Extreme points of two-dimensional discrete Gaussian Free Field

Marek Biskup

(UCLA)

Joint work with

Oren Louidor (Technion, Haifa)

Discrete Gaussian Free Field (DGFF)

$D \subset \mathbb{R}^d$ (or $\mathbb{C}$ in $d = 2$) bounded, open, "nice" boundary
$D_N := \{x \in \mathbb{Z}^d : x/N \in D\}$
$G_N^D :=$ Green function of SRW on D_N killed upon exit

$$G_N^D(x,y) = E^x\left(\sum_{k=0}^{\tau-1} 1_{\{X_k = y\}}\right)$$

where $\tau :=$ first exit time from D_N

Definition

DGFF on $D_N :=$ Gaussian process $\{h_x : x \in D_N\}$ with

$$E(h_x) = 0 \quad \text{and} \quad E(h_x h_y) = G_N^D(x,y)$$

N.B. Zero values outside D_N

Gibbs-Markov property

Law of DGFF = finite-volume Gibbs measure characterized by

The Gibbs-Markov property:

Assume $\widetilde{D} \subset D$. Then

$$h^D \overset{\text{law}}{=} h^{\widetilde{D}} + \varphi^{D,\widetilde{D}}$$

where

(1) $h^{\widetilde{D}}$ and $\varphi^{D,\widetilde{D}}$ independent Gaussian fields

(2) $x \mapsto \varphi^{D,\widetilde{D}}(x)$ discrete harmonic on $\widetilde{D}_N$

Set $\varphi^{D,\widetilde{D}}(x) := E\big(h^D(x) \,\big|\, h^D(z) \colon z \in D_N \smallsetminus \widetilde{D}_N\big)$

Then $h^D - \varphi^{D,\widetilde{D}} \perp\!\!\!\perp \varphi^{D,\widetilde{D}}$ and $h^D - \varphi^{D,\widetilde{D}} \overset{\text{law}}{=} h^{\widetilde{D}}$

Why 2D?

For x with $\operatorname{dist}(x, D_N^{\mathrm{c}}) > \delta N$,

$$\operatorname{Var}(h_x) = G_N(x,x) \asymp \begin{cases} N, & \text{if } d = 1, \\ \log N, & \text{if } d = 2, \\ 1, & \text{if } d \geq 3. \end{cases}$$

In $d = 2$:

$$\begin{aligned} G_N(x,y) &= g \log N - a(x,y) + o(1), && N \gg 1 \\ a(x,y) &= g \log|x-y| + c_0 + o(1), && |x-y| \gg 1 \end{aligned}$$

where

$$g := \frac{2}{\pi} \qquad \text{(In physics, } g := \tfrac{1}{2\pi}\text{)}$$

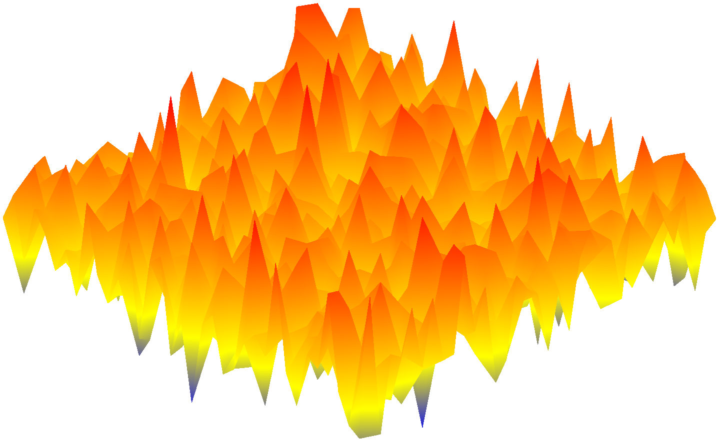
The model is **asymptotically scale invariant:**

$$G_{2N}(2x, 2y) = G_N(x,y) + o(1)$$

In fact: **conformally invariant** (S. Andres' talk)

