Homogenization for random walks in random environments

Marek Biskup (UCLA)

New trends in homogenization

Roscoff, France

November 28, 2023

Disclaimer

This is a probability talk by a probabilist!

Expect different notation, very little “continuum” business, etc

The talk is an overview “panorama” lecture

Don’t hesitate to ask questions

Integer lattice: $\mathbb{Z}^d := \{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1,\dots,x_d \in \mathbb{Z}\}$.

Markov chain on $\mathbb{Z}^d$ prescribed by $\mathsf{P}\colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ such that

$$\forall x \in \mathbb{Z}^d\colon \qquad \sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y) = 1$$

Notation:

- $X = \{X_n\}_{n\geqslant 0}$ **sample path** of the Markov chain
- $P^x :=$ **law of** X on $(\mathbb{Z}^d)^{\mathbb{N}}$ such that $P^x(X_0 = x) = 1$

Natural choices for transition probability:

- P **constant** … leads to "ordinary" random walks
- P **periodic** … basically the same
- P **aperiodic** (deterministic) … ???
- P **random** … defines a **random walk in random environment** (RWRE)

In all cases we refer to the choice of P as **environment**

Generator $\mathsf{L} := \mathsf{P} - 1$ acts as

$$\mathsf{L}f(x) = \sum_{y \in \mathbb{Z}^d} \mathsf{P}(x,y)\big[f(y) - f(x)\big]$$

Continuum analogue of X: **diffusion given via an SDE**

$$\mathrm{d}X_t = v(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

This has generator in **non-gradient form**

$$\mathcal{L}f(x) := \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^{\mathrm{T}})_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{d} v_i(x)\frac{\partial}{\partial x_i} f(x)$$

which, denoting $\tilde{v} := v - \frac{1}{2}\nabla(\sigma\sigma^{\mathrm{T}})$, can also be written in **gradient form**

$$\mathcal{L}f(x) := \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial_{x_i}}(\sigma\sigma^{\mathrm{T}})_{ij}(x)\frac{\partial}{\partial x_j} f(x) + \sum_{i=1}^{d} \tilde{v}_i(x)\frac{\partial}{\partial x_i} f(x)$$

Discrete world is more complex ...

Overall goals

- Describe the long time statistical properties of sample paths $n \mapsto X_n$
- Track the effects caused by various irregularities of P
- Hopefully prove that **averaging occurs** and most of these wash out

Homogeneous case: If $y \mapsto \mathsf{P}(x,y)$ independent of x then X is "ordinary" random walk on $\mathbb{Z}^d$. Namely $X_n = Z_1 + \cdots + Z_n$ where $\{Z_k \colon k \geqslant 1\}$ are i.i.d.

Theorem (Strong Law of Large Numbers)

Assume $X_n := Z_1 + \cdots + Z_n$ for $\{Z_k \colon k \geqslant 1\}$ i.i.d. $\mathbb{R}^d$-valued with $Z_1 \in L^1$. Then

$$\lim_{n\to\infty} \frac{X_n}{n} = E(Z_1) \quad \text{a.s.}$$

Theorem (Multivariate Central Limit Theorem)

Assume $X_n := Z_1 + \cdots + Z_n$ for $\{Z_k \colon k \geqslant 1\}$ i.i.d. $\mathbb{R}^d$-valued with $Z_1 \in L^2$. Then

$$\frac{X_n - nE(Z_1)}{\sqrt{n}} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{N}(0,C)$$

where C is a $d \times d$ matrix determined by

$$v \cdot Cv = E\big((v \cdot Z_1)^2\big), \quad v \in \mathbb{R}^d$$

A (d-dimensional) **standard Brownian motion** $\{B_t : t \geqslant 0\}$ is a process with

- continuous sample paths, and
- independent increments such that $B_t - B_s = \mathcal{N}(0, (t-s)1)$ for all $0 \leqslant s \leqslant t$

Wiener measure := law of B on $C(\mathbb{R}_+, \mathbb{R}^d) := \{f : \mathbb{R}_+ \to \mathbb{R}^d : \text{ continuous}\}$

Theorem (Donsker's Invariance Principle)

Assume $X_n := Z_1 + \cdots + Z_n$ *for* $\{Z_k : k \geqslant 1\}$ *i.i.d.* $\mathbb{R}^d$*-valued with*

$$Z_1 \in L^2 \quad \text{and} \quad E(Z_1) = 0$$

Set

$$B_t^{(n)} := \frac{1}{\sqrt{n}}\Big(X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor})\Big), \quad t \geqslant 0$$

Then the law of $t \mapsto B^{(n)}$ *on* $C(\mathbb{R}_+, \mathbb{R}^d)$ *tends to that* $t \mapsto AB_t$ *where* B *is standard Brownian motion and* A *is a* $d \times d$*-matrix s.t.* $\text{Cov}(Z_1) = AA^{\mathrm{T}}$.

Key question and plan

Main question of interest:

How robust are the above limit results w.r.t. perturbations of P?

Plan of discussion:

- Periodic environments
- Stochastic reversible environments
- Deterministic reversible environments

Periodic environments

A Probability-101 approach

Assume that P is **periodic in all directions**:

$$\exists L \geqslant 1 \, \forall x,y,z \in \mathbb{Z}^d \colon \; \mathsf{P}(x+Lz, y+Lz) = \mathsf{P}(x,y)$$

Key fact: X has conditionally independent increments

Define $\Lambda_L := [0,L)^d \cap \mathbb{Z}^d$ and $\mathsf{Q}_L \colon \Lambda_L \times \Lambda_L \to [0,1]$ by

$$\mathsf{Q}_L(x,y) := \sum_{z \in \mathbb{Z}^d} \mathsf{P}(x, y+Lz)$$

and, for each $y \in \Lambda_L$, set

$$\mu_{x,y}(z) := \frac{\mathsf{P}(x,y+Lz)}{\mathsf{Q}_L(x,y)}$$

(Make an arbitrary choice if $\mathsf{Q}_L(x,y) = 0$.) Then

- Q_L is a transition probability of a Markov chain on Λ_L
- $\mu_{x,y}$ is a probability on $\mathbb{Z}^d$ for each $x,y \in \Lambda_L$

We then have …

Lemma (Factorization)

Given $x \in \Lambda_L$, sample:

- $Y :=$ *a path of Markov chain on* Λ_L *with transition probability* Q_L *started from* x
- *independent* $\{Z_k^{(y,y')} : k \geqslant 1,\, y, y' \in \Lambda_L\}$ *with* $\mathrm{law}(Z_k^{(y,y')}) = \mu_{y,y'}$

