Eigenvalue order statistics for random Schrödinger operators

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Random Schrödinger operator

Transport theory in disordered quantum systems

(Anderson tight-binding) **Hamiltonian** on lattice ($x \in \mathbb{Z}^d$):

$$(H_\xi f)(x) := \sum_{y:\,|y-x|=1} \big[f(y) - f(x)\big] + \xi(x) f(x)$$

In short $H_\xi := \Delta + \xi$.

Objects:

$$\begin{aligned} f(x) &= \text{wave function at site } x \\ \xi(x) &= \text{potential energy at } x \end{aligned}$$

Disordered system: $\xi(x)$ random, i.i.d.

NOTE: Sign convention different from physics.

Main question
Eigenvalue order statistics

Dirichlet eigenvalues: For $D \subset \mathbb{Z}^d$ finite,

$$\lambda_D^{(1)}(\xi) \geq \lambda_D^{(2)}(\xi) \geq \cdots \geq \lambda_D^{(|D|)}(\xi)$$

eigenvalues of H_ξ on functions $f\colon \mathbb{Z}^d \to \mathbb{C}$ with $f := 0$ on D^{c}

Scaled volume: $D_L := (LD) \cap \mathbb{Z}^d$ where $D \subset \mathbb{R}^d$ open

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Question: Are there a_L and b_L such that

$$\Xi_L := \left\{ \frac{1}{b_L}\big(\lambda_{D_L}^{(k)}(\xi) - a_L\big) \colon k = 1, \dots, |D_L| \right\}$$

scales, as $L \to \infty$, to a non-degenerate point process?

Some answers known

$\mathrm{spec}(H_\xi) = [-2d, 0] + \mathrm{supp}\,\xi$

Answer: In localization regime (a_L constant, $b_L := L^d$)

$$\Xi_L \underset{L\to\infty}{\Longrightarrow} \text{homogeneous Poisson point process}$$

Killip-Nakano, Germinet-Klopp ("localization" technology)

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Our interest: **spectral edge**

Some results there as well:

Germinet-Klopp ($d = 1$ or $d > 1$ with long-range Δ)

"Heavy-tailed" potentials: Grenkova-Molchanov-Sudarev,
van der Hofstad-Mörters-Sidorova, König-Locoin-Mörters-Sidorova, Austrauskas

Note: Exponential, even Gaussian tails are heavy!

Motivation

Diffusive dynamics

Parabolic Anderson model (PAM):

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) + \xi(x) u(t,x) \\ u(0,x) = \delta_0(x). \end{cases}$$

Applications:

- chemical kinetics (Zel'dovich *et al*)
- hydrodynamics (Carmona and Molchanov)
- magnetic phenomena (Molchanov and Ruzmaikin)

In probability:

- population dynamics w/ inhomogeneous rates
- Brownian motion among obstacles
- interacting random polymers

Time evolution

"To i or not to i"

Quantum evolution:

$$f(t,x) = \langle \delta_x, \mathrm{e}^{\mathrm{i}tH_\xi} f(0,\cdot)\rangle_{\ell^2(\mathbb{Z}^d)}$$

$f(t,x)$ = wave function of an electron in a disordered metal

Diffusive dynamics:

$$f(t,x) = \langle \delta_x, \mathrm{e}^{tH_\xi} f(0,\cdot)\rangle_{\ell^2(\mathbb{Z}^d)}$$

$f(t,x)$ = density profile in landscape of sources & sinks

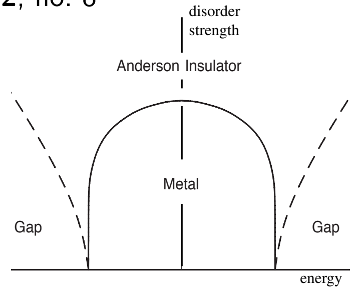
Diffusive case "easier:" Only top of spec(H_ξ) matters

Anderson localization

Nobel prize for physics 1977: Anderson, van Vleck, Mott

A. Lagendijk, B. van Tiggelen, D.S. Wiersma (2009): Fifty years of Anderson localization, *Physics Today* **62**, no. 8

- **Conductor** ⇔ continuous spectrum
  a.k.a. **metal** or **extended** state
- **Insulator** ⇔ discrete spectrum
  a.k.a. **localized** state
- **Mott transition** = line in-between
  also called **mobility edge**



Rigorous work on localization: Fröhlich-Spencer, Martinelli-Scoppola, Simon-Wulff, Aizenman-Molchanov, . . .

Delocalization on Cayley tree: AbouChacra-Anderson-Thouless, Klein, Aizenman-Sims-Warzel, Aizenman-Warzel, . . .

Main result

Assumptions on potential law

Gärtner & Molchanov (1998), Gärtner & den Hollander (1999)

$$\mathrm{Prob}\big(\xi(0) > r\big) = \exp\big\{-\mathrm{e}^{r/\rho}\big\}$$

Definition: We say that ξ is in **doubly-exponential class** if

$$F(r) := \log\log[\mathbb{P}(\xi(0) > r)^{-1}]$$

is C^1 on its domain $(r > \mathrm{essinf}\xi(0))$ and the limit

$$\frac{1}{\rho} := \lim_{r\to\infty} F'(r) \quad \text{exists in } (0,\infty)$$

One out of **4 universality classes** in PAM

Main result

Localization