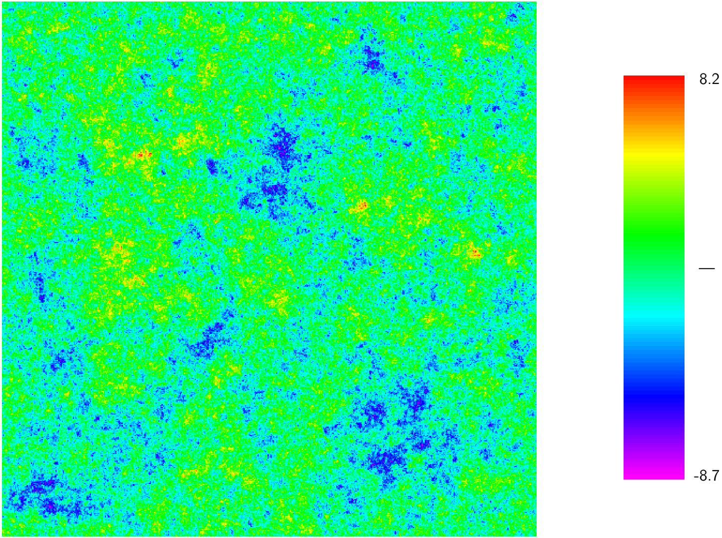
DGFF: a sample figure

Box 35×35, range of values $\approx [-5, 5]$



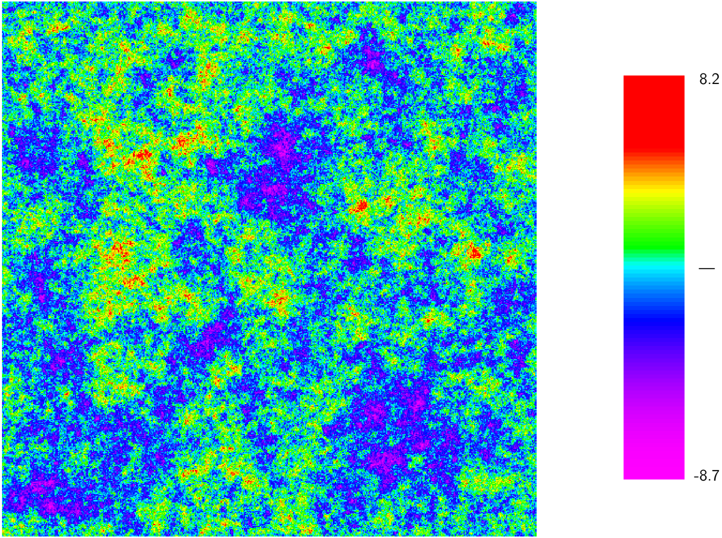
DGFF on 500×500 square

Uniform color system



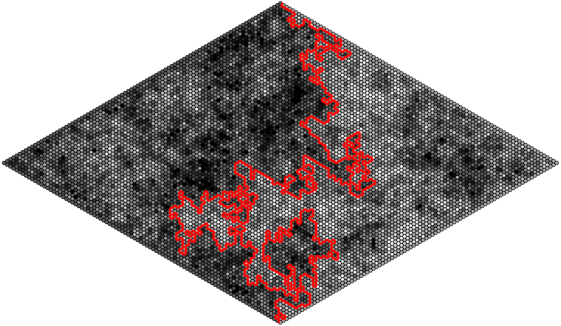
DGFF on 500×500 square

Emphasizing the extreme values



Level set (fractal) geometry
Some known facts

- $O(1)$-**level sets:** SLE_4 curves (Schramm & Sheffield)



- $O(\log N)$-**level sets:** Hausdorff dimension (Daviaud)

$$\left\{x \in D_N \colon h(x) \geq 2\sqrt{g}\,\gamma \log N\right\}, \qquad 0 < \gamma < 1$$

has $N^{2(1-\gamma^2)+o(1)}$ vertices. Note: Maximum order $\log N$!