For each $n \geqslant 0$ *define*

$$X_n := Y_n + L \sum_{k=1}^{n} Z_k^{(Y_k, Y_{k+1})}$$

Then $X = \{X_n\}_{n \geqslant 1}$ *has the law* P^x

Proof: Fix $y, y' \in \Lambda_L$ and $z, z' \in \mathbb{Z}^d$. Then

$$\begin{aligned} &P\big(X_{n+1} = y' + Lz' \,\big|\, X_n = y + Lz\big) \\ &\quad = P\big(Y_{n+1} = y', Z_{n+1}^{(y,y')} = z' - z \,\big|\, X_n = y + Lz\big) \\ &\quad = \mathsf{Q}_L(y,y')\mu_{y,y'}(z'-z) = \mathsf{P}\big(x, y + L(z'-z)\big) = \mathsf{P}(x + Lz, y + Lz') \qquad \square \end{aligned}$$

As it turns out, all we need from Y are the **jump counts**

$X_n := Y_n + L \sum_{k=1}^{n} Z_k^{(Y_k, Y_{k+1})}$

$$N_n(y,y') := \sum_{k=1}^{n} 1_{\{Y_{k-1}=y\}} 1_{\{Y_k=y'\}}$$

Assume henceforth Y **irreducible**; meaning $\forall x,y \in \Lambda_L:\ \inf_{n\geqslant 1} P^x(X_n = y) > 0$

Lemma (SLLN for jump counts)

For all $y,y' \in \Lambda_L$ *there is* $\mathfrak{q}(y,y') \in [0,1]$ *such that, for all* $x,y \in \Lambda_L$,

$$\lim_{n\to\infty} \frac{N_n(y,y')}{n} = \mathfrak{q}(y,y') \quad P^x\text{-a.s.}$$

Proof: Renewal theorem for times between visits to y ... □

Corollary (SLLN for random walk)

For all $x \in \Lambda_L$,

$$\lim_{n\to\infty} \frac{X_n}{n} = L \sum_{y,y'\in\Lambda_L} \mathfrak{q}(y,y') E(Z_1^{(y,y')}) \quad P^x\text{-a.s.}$$

Lemma (CLT for jump counts)

There exists a covariance matrix C such that, for all $x \in \Lambda_L$,

$$\text{law of } \left\{ \frac{N_n(y,y') - \mathfrak{q}(y,y')n}{\sqrt{n}} : y, y' \in \Lambda_L \right\} \text{ under } P^x \xrightarrow[n\to\infty]{\text{w}} \mathcal{N}(0,C)$$

Proof: CLT for additive functionals of Markov chains (Gordin and Lifšic 1981)

Corollary (CLT for random walk)

Abbreviate $v := L \sum_{y,y' \in \Lambda_L} \mathfrak{q}(y,y') E(Z_1^{(y,y')})$. *Then*

$$\frac{X_n - nv}{\sqrt{nL}} \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0,\Sigma)$$

where

$$\Sigma := \sum_{y,y' \in \Lambda_L} \mathfrak{q}(y,y') \operatorname{Cov}(Z^{(y,y')}) + \sum_{y,y' \in \Lambda_L} \sum_{\tilde{y},\tilde{y}' \in \Lambda_L} C\big((y,y')(\tilde{y},\tilde{y}')\big) E(Z_1^{(y,y')}) \otimes E(Z_1^{(\tilde{y},\tilde{y}')})$$

All random walks in periodic environments are explicitly treatable

The key step is a decomposition of the Markov chain into

- a finite state Markov chain representing “microscopic” steps
- a sequence of conditionally independent “macroscopic” steps

This is an example of **two-scale decomposition**

A crucial input is a **CLT for additive functionals of Markov chains**

Problem: Explicit decomposition unclear for non-periodic environments.

Periodic environments

Homogenization approach

Keep assuming

$$\exists L \geqslant 1\, \forall x, y, z \in \mathbb{Z}^d \colon \quad \mathsf{P}(x + Lz, y + Lz) = \mathsf{P}(x, y)$$

but consider first the **symmetric** (a.k.a. **balanced**) case:

$$\forall x \in \mathbb{Z}^d \colon \quad \sum_{y \in \mathbb{Z}^d} \mathsf{P}(x, y)(y - x) = 0$$

Then $E(X_{n+1}|X_n = x) = x + E(X_{n+1} - x|X_n = x) = x$ and so X conforms to:

Definition (Martingale)

A sequence $\{X_n\}_{n\geqslant 0}$ of $\mathbb{R}^d$-valued random variables endowed with a filtration $\{\mathcal{F}_n\}_{n\geqslant 0}$ is a **martingale** if, for all $n \geqslant 0$,

$$X_n \in L^1 \quad \text{and} \quad X_n \text{ is } \mathcal{F}_n\text{-measurable}$$

and

$$E(X_{n+1} \,|\, \mathcal{F}_n) = X_n \quad \text{a.s.}$$

Natural filtration: $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$

Theorem (SLLN for martingales)

For each martingale X,

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \quad \text{a.s.}$$

Theorem (CLT for martingales)

Let M be an $\mathbb{R}^d$-valued martingale such that $M_n \in L^2$ for each $n \geqslant 0$. Assume that there exists a covariance matrix C such that

$$\frac{1}{n}\sum_{k=1}^{n} E\left((v\cdot(M_k - M_{k-1}))^2 \,|\mathcal{F}_{k-1}\right) \xrightarrow[n\to\infty]{P} v\cdot Cv, \qquad v\in\mathbb{R}^d$$

and

$$\frac{1}{n}\sum_{k=1}^{n} E\left(|M_k - M_{k-1}|^2 1_{\{|M_k-M_{k-1}|>\epsilon/\sqrt{n}\}} \,|\mathcal{F}_{k-1}\right) \xrightarrow[n\to\infty]{P} 0, \qquad \epsilon > 0$$

Then

$$\frac{1}{\sqrt{n}} M_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, C)$$

For each $y \in \Lambda_L$, set
$$N_n(y) := \sum_{k=0}^{n} 1_{\{X_k = y \bmod L\}}$$

By Renewal Theorem we know that, assuming Y irreducible,
$$\mathfrak{q}(y) := \lim_{n\to\infty} \frac{N_n(y)}{n}$$
exists P^x-a.s. for all $y \in \Lambda_L$ (independently of x)

Theorem (CLT for balanced periodic environments)

Suppose P *is L-periodic, balanced with the underlying chain* Y *irreducible. Assume*
$$\forall x \in \mathbb{Z}^d: \quad \sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)|y-x|^2 < \infty$$
Then
$$\frac{1}{\sqrt{n}} X_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, C)$$
where
$$C := \sum_{x\in\Lambda_L} \mathfrak{q}(x) \sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)(y-x)\otimes(y-x)$$

Proof — key calculation

Note that, by the Markov property,

$$E\big((v\cdot(X_k-X_{k-1}))^2\,|\mathcal{F}_{k-1}\big)=f(X_{k-1})$$

where

$$f(x):=\sum_{y\in\mathbb{Z}^d}\mathsf{P}(x,y)\big(v\cdot(y-x)\big)^2$$

Hence we get

$$\sum_{k=1}^{n}E\big((v\cdot(X_k-X_{k-1}))^2\,|\mathcal{F}_{k-1}\big)=\sum_{x\in\Lambda_L}N_{n-1}(x)f(x)$$

