal of the analysis is to characterize the distributions μ for which the resulting avalanche has the properties usually attributed to self-organized critical states: no infinite avalanches but a power-law decay of avalanche distributions. I will sketch a complete probabilistic solution of the problem, including the characterization of the critical exponents which turn out to be in the mean-field percolation universality class.

Phase transitions by comparison to mean-field theory

A fair number of systems arising in solid state physics are represented by a collection of vector-valued spin variables $(\mathbf{S}_x)$ on the cubic lattice which interact via a dot-product coupling. (In particular, this includes the well known Ising and Potts models.) The thermal equilibrium of these systems is described by the Gibbs-Boltzmann distribution $\mu(\mathrm{d}\mathbf{S}) \propto e^{-\beta H(\mathbf{S})}\mathrm{d}\mathbf{S}$, where $H(\mathbf{S})$ is the sum of terms $-\mathbf{S}_x \cdot \mathbf{S}_y$ over all nearest neighbors x and y. At low temperatures ($\beta \gg 1$), these measures strongly prefer alignment among spins regardless of their spatial separation, while at high temperatures ($\beta \ll 1$) the spins are more or less uncorrelated. The ordered and disordered regimes generically meet at a single point of phase-transition where the highly ordered ($\beta \gg 1$) states coexist with the ($\beta \ll 1$) disordered state.

The above phase transition is hard to prove mathematically, particularly when the spins have a continuous *a priori* distribution. I will describe a novel technique, based on a comparison to mean-field theory, that reduces such proofs to an optimization problem in multivariable calculus. (Mean-field theory is an approximation method, very common in theoretical physics, in which the influence of all spins on a single spin is replaced by that of an effective external field.) As an application it can be shown that the 3-state Potts model on $\mathbb{Z}^d$ undergoes a first-order phase transition (as β varies) provided the spatial dimension d is sufficiently large.