Goal of today's talk

Extreme values of DGFF

Main goal: Describe statistics of extreme values of h^D

Question of interest: What's the role of conformal invariance?

N.B.: Continuum GFF **not** a function!

Surprise connections:

- Invariant measures for independent particle systems
- Liouville Quantum Gravity (S. Andres' talk)

Further known facts
Absolute maximum

Setting and notation: $D_N := (0,N)^2 \cap \mathbb{Z}^2$

$$M_N := \max_{x \in D_N} h_x \quad \text{and} \quad m_N := EM_N$$

Leading scale (Bolthausen, Deuschel & Giacomin):

$$m_N \sim 2\sqrt{g} \log N$$

Tightness for a subsequence (Bolthausen, Deuschel & Zeitouni):

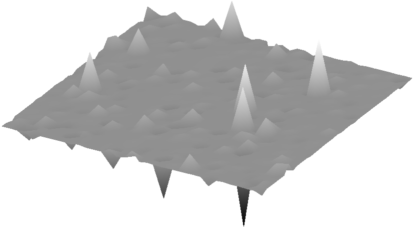
$$2E\big|M_N - m_N\big| \le m_{2N} - m_N$$

Full tightness (Bramson & Zeitouni):

$$EM_N = 2\sqrt{g} \log N - \frac{3}{4}\sqrt{g} \log\log N + O(1)$$

Convergence in law (Bramson, Ding & Zeitouni)

Some known facts
Extremal process tightness



Ding & Zeitouni, Ding established
extremal process tightness:

Extreme level set:

$$\Gamma_N(t) := \{x \in D_N \colon h_x \geq m_N - t\}$$

$$\exists c, C \in (0,\infty): \qquad \lim_{\lambda\to\infty} \liminf_{N\to\infty} P\left(\mathrm{e}^{c\lambda} \leq |\Gamma_N(\lambda)| \leq \mathrm{e}^{C\lambda}\right) = 1$$

and $\exists c > 0$ s.t.

$$\lim_{r\to\infty} \limsup_{N\to\infty} P\Big(\exists x, y \in \Gamma_N(c \log\log r) \colon r \leq |x-y| \leq N/r\Big) = 0$$

Setup for extreme order theory

Extremal point process

Full process: Measure η_N on $\overline{D} \times \mathbb{R}$

$$\eta_N := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x - m_N}$$

Problem: Values in each "peak" strongly correlated

Local maxima only: $\Lambda_r(x) := \{z \in \mathbb{Z}^2 \colon |z - x| \le r\}$

$$\eta_{N,r} := \sum_{x \in D_N} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N}$$

Main result
Scaling limit

Theorem (Convergence to Cox process)

There is a random Borel measure Z^D on $\overline{D}$ with $0 < Z^D(\overline{D}) < \infty$ a.s. such that for any $r_N \to \infty$ and $N/r_N \to \infty$,

$$\eta_{N,r_N} \xrightarrow[N\to\infty]{\text{law}} \mathrm{PPP}\Big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\Big)$$

where $\alpha := 2/\sqrt{g} = \sqrt{2\pi}$.

Corollaries

Asymptotic law of maximum: Setting $Z := Z^D(\overline{D})$,

$$P\big(M_N \le m_N + t\big) \underset{N\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

N.B.: Laplace transform of Z

Joint law position/value: $A \subset D$ open, $\widehat{Z}(A) = Z^D(A)/Z^D(\overline{D})$

$$P\Big(M_N \le m_N + t,\, N^{-1}\text{argmax}\, h^D \in A\Big) \underset{N\to\infty}{\longrightarrow} E\big(\widehat{Z}(A)\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