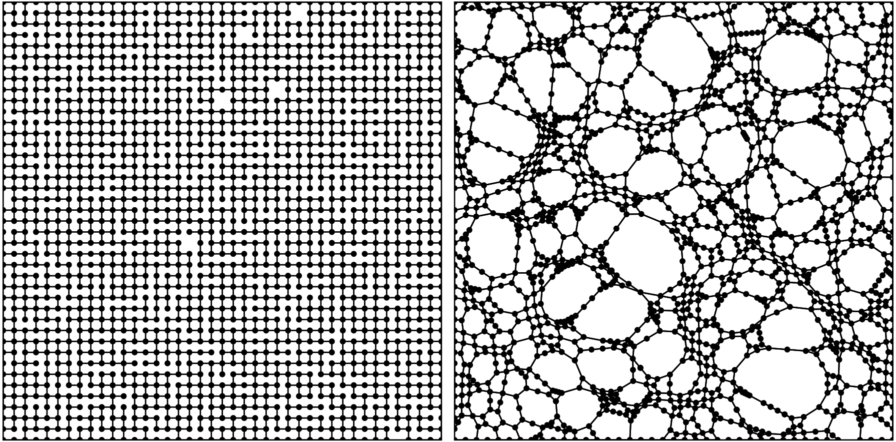
and so, since $\frac{1}{n}N_n(x)\to\mathfrak{q}(x)$ a.s.,

$$\frac{1}{n}\sum_{k=1}^{n}E\big((v\cdot(X_k-X_{k-1}))^2\,|\mathcal{F}_{k-1}\big)\underset{n\to\infty}{\longrightarrow}\sum_{x\in\Lambda_L}\mathfrak{q}(x)f(x)\quad\text{a.s.}$$

Now write the r.h.s. as $v\cdot Cv$ and apply Martingale CLT! □

If P is not balanced, then we will make it to be one!

Key idea: Change/deform the **embedding of** $\mathbb{Z}^d$ by a function $x \mapsto \psi(x)$ so that $\{\psi(X_k) - kv\}_{k\geqslant 1}$ is a martingale. We need v to compensate for **global drift**.



Introducing corrector

Write $\psi(x) = x + \chi(x)$, where χ will be the **corrector**

The Markov property gives

$$E\big(\psi(X_{n+1})\,|\mathcal{F}_n\big) = \psi(X_n) + V(X_n) + \sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)\big[\chi(y) - \chi(X_n)\big]$$

for the **local drift** V given by

$$V(x) := \sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)(y-x)$$

This shows that χ must solve the **Poisson equation**

$$\sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)\big[\chi(y) - \chi(x)\big] = v - V(x)$$

Since V is L-periodic, we can look for L-periodic solutions!

How to find v? Recall $Y_n := X_n \bmod L$. Then

$\sum_{y\in\mathbb{Z}^d} \mathsf{P}(x,y)[\chi(y)-\chi(x)] = v - V(x)$

$$\chi(x) = V(x) - v + \sum_{y\in\Lambda_L} \mathsf{Q}_L(x,y)\chi(y) = \cdots = E^x\Big(\sum_{k=0}^{n}[V(Y_k)-v]\Big) + E^x\big(\chi(Y_{n+1})\big)$$

so, using $N_n(y) := \sum_{k=0}^{n} 1_{\{Y_k=y\}}$,

$$\chi(x) = \sum_{y\in\Lambda_L} E^x\big(N_n(y)\big)\big[V(y)-v\big] + E^x\big(\chi(Y_n)\big)$$

But $N_n(y)/n \to \mathfrak{q}(y)$ a.s. so the first term will grow linearly unless we set

$$v := \sum_{y\in\Lambda_L} \mathfrak{q}(y)V(y)$$

Here we used that $\sum_{y\in\Lambda_L} \mathfrak{q}(y) = 1$. (In fact, $\mathfrak{q}$ is the invariant distribution of Y)

Poisson equation reads: $(1 - \mathsf{Q}_L)\chi = V - v$

Natural L^2-space: $\ell^2(\Lambda_L, \mathfrak{q})$ with $\langle f, g \rangle := \sum_{x \in \Lambda_L} \mathfrak{q}(x) f(x) g(x)$

Kernel-Range duality: $\mathrm{Ran}(1 - \mathsf{Q}_L) = \mathrm{Ker}(1 - \mathsf{Q}_L^+)^\perp$ where

$$\mathsf{Q}_L^+(x,y) = \frac{\mathfrak{q}(y)}{\mathfrak{q}(x)} \mathsf{Q}_L(y,x)$$

Now $f \in \mathrm{Ker}(1 - \mathsf{Q}_L^+)$ along with irreducibility implies

$$f(x) = \sum_{y \in \Lambda_L} \mathsf{Q}_L^+(x,y) f(y) = \cdots = \frac{1}{n} \sum_{k=0}^{n-1} (\mathsf{Q}_L^+)^k(x,y) f(y)$$

$$= \frac{1}{\mathfrak{q}(x)} \sum_{y \in \Lambda_L} \mathfrak{q}(y) \frac{E^y\big(N_n(x)\big)}{n} f(y) \underset{n \to \infty}{\longrightarrow} \sum_{y \in \Lambda_L} \mathfrak{q}(y) f(y)$$

meaning that f **is a constant**. But $v - V$ is orthogonal to constants in $\ell^2(\Lambda_L, \mathfrak{q})$ so $v - V \in \mathrm{Ran}(1 - \mathsf{Q}_L)$. **A solution χ thus exists!**

Periodic environments finished

Since $X_n = \psi(X_n) + O(1)$, exactly the same proof as before now gives:

Theorem (CLT for periodic environments)

Suppose P *is L-periodic with the underlying chain Y irreducible. Assume*

$$\forall x \in \mathbb{Z}^d: \quad \sum_{y \in \mathbb{Z}^d} \mathsf{P}(x,y)|y-x|^2 < \infty$$

and set

$$v := \sum_{y \in \Lambda_L} \mathfrak{q}(y) V(y) \quad \textit{for} \quad V(x) := \sum_{y \in \mathbb{Z}^d} \mathsf{P}(x,y)(y-x)$$

and

$$\mathsf{C} := \sum_{x \in \Lambda_L} \mathfrak{q}(x) \sum_{y \in \mathbb{Z}^d} \mathsf{P}(x,y) \big(\psi(y) - \psi(x) - v\big) \otimes \big(\psi(y) - \psi(x) - v\big)$$

Then

$$\frac{X_n - nv}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, C)$$

Note: $\mathfrak{q}$ is the **stationary distribution** of Y

We proved a CLT by employing these ideas:

- X becomes a martingale once we embed $\mathbb{Z}^d$ harmonically
- Underlying chain on periodic cell is ergodic
- Martingale CLT requires only a LLN-type of condition

Note: The same argument (supplied with **Martingale Functional CLT**) proves convergence to Brownian motion

Random environments

A natural next step: Make P **random**

Main cases studied:

- $\{\mathsf{P}(x,\cdot)\colon x\in\mathbb{Z}^d\}$ **i.i.d.** ... (true) **random walk in random environment**
- **balanced** random walks ... $\sum_{z\in\mathbb{Z}^d}\mathsf{P}(x,x+z)z=0$
- **divergence free** random walks ... $\sum_{z\in\mathbb{Z}^d}\mathsf{P}(x,x+z)=\sum_{z\in\mathbb{Z}^d}\mathsf{P}(x+z,z)$
- **conductance models** ... to be defined next

Consider a Markov chain on $\mathbb{Z}^d$ with transition probability P

A measure π on $\mathbb{Z}^d$, identified with $\pi\colon \mathbb{Z}^d \to [0,\infty)$, is said to be

- **invariant** for P if $\forall x \in \mathbb{Z}^d\colon\ \pi(x) = \sum_{y\in\mathbb{Z}^d} \pi(y)\mathsf{P}(y,x)$
- **reversible** for P if $\forall x,y \in \mathbb{Z}^d\colon\ \pi(x)\mathsf{P}(x,y) = \pi(y)\mathsf{P}(y,x)$

Reversibility natural: a statement of **detailed balance** condition

Conductance of (x,y):

$$\mathsf{c}(x,y) := \pi(x)\mathsf{P}(x,y)$$

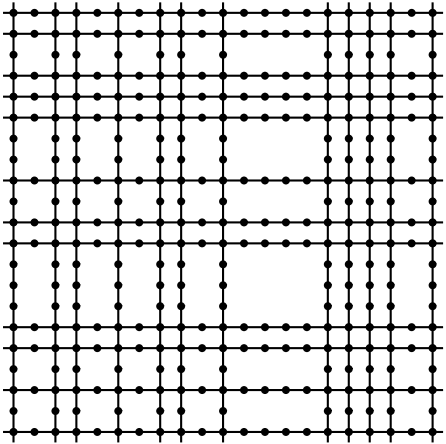
Note: π reversible $\Leftrightarrow$ $\forall x,y \in \mathbb{Z}^d\colon \mathsf{c}(x,y) = \mathsf{c}(y,x)$

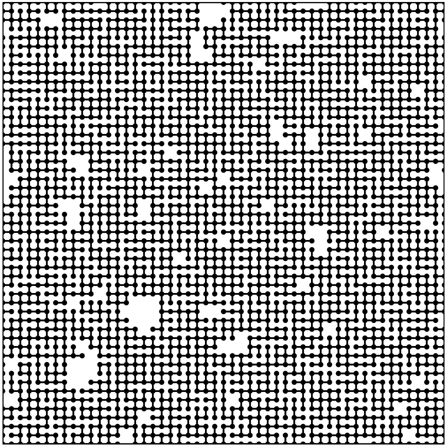
Conversely, given conductances $\{\mathsf{c}(x,y) = \mathsf{c}(y,x)\colon x,y \in \mathbb{Z}^d\}$ we define

$$\mathsf{P}(x,y) := \frac{\mathsf{c}(x,y)}{\pi(x)} \quad \text{where} \quad \pi(x) := \sum_{y\in\mathbb{Z}^d} \mathsf{c}(x,y)$$

(We will always assume these are not singular.)

We refer to chains prescribed this way as **conductance models**





Formal setup for (random) conductance models

Let $\Omega :=$ set of all non-negative conductances, ω an element of Ω

Write $\mathsf{c}_\omega(x,y)$ for **coordinate projection** of ω for pair (x,y). By assumption

$$\forall \omega \in \Omega\colon\; \mathsf{c}_\omega(x,y) = \mathsf{c}_\omega(y,x)$$

Denote $P^x_\omega :=$ law of X in environment ω with expectation E^x_ω

Write P_ω for transition probability, π_ω reversible measure

Denote $\tau_x :=$ **shift by** x acting on Ω as

$$\mathsf{c}_{\tau_x(\omega)}(y,z) = \mathsf{c}_\omega(y+x, z+x)$$

We may drop ω when these regarded as random variables

Note: ω can be a deterministic configuration!

First martingale decomposition

We want to prove a LLN. For this write

$$X_n = X_0 + \sum_{k=1}^{n} (X_k - X_{k-1})$$
$$= X_0 + \sum_{k=0}^{n-1} V_\omega(X_k) + \sum_{k=1}^{n} \Big[(X_k - X_{k-1}) - E^0_\omega\big(X_k - X_{k-1} \,\big|\, \mathcal{F}_{k-1}\big) \Big]$$

Denote second term by M_n. Then $\{M_n\}_{n \geqslant 0}$ is a martingale with

$$|M_k - M_{k-1}| \leqslant |X_k - X_{k-1}| + E^0_\omega\big(|X_k - X_{k-1}| \,\big|\, \mathcal{F}_{k-1}\big)$$

First term is an **additive functional of** X so it perhaps averages out

An issue: The chain X definitely not ergodic

Need to find a version of the chain on periodic cell . . .

Lemma

Given a sample X from P^0_ω, the sequence $\{\tau_{X_k}(\omega)\}_{k\geqslant 0}$ is a Markov chain on Ω with transition probability $\Pi\colon \Omega\times\mathcal{F}\to[0,1]$ defined by

$$\Pi(\omega',\cdot) := \sum_{x\in\mathbb{Z}^d} \mathsf{P}_{\omega'}(0,x)\delta_{\tau_x(\omega')}(\cdot)$$

We will call $\{\tau_{X_k}(\omega)\}_{k\geqslant 0}$ the **environment chain**. We now allow ω random …

Endow Ω with product sigma algebra $\mathcal{F}$ and let $\mathbb{P}$ be a probability on $(\Omega,\mathcal{F})$ with expectation denoted $\mathbb{E}$. Then:

Lemma

If $\mathbb{P}$ is translation invariant, i.e., $\mathbb{P}\circ\tau_x^{-1} = \mathbb{P}$ for all $x\in\mathbb{Z}^d$, then

$$\mathbb{Q}(\mathrm{d}\omega) := \frac{\pi_\omega(0)}{\mathbb{E}\pi(0)}\mathbb{P}(\mathrm{d}\omega)$$

is reversible and thus invariant for environment chain. Moreover, $\mathbb{P}\ll\mathbb{Q}$ and $\mathbb{Q}\ll\mathbb{P}$