In fact: Key steps of the proof

As $\{\eta_{N,r_N}\colon N \geq 1\}$ tight, we can extract converging subsequences
Denote

$$\langle \eta, f \rangle := \int \eta(\mathrm{d}x, \mathrm{d}h) f(x,h)$$

Proposition (Distributional invariance)

Suppose $\eta :=$ a weak-limit point of some $\{\eta_{N_k, r_{N_k}}\}$. Then for any $f\colon D \times \mathbb{R} \to [0,\infty)$ continuous, compact support,

$$E\left(\mathrm{e}^{-\langle \eta, f \rangle}\right) = E\left(\mathrm{e}^{-\langle \eta, f_t \rangle}\right), \qquad t > 0,$$

where

$$f_t(x,h) := -\log E\mathrm{e}^{-f(x, h + B_t - \frac{\alpha}{2} t)}$$

with $B_t :=$ standard Brownian motion.

Proposition explained

We may write

$$\eta = \sum_{i\geq 1} \delta_{(x_i,h_i)}$$

Let $\{B_t^{(i)}\} :=$ i.i.d. standard Brownian motions. Set

$$\eta_t := \sum_{i\geq 1} \delta_{(x_i,h_i+B_t^{(i)} - \frac{\alpha}{2}t)}$$

Well defined as $t \mapsto |\Gamma_N(t)|$ grows only exponentially. Then

$$E\big(\mathrm{e}^{-\langle \eta_t, f\rangle}\big) = E\big(\mathrm{e}^{-\langle \eta, f_t\rangle}\big)$$

and so Proposition says

$$\eta_t \overset{\text{law}}{=} \eta, \qquad t > 0$$

Proof of Proposition I

Gaussian interpolation: $h', h'' \overset{\text{law}}{=} h$, independent

$$h \overset{\text{law}}{=} \underbrace{\left(1 - \frac{t}{g \log N}\right)^{1/2} h'}_{\text{main term}} + \underbrace{\left(\frac{t}{g \log N}\right)^{1/2} h''}_{\text{perturbation}}$$

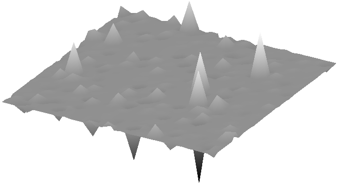
Denote

$$\Theta_{N,r}(\lambda) := \{x \in D_N \colon h'_x \geq m_N - \lambda,\, h'_x = \max_{z \in \Lambda_r(x)} h'_z\}$$

For $x \in \Theta_{N,r}(\lambda)$ ("large r-local maximum"):

$$\begin{aligned}\left(1 - \frac{t}{g \log N}\right)^{1/2} h'_x &= h'_x - \frac{1}{2} \frac{t}{g \log N} h'_x + o(1) \\ &= h'_x - \frac{t}{2} \frac{m_N}{g \log N} + o(1) \\ &= h'_x - \frac{\alpha}{2} t + o(1)\end{aligned}$$

Proof of Proposition II



Concerning h'', abbreviate

$$\widetilde{h}''_x := \left(\frac{t}{g \log N}\right)^{1/2} h''_x$$

Properties of Green function:

$$\mathrm{Cov}\left(\widetilde{h}''_x, \widetilde{h}''_y\right) = \begin{cases} t + o(1), & \text{if } |x-y| \le r \quad \text{nearly constant} \\ o(1), & \text{if } |x-y| \ge N/r \quad \text{nearly independent} \end{cases}$$

So we conclude: The law of

$$\left\{\widetilde{h}''_x \colon x \in \Theta_{N,r}(\lambda)\right\}$$

is asymptotically that of independent B.M.'s □

Problem: Characterize point processes on $\overline{D} \times \mathbb{R}$ are invariant under independent Dysonization

$$(x, h) \mapsto \left(x, h + B_t - \frac{\alpha}{2} t\right)$$

of (the second coordinate of) its points

Easy to check: $\mathrm{PPP}(\nu(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h} \mathrm{d}h)$ okay for any ν (even random)

Any other solutions?