Reversibility is equivalent to self-adjointness of transition probability

For $f, g \in L^2(\mathbb{Q})$ with $\langle f, g\rangle_{L^2(\mathbb{Q})} := \int f(\omega)g(\omega)\mathbb{Q}(\mathrm{d}\omega)$,

$$\mathbb{E}(\pi(0))\langle f, \Pi g\rangle_{L^2(\mathbb{Q})} = \mathbb{E}\left(\pi_\omega(0)\sum_{x\in\mathbb{Z}^d}\mathsf{P}_\omega(0,x)f(\omega)g\circ\tau_x(\omega)\right)$$

Next note that

$$\pi_\omega(0)\mathsf{P}_\omega(0,x) = \mathsf{c}_\omega(0,x) = \mathsf{c}_{\tau_x(\omega)}(-x,0) = \mathsf{c}_{\tau_x(\omega)}(0,-x) = \pi_{\tau_x(\omega)}(0)\mathsf{P}_{\tau_x(\omega)}(0,-x)$$

to continue the calculation as

$$\dots = \mathbb{E}\left(\pi_{\tau_x(\omega)}(0)\sum_{x\in\mathbb{Z}^d}\mathsf{P}_{\tau_x(\omega)}(0,-x)f(\omega)g\circ\tau_x(\omega)\right)$$

Finally, take out the sum and shift by $-x$ to get

$$\dots = \mathbb{E}\left(\pi_\omega(0)\sum_{x\in\mathbb{Z}^d}\mathsf{P}_\omega(0,-x)f\circ\tau_{-x}(\omega)g(\omega)\right) = \mathbb{E}(\pi(0))\langle \Pi f, g\rangle_{L^2(\mathbb{Q})}$$

by relabeling $-x$ for x $\square$

Lemma

Suppose $\mathbb{P}$ jointly ergodic with respect to shifts $\{\tau_x : x \in \mathbb{Z}^d\}$. Then environment chain is ergodic (in time) in the sense that, for all $f \in L^1(\mathbb{Q})$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ \tau_{X_k}(\omega) \underset{n\to\infty}{\longrightarrow} E_{\mathbb{Q}}(f), \quad P^0_\omega\text{-a.s.}$$

for $\mathbb{Q}$-a.e. $\omega \in \Omega$

Proof: Limit exists by Wiener's Ergodic Theorem (and stationarity of $\mathbb{Q}$)

Need to show: $\mathbb{Q}(A) \in \{0,1\}$ if $\omega \in A \Leftrightarrow \tau_{X_1}(\omega) \in A$. Expectation conditional on ω gives

$$1_A(\omega) = \sum_{x\in\mathbb{Z}^d} \mathsf{P}_\omega(0,x) 1_A \circ \tau_x(\omega)$$

and so $1_A(\omega) = 0$ forces $1_A(\tau_x(\omega)) = 0$ whenever $\mathsf{P}_\omega(0,x) > 0$. Iterating using irreducibility, $1_A \circ \tau_x(\omega) = 0$ for all $x \in \mathbb{Z}^d$. By ergodicity $\mathbb{P}(A) \in \{0,1\}$. □

Theorem (SLLN)

Suppose

$$\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d} c(0,x)|x|\Big) < \infty$$

Then for $\mathbb{P}$*-a.e.* $\omega \in \Omega$*,*

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} 0, \quad P_\omega^0\text{-a.s.}$$

Proof: Recall

$$X_n = X_0 + \sum_{k=0}^{n-1} V_\omega(X_k) + M_n$$

where M_n is as above. Markov property and Ergodic Theorem give

$$\frac{M_n}{n} \underset{n\to\infty}{\longrightarrow} E_{\mathbb{Q}} E_\omega^0(M_1) = 0$$

Similarly, $V_\omega(X_k) = V_{\tau_{X_k}(\omega)}(0)$ and $V(0) \in L^1(\mathbb{Q})$. Ergodic Theorem:

$$\frac{1}{n}\sum_{k=0}^{n-1} V_\omega(X_k) \underset{n\to\infty}{\longrightarrow} E_{\mathbb{Q}}\big(V(0)\big) = \mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d} c(0,x)x\Big) = 0 \qquad \square$$

Insofar, we have

- identified an analogue of the periodic-cell chain, and
- proved its ergodicity

To complete the plan, we need to find the corrector

Idea: Reversibility reduces this to convex optimization problem

$$\inf_{\varphi\in L^\infty(\mathbb{P})} \mathbb{E}\Bigl(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|x+\varphi\circ\tau_x-\varphi|^2\Bigr)$$

Any minimizing sequence converges (in the sense of discrete gradients) in L^2 weighted by conductance. This gives:

Lemma

Suppose

$$\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|x|^2\Big) < \infty$$

Then there exists $\chi\colon \Omega\times\mathbb{Z}^d \to \mathbb{R}^d$ *such that* $\psi(\omega,x) := x + \chi(\omega,x)$ *obeys*

$$\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|\psi(\cdot,x)|^2\Big) < \infty$$

and, for all $x\in\mathbb{Z}^d$,

$$\sum_{y\in\mathbb{Z}^d} \mathrm{c}(x,y)\psi(\cdot,y) = \pi(x)\psi(\cdot,x), \qquad \mathbb{P}\text{-a.s.}$$

Furthermore, for $\mathbb{P}$*-a.e.* ω *and all* $x,z\in\mathbb{Z}^d$,

$$\psi(\omega,x+z) - \psi(\omega,x) = \psi(\tau_x(\omega),z)$$

Proof: First two properties follow directly from optimization. The last property comes from the fact that $\psi(x,\cdot)$ is L^2-limit of $\varphi\circ\tau_x - \varphi$ $\square$

Corollary

Suppose

$$\mathbb{E}\Bigl(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|x|^2\Bigr) < \infty$$

Then for $\mathbb{P}$*-a.e.* $\omega \in \Omega$*,*

$$\frac{\psi(\omega,X_n)}{\sqrt{n}} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{N}(0,C)$$

where

$$C := \frac{1}{\mathbb{E}(\pi(0))}\mathbb{E}\Bigl(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)\psi(\cdot,x)\otimes\psi(\cdot,x)\Bigr)$$

Same applies to **Functional CLT** associated with $\{\psi(\omega,X_k)\}_{k\geqslant 0}$

New issue: The **corrector** $\chi(\cdot,x) := \psi(\cdot,x) - x$ is not bounded!

For CLT need to show that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\frac{|\chi(\omega, X_n)|}{\sqrt{n}} \xrightarrow[n\to\infty]{P} 0$$

For Functional CLT we even need

$$\frac{1}{\sqrt{n}} \max_{k \leqslant n} |\chi(\omega, X_k)| \xrightarrow[n\to\infty]{P} 0$$

Meaning of P matters a lot for how much we need to work!

Theorem (Kipnis and Varadhan 1986)

Suppose

$$\mathbb{E}\left(\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|x|^2\right) < \infty$$

Let $P(\cdot) := E_{\mathbb{Q}} P^0_\omega(\cdot)$ *be so called* **annealed law**. *Then*

$$\frac{1}{\sqrt{n}} \max_{k\leqslant n} |\chi(\omega, X_k)| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in probability w.r.t. } P$$

In particular, a Functional CLT holds for X under P

We get even a bit better: For all $F\colon C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ bounded continuous

$$E^0_\omega\big(F(B^{(n)})\big) \xrightarrow[n\to\infty]{\mathbb{P}} E\big(F(B)\big)$$

This is sometimes called **Invariance Principle in probability**

Key idea: **Forward/backward martingale decomposition**

Let X be a Markov chain started from a reversible measure and let f be an integrable function. Reversibility implies

$$E\big(f(X_{k+1})-f(X_k)\,\big|\,\sigma(X_k)\big)=E\big(f(X_{k-1})-f(X_k)\,\big|\,\sigma(X_k)\big)=A_f(X_k)$$

for

$$A_f(x):=E\big(f(X_1)-f(X_0)\,\big|\,X_0=x\big).$$

Hence we get

$$\begin{aligned}f(X_n)-f(X_0)&=\frac{1}{2}\big[A_f(X_0)-A_f(X_n)\big]\\&\quad+\frac{1}{2}\sum_{k=0}^{n-1}\Big[f(X_{k+1})-f(X_k)-A_f(X_k)\Big]-\frac{1}{2}\sum_{k=1}^{n}\Big[f(X_{k-1})-f(X_k)-A_f(X_k)\Big]\end{aligned}$$

Note: both sums are martingales (albeit for different filtrations), same law!

Returning to our RWRE, pick $\varphi \in L^\infty(\mathbb{P})$ and write S_n for the first sum with

$$f(x) := \chi(\omega,x) - \big(\varphi\circ\tau_x(\omega) - \varphi(\omega)\big)$$

Since this is in $L^2(P)$ and $A_f(X_k) = A_f(0)\circ\tau_{X_k}$, stationarity implies

$$P\Big(\max_{0\leqslant k\leqslant n}\big|A_f(X_k)\big| > \delta\sqrt{n}\Big) \underset{n\to\infty}{\longrightarrow} 0$$

Next, by Doob's L^2-inequality

$$P\Big(\max_{0\leqslant k\leqslant n}\big|S_k\big| > \delta\sqrt{n}\Big) \leqslant \frac{4}{\delta^2}\frac{1}{n}E\big(|S_n|^2\big)$$

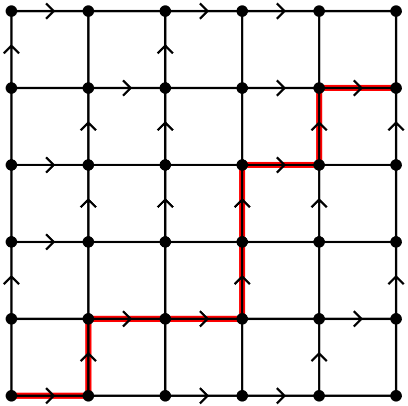
Since S_n is a martingale, stationarity then gives

$$\begin{aligned}\frac{1}{n}E\big(|S_n|^2\big) &= E\big[|f(X_1) - f(X_0) - A_f(X_0)|^2\big)\\ &\leqslant 4E\Big(\big|\chi(\omega,X_1) - \big(\varphi\circ\tau_{X_1}(\omega) - \varphi(\omega)\big)\big|^2\Big)\end{aligned}$$

Optimizing over φ, the r.h.s. tends to zero □

Annealed CLT quite useful in applications

Unfortunately, little can be said about limit law under **quenched law** P_ω^0



We should have been here before lunch, grr!

Back to non-random environments

Significant effort to prove a **Quenched Invariance Principle**; i.e., Functional CLT under P^0_ω for $\mathbb{P}$-a.e. $\omega \in \Omega$ over 20 years. Completed for:

- Uniformly elliptic environments (Sidoravicius & Sznitman 2004)
- Supercritical percolation (Berger & B. 2007, Mathieu & Piatnitski 2007)
- i.i.d. nearest neighbor conductances (Mathieu 2008, B. & Prescott 2008, Barlow& Deuschel 2012, Andres, Barlow, Deuschel, Hambly 2013)
- Under moment conditions for nearest neighbor conductances:

$$\forall x \sim 0\colon\ \ \mathrm{c}(0,x) \in L^p(\mathbb{P}) \quad \text{and} \quad \mathrm{c}(0,x)^{-1} \in L^q(\mathbb{P})$$

  for some $p,q > 1$ with $1/p + 1/q < 2/d$ (Andres, Deuschel, Slowik 2015). Generalized (B., Chen, Kumagai, Wang 2021) to long-range models with

$$\sum_{x\in\mathbb{Z}^d} \mathrm{c}(0,x)|x|^2 \in L^p \quad \text{and} \quad \forall x \sim 0\colon\ \mathrm{c}(0,x)^{-1} \in L^q(\mathbb{P})$$

- For nearest-neighbor models improved to $1/p + 1/q < 2/(d-1)$ by Bella & Schäffner (2020). In $d = 1,2$ enough to have $p = 1$ and $q = 1$ (B. 2011).

Stationarity and ergodicity assumed throughout!

Key new ingredient: elliptic regularity theory

All known proofs of Quenched Invariance Principle (except for n.n. in $d = 1, 2$) use some input from **elliptic regularity** theory.

Usually, this is in the form of **heat-kernel upper bounds**, i.e.,

$$P_\omega^x(X_n = y) \leqslant \frac{c_1}{n^{d/2}} \mathrm{e}^{-c_2|x-y|^2/n}$$

or **Moser iteration** etc.

Key objective: prove **(everywhere) sublinearity of the corrector**

$$\max_{|x|\leqslant n} \frac{|\chi(\cdot, x)|}{n} \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}$$

Known to **fail** unless p, q-condition holds with $1/p + 1/q \leqslant 2/(d-1)$ (n.n) and $1/p + 1/q \leqslant 2/d$ (long range)!

The elliptic regularity input is inherently **deterministic**

The stochasticity of environment enters in two crucial non-constructive steps

- limits using the Spatial Ergodic Theorem
- extraction and control of the corrector

These actually muddle the conclusion!

Indeed, for each translation-invariant, ergodic law ν on $(\Omega, \mathcal{F})$, stochastic homogenization outputs an **inexplicit set** $\Omega_\nu \in \mathcal{F}$ which is **full-measure**, $\nu(\Omega_\nu) = 1$, and such that an invariant principle holds for all $\omega \in \Omega_\nu$.

Issues with this:

- If μ and ν distinct (ergodic), then we could have $\Omega_\mu \cap \Omega_\nu = \emptyset$
- Barring the periodic cases, $\nu(\{\omega\}) = 0$ for each $\omega \in \Omega_\nu$

We thus **cannot claim** an Invariance Principle for any single $\omega \in \Omega$!