Liggett's "folk" theorem

Setting:

- Markov chain on (nice space) $\mathscr{X}$ w/ transition kernel P
- System of particles evolving independently by P
- $\mathscr{I} :=$ loc. finite **invariant measures** on particle systems

Theorem (Liggett 1977)

Assume **uniform dispersivity property:**

$$\sup_{x \in \mathscr{X}} \mathrm{P}^n(x, C) \underset{n \to \infty}{\longrightarrow} 0 \qquad \forall C \subset \mathscr{X} \text{ compact}$$

Then each $\mu \in \mathscr{I}$ takes the form $\mathrm{PPP}(M(\mathrm{d}x))$, where M is a random measure satisfying

$$M\mathrm{P} \overset{\text{law}}{=} M$$

Liggett's 1977 derivation

For $t > 0$ define Markov kernel P on $\overline{D} \times \mathbb{R}$ by

$$(\mathrm{P}g)(x,h) := E^0 g\big(x, h + B_t - \tfrac{\alpha}{2} t\big)$$

Set $g(x,h) := \mathrm{e}^{-f(x,h)}$ for $f \geq 0$ continuous with compact support. Proposition implies

$$E\big(\mathrm{e}^{-\langle \eta, f \rangle}\big) = E\big(\mathrm{e}^{-\langle \eta, f^{(n)} \rangle}\big)$$

where

$$f^{(n)}(x,h) = -\log(\mathrm{P}^n \mathrm{e}^{-f})(x,h)$$

P has **uniform dispersivity property** and so $\mathrm{P}^n \mathrm{e}^{-f} \to 1$ uniformly on $\overline{D} \times \mathbb{R}$. Expanding the log,

$$f^{(n)} \sim 1 - \mathrm{P}^n \mathrm{e}^{-f} \qquad \text{as } n \to \infty$$

Liggett's 1977 derivation (continued)

Hence

$$E\left(\mathrm{e}^{-\langle \eta, f\rangle}\right) = \lim_{n\to\infty} E\left(\mathrm{e}^{-\langle \eta, 1-\mathrm{P}^n \mathrm{e}^{-f}\rangle}\right) \qquad (*)$$

But, as P is Markov,

$$\langle \eta, 1-\mathrm{P}^n \mathrm{e}^{-f}\rangle = \langle \eta \mathrm{P}^n, 1-\mathrm{e}^{-f}\rangle$$

$(*)$ shows that $\{\eta \mathrm{P}^n \colon n \geq 1\}$ is tight. Along a subsequence

$$\eta \mathrm{P}^{n_k}(\mathrm{d}x, \mathrm{d}h) \xrightarrow[k\to\infty]{\text{law}} M(\mathrm{d}x, \mathrm{d}h)$$

and so

$$E\left(\mathrm{e}^{-\langle \eta, f\rangle}\right) = E\left(\mathrm{e}^{-\langle M, 1-\mathrm{e}^{-f}\rangle}\right)$$

i.e., $\eta = \mathrm{PPP}(M(\mathrm{d}x, \mathrm{d}h))$. Clearly,

$$M\mathrm{P} \overset{\text{law}}{=} M$$

Question: What M can we get in our case?

Theorem (Liggett 1977)

$MP \overset{\text{law}}{=} M$ implies $MP = M$ a.s. when P is a kernel of

(1) *an irreducible, recurrent Markov chain*

(2) *a random walk on a closed abelian group w/o proper closed invariant subset*

N.B.:**(2) covers our case** and

$$MP = M \text{ a.s.} \quad \Leftrightarrow \quad M \text{ random mixture of P-invariant laws}$$

For our chain Choquet-Deny (or $t \downarrow 0$) shows

$$M(\mathrm{d}x, \mathrm{d}h) = Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h + \widetilde{Z}^D(\mathrm{d}x) \otimes \mathrm{d}h$$

Tightness of maximum forces $\widetilde{Z}^D = 0$ a.s. □

Proof of Theorem completed

Uniqueness of the limit

We thus know $\eta_{N_k, r_{N_k}} \xrightarrow{\text{law}} \eta$ implies

$$\eta = \mathrm{PPP}\big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\big)$$

for some random Z^D — albeit **possibly depending** on $\{N_k\}$.