We will thus follow a **fully deterministic** approach:

- Impose mixing conditions directly on conductance configurations
- Effectively eliminate the above non-constructive steps by working with **spatial averaging**, rather then ensemble averaging

We will work with non-degenerate nearest-neighbor models, so

$$\Omega := (0,\infty)^{E(\mathbb{Z}^d)}$$

where $E(\Lambda) := \{\text{n.n. edges incident with } \Lambda\}$

$\mathcal{F} :=$ product Borel sigma algebra on Ω

Blocks denoted by $\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d$

As before $\mathrm{c}_\omega(x,y) = \mathrm{c}_\omega(y,x) :=$ coordinate projection of ω on edge (x,y)

Let $C_{\mathrm{loc}}(f)$:= continuous local functions with compact support

Definition

An $\omega \in \Omega$ is said to be **averaging** if the limit

$$\ell_\omega(f) := \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} f \circ \tau_x(\omega)$$

exists for each $f \in C_{\mathrm{loc}}(\Omega)$

Lemma

Denote

$$\Omega' := \Big\{ \omega \in \Omega \colon \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n^d} \sum_{e\in E(\Lambda_n)} 1_{\mathbb{R}\smallsetminus[\epsilon,1/\epsilon]}(c_\omega(e)) = 0 \Big\}.$$

For each averaging $\omega \in \Omega'$, there exists a unique probability measure $\mathbb{P}_\omega$ on $(\Omega, \mathcal{F})$ such that

$$\forall f \in C_{\mathrm{loc}}(\Omega)\colon \; E_{\mathbb{P}_\omega}(f) = \ell_\omega(f).$$

Moreover, $\mathbb{P}_\omega$ is translation invariant, i.e., $\forall x \in \mathbb{Z}^d \colon \mathbb{P}_\omega \circ \tau_x^{-1} = \mathbb{P}_\omega$

Proof: Riesz Representation Theorem

Definition

An averaging $\omega \in \Omega'$ is said to be **ergodic** if $\mathbb{P}_\omega$ is jointly ergodic with respect to the shifts $\{\tau_x : x \in \mathbb{Z}^d\}$ on $(\Omega, \mathcal{F})$

Not a superfluous concept: Take $d = 1$ and let

$$c_\omega(x, x+1) := \begin{cases} 1, & \text{if } |x| \in [n^{3/2}, (n+1)^{3/2}) \text{ for } n \text{ even,} \\ 2, & \text{else,} \end{cases}$$

Then ω is averaging but, since $\mathbb{P}_\omega = \frac{1}{2}(\delta_1 + \delta_2)$, **not ergodic**!

Lemma

The set

$$\Omega^\star := \{\omega \in \Omega' : \text{ averaging and ergodic}\}$$

is $\mathcal{F}$-measurable and translation invariant. We have

$$\nu(\Omega^\star) = 1$$

for each translation-invariant, ergodic probability measure ν on $(\Omega, \mathcal{F})$

Main theorem

For $p, q > 0$ denote

$$\Omega_{p,q} := \left\{ \omega \in \Omega \colon \sup_{n \geqslant 1} \frac{1}{n^d} \sum_{e \in E(\Lambda_n)} c_\omega(e)^p < \infty \wedge \sup_{n \geqslant 1} \frac{1}{n^d} \sum_{e \in E(\Lambda_n)} c_\omega(e)^{-q} < \infty \right\}$$

Theorem (Homogenization of deterministic conductance models)

Let $p, q \in (1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d} \quad \text{if} \quad d \geqslant 2$$

An Invariance Principle then holds under P^0_ω for all $\omega \in \Omega^\star \cap \Omega_{p,q}$. The covariance of the limiting Brownian motion is given for each $v \in \mathbb{R}^d$ by

$$v \cdot Cv = \frac{1}{E_{\mathbb{P}_\omega} \pi(0)} \inf_{\varphi \in C_{\mathrm{loc}}(\Omega)} E_{\mathbb{P}_\omega} \Bigg(\sum_{\substack{x \in \mathbb{Z}^d \\ (0,x) \in E(\mathbb{Z}^d)}} c(0, x) |v \cdot x + \varphi \circ \tau_x - \varphi|^2 \Bigg)$$

We have spatial averages, but for control of X we **need averaging in time**

So our first concern is the proof of:

Theorem (Ergodicity in time)

Let $p, q > 1$ be as above. Then for all $\omega \in \Omega^\star \cap \Omega_{p,q}$ and all $f \in C_b(\Omega)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ \tau_{X_k}(\omega) \underset{n\to\infty}{\longrightarrow} E_{\mathbb{Q}_\omega}(f) \qquad \text{in } L^1(P_\omega^0),$$

where

$$\mathbb{Q}_\omega(\mathrm{d}\omega') := \frac{\pi_{\omega'}(0)}{E_{\mathbb{P}_\omega}\pi(0)}\mathbb{P}_\omega(\mathrm{d}\omega')$$

Let $Y_t := X_{N_t}$ for $N_t :=$ homogeneous Poisson process

Proposition (Andres, Deuschel, Slowik 2016)

Let $p, q > 1$ be as above. Then for each $\omega \in \Omega_{p,q}$ there exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that for all $x, y \in \mathbb{Z}^d$ and all $t \geqslant 0$ with $\sqrt{t} \geqslant c_3(1 + \min\{|x|_1, |y|_1\})$:

(1) *if $|x - y|_1 \leqslant c_4 t$, then*

$$P^x_\omega(Y_t = y) \leqslant c_1 t^{-d/2} \exp\left\{-c_2 \frac{|x-y|_1^2}{t}\right\} \pi_\omega(y),$$

(2) *while if $|x - y|_1 > c_4 t$, then*

$$P^x_\omega(Y_t = y) \leqslant c_1 t^{-d/2} \exp\left\{-c_2 |x-y|_1 \Big(1 + \log\big(1 + \tfrac{|x-y|_1}{\sqrt{t}}\big)\Big)\right\} \pi_\omega(y)$$

Here $|\cdot|_1$ is the ℓ^1-norm on $\mathbb{R}^d$.

Note: $c_1, \dots, c_4$ may depend on ω!

Proof: Elliptic regularity theory (explicitly: Moser iteration)

Lemma (Conversion Lemma)

Let $p, q > 1$ be as above. For each $\omega \in \Omega_{p,q}$ there exists $c(\omega) \in (0,\infty)$ such that for all bounded $f : \mathbb{Z}^d \to [0,\infty)$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} E_\omega^0\big(f(X_k)\big) \leqslant c(\omega) \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} f(x)\pi_\omega(x)$$

Proof: Fix $\epsilon > 0$. Then, roughly,

$$\sum_{k=0}^{n-1} E_\omega^0\big(f(X_k)\big) \leqslant \epsilon n \|f\|_\infty + c_1 \sum_{k=\epsilon n}^{n} \sum_{x\in\mathbb{Z}^d} f(x)\pi_\omega(x)\mathrm{e}^{-c_2|x|^2/n}$$

Now divide by n and take $n \to \infty$ □

Want to show

$$\frac{1}{n}\sum_{k=0}^{n-1} f\circ\tau_{X_k}(\omega) \underset{n\to\infty}{\longrightarrow} E_{\mathbb{Q}_\omega}(f) \qquad \text{in } L^1(P^0_\omega),$$

Suffices to prove this for f with $E_{\mathbb{Q}_\omega}(f) = 0$

Recall that the transition probability of environment chain acts as

$$(\Pi f)(\omega) := \frac{1}{\pi_\omega(0)} \sum_{\substack{x\in\mathbb{Z}^d \\ (0,x)\in E(\mathbb{Z}^d)}} c_\omega(0,x) f\circ\tau_x(\omega)$$

The generator is given by $\mathcal{L} = \Pi - 1$.

We proceed by martingale approximation: Given $\epsilon > 0$, set

$$h_\epsilon(\omega') := \sum_{n\geqslant 0} \frac{1}{(1+\epsilon)^{n+1}} (\Pi^n f)(\omega')$$

Then

$$(\epsilon - \mathcal{L})h_\epsilon(\omega') = f(\omega')$$

Now use this to write …

$$\sum_{k=0}^{n-1} f \circ \tau_{X_k}(\omega) = \sum_{k=0}^{n-1} (\epsilon + 1 - \Pi) h_\epsilon \circ \tau_{X_k}(\omega)$$
$$= \epsilon \sum_{k=0}^{n-1} h_\epsilon \circ \tau_{X_k}(\omega) + \sum_{k=0}^{n-1} \left[h_\epsilon \circ \tau_{X_k}(\omega) - \Pi h_\epsilon \circ \tau_{X_k}(\omega) \right].$$

Since

$$\Pi h_\epsilon \circ \tau_{X_k}(\omega) = E_\omega^0 \left(h_\epsilon \circ \tau_{X_{k+1}}(\omega) \middle| \mathcal{F}_k \right)$$

we have

$$\sum_{k=0}^{n-1} \left[h_\epsilon \circ \tau_{X_k}(\omega) - \Pi h_\epsilon \circ \tau_{X_k}(\omega) \right]$$
$$= h_\epsilon(\omega) - E\left(h_\epsilon \circ \tau_{X_n}(\omega) \middle| \mathcal{F}_{n-1} \right)$$
$$+ \sum_{k=1}^{n} \left[h_\epsilon \circ \tau_{X_k}(\omega) - E\left(h_\epsilon \circ \tau_{X_k}(\omega) \middle| \mathcal{F}_{k-1} \right) \right].$$

As h_ϵ bounded, the first two terms bounded. The sum is a martingale with bounded increments so it's sublinear in n by Azuma-Hoeffding inequality

Using Conversion Lemma, the averaging property of ω (along with $\omega \in \Omega_{p,q}$) and Cauchy-Schwarz,

$$\epsilon \sum_{k=0}^{n-1} h_\epsilon \circ \tau_{X_k}(\omega)$$

is bounded as

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\epsilon h_\epsilon \circ \tau_{X_k}(\omega)\right| \leqslant c(\omega) \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} \left|\epsilon h_\epsilon \circ \tau_x(\omega)\right| \pi_\omega(x)$$

$$= c(\omega) E_{\mathbb{P}_\omega}\big(\pi(0)\epsilon|h_\epsilon|\big) \leqslant c(\omega) E_{\mathbb{P}_\omega}\big(\pi(0)\big) \|\epsilon h_\epsilon\|_{L^2(\mathbb{Q}_\omega)}$$

We are now in a probability space so **back to stochastic homogenization**!

Let $\mu_f :=$ **spectral measure** of $-\mathcal{L}$ on $L^2(\mathbb{Q}_\omega)$. Concentrated on $[0,2]$

Spectral Theorem gives:

$$\|\epsilon h_\epsilon\|^2_{L^2(\mathbb{Q}_\omega)} = \langle \epsilon h_\epsilon, \epsilon h_\epsilon \rangle_{L^2(\mathbb{Q}_\omega)}$$

$$= \big\langle \epsilon(\epsilon - \mathcal{L})^{-1} f, \epsilon(\epsilon - \mathcal{L})^{-1} f \big\rangle_{L^2(\mathbb{Q}_\omega)} = \int_{[0,2]} \Big(\frac{\epsilon}{\epsilon + \lambda}\Big)^2 \mu_f(\mathrm{d}\lambda),$$

Proof, part IV

Bounded Convergence shows

$$\|\epsilon h_\epsilon\|^2_{L^2(\mathbb{Q}_\omega)} \underset{\epsilon\downarrow 0}{\longrightarrow} \mu_f(\{0\})$$

Spatial ergodicity of $\mathbb{P}_\omega$ induces ergodicity of $\mathbb{Q}_\omega$ w.r.t. environment chain.
This implies that $\mathrm{Ker}(\mathcal{L})$ is **just constants!**

But $E_{\mathbb{Q}_\omega}(f) = 0$ means $f \perp \mathrm{Ker}(\mathcal{L})$ so

$$\mu_f(\{0\}) = 0$$

We have thus shown that $E_{\mathbb{Q}_\omega}(f) = 0$ implies

$$\frac{1}{n}\sum_{k=0}^{n-1} f\circ\tau_{X_k}(\omega) \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } L^1(P^0_\omega),$$

Shifting by a constant, this is the claim □

With averaging along the path in hand, we need to do corrector method

- Define approximate corrector via $(\epsilon - \mathcal{L})\chi_\epsilon = V$. Solution:

$$\chi_\epsilon(\omega) := \sum_{n\geqslant 0} \frac{1}{(1+\epsilon)^{n+1}} \Pi^n V(\omega)$$

- Use martingale approximation **trice** in a row

$$\begin{aligned} X_n &= X_0 + \sum_{k=1}^{n} V \circ \tau_{X_{k-1}}(\omega) + M_n \\ &= X_0 + \chi_\epsilon(\omega) - \chi_\epsilon \circ \tau_{X_n} + \sum_{k=1}^{n} \epsilon\chi_\epsilon \circ \tau_{X_{k-1}}(\omega) + \widetilde{M}_n \end{aligned}$$

  and once again, this time with an actual corrector albeit in box Λ_r with Dirichlet b.c. This kills the pesky additive term!

- Take $r := \epsilon^{-1}\sqrt{n}$ and control $n \to \infty$ followed by $\epsilon \downarrow 0$ by homogenization of finite-box Dirichlet energy

Details: arXiv:2303.08382

THANK YOU!