But for $Z := Z^D(\overline{D})$, this yields

$$P\big(M_{N_k} \le m_{N_k} + t\big) \underset{k\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

Hence: **law of $Z^D(\overline{D})$ unique if limit law of maximum unique** (and we know this from Bramson & Ding & Zeitouni)

Existence of joint limit of maxima in finite number of disjoint subsets of $D \Rightarrow$ **uniqueness of law of $Z^D(\mathrm{d}x)$** □

Some literature

Details for above derivation for $D := (0,1)^2$:
Biskup-Louidor (arXiv:1306.2602)

Maxima for log-correlated fields:
Madaule (arXiv:1307.1365), Acosta (arXiv:1311.2000)
Ding, Roy and Zeitouni (in preparation)

Properties of Z^D-measure

Theorem

The measure Z^D satisfies:

(1) $Z^D(A) = 0$ *a.s. for any Borel* $A \subset \overline{D}$ *with* $\mathrm{Leb}(A) = 0$

(2) $\mathrm{supp}(Z^D) = \overline{D}$ *and* $Z^D(\partial D) = 0$ *a.s.*

(3) Z^D *is non-atomic a.s.*

Property (3) is only barely true:

Conjecture

Z^D *is supported on a set of zero Hausdorff dimension*

Fancy properties

Gibbs-Markov for Z^D measure

Recall $\widetilde{D} \subseteq D$ yields $h^D \overset{\text{law}}{=} h^{\widetilde{D}} + \varphi^{D,\widetilde{D}}$

Fact: $\varphi^{D,\widetilde{D}} \xrightarrow{\text{law}} \Phi^{D,\widetilde{D}}$ on $\widetilde{D}$ where

(1) $\{\Phi^{D,\widetilde{D}}(x) \colon x \in \widetilde{D}\}$ mean-zero **Gaussian** field with

$$\text{Cov}\big(\Phi^{D,\widetilde{D}}(x), \Phi^{D,\widetilde{D}}(y)\big) = G^D(x,y) - G^{\widetilde{D}}(x,y)$$

continuum Green functions

(2) $x \mapsto \Phi^{D,\widetilde{D}}(x)$ **harmonic** on $\widetilde{D}$ a.s.

Theorem (Gibbs-Markov property)

Suppose $\widetilde{D} \subseteq D$ *be such that* $\text{Leb}(D \smallsetminus \widetilde{D}) = 0$. *Then*

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha \Phi^{D,\widetilde{D}}(x)} Z^{\widetilde{D}}(\mathrm{d}x)$$

Theorem (Conformal symmetry)

Suppose $f : D \to f(D)$ analytic bijection. Then

$$Z^{f(D)} \circ f(\mathrm{d}x) \stackrel{\text{law}}{=} |f'(x)|^4 Z^D(\mathrm{d}x)$$

In particular, for D simply connected and $rad_D(x)$ conformal radius

$$\mathrm{rad}_D(x)^{-4} Z^D(\mathrm{d}x)$$

is invariant under conformal maps of D.

Note:

(1) $\mathrm{Leb} \circ f(\mathrm{d}x) = |f'(x)|^2 \mathrm{Leb}(\mathrm{d}x)$ and so $\mathrm{rad}_D(x)^{-2}\mathrm{Leb}(\mathrm{d}x)$ is invariant under conformal maps.

(2) By GM property it suffices to know law$(Z^{\mathbb{D}})$ for $\mathbb{D} :=$ unit disc. So this is a **statement of universality**

Unifying scheme?

Continuum Gaussian Free Field

Continuum GFF := Gaussian on $\mathrm{H}_0^1(D)$ w.r.t. norm $f \mapsto \pi\|\nabla f\|_2^2$

Formal expression: $h(x) = \sum_{n\geq 1} Z_n \varphi_n(x)$ $\{\varphi_n\}$ ONB

Exists only as a linear functional on $\mathrm{H}_0^1(D)$:

$$h(f) = \sqrt{\pi} \sum_{n\geq 1} Z_n \langle \nabla f, \nabla \varphi_n \rangle_{L^2(D)}$$

Derivative martingale: ($\gamma = 2$ in S. Andres' talk)

$$M'(\mathrm{d}x) = \left[2\mathrm{Var}(h(x)) - h(x)\right] \mathrm{e}^{2h(x) - 2\mathrm{Var}(h(x))} \mathrm{d}x$$

Defined by smooth approximations to h or expansion in ONB (Duplantier, Sheffield, Rhodes, Vargas)

KPZ relation links M'-measure of sets to Lebesgue measure

Unifying scheme?

Liouville Quantum Gravity/Multiplicative Chaos

Liouville Quantum Gravity (LQG):

$$M^D(\mathrm{d}x) := \mathrm{rad}_D(x)^2\, M'(\mathrm{d}x)$$

Theorem (B-Louidor, in progress)

There is constant $c_\star \in (0,\infty)$ s.t. for all D

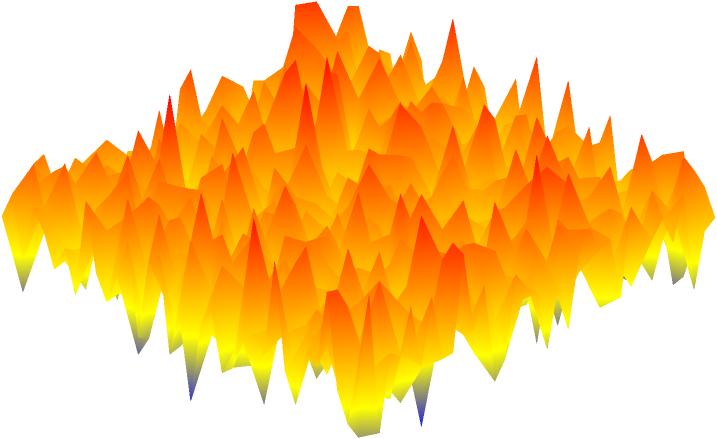
$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} c_\star\, M^D(\mathrm{d}x)$$

Current status: Law of Z^D characterized by

- Gibbs-Markov property — okay for LQG
- shift and dilation symmetry — okay for LQG
- precise upper tails of $Z^D(A)$ — so far open

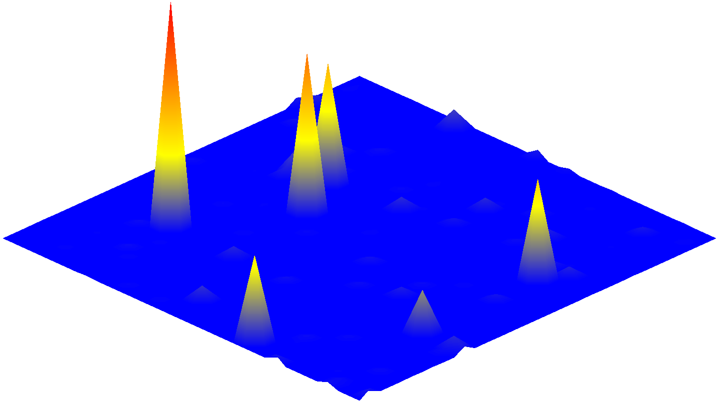
LQG in pictures

Naked GFF:



LQG in pictures

Derivative martingale:



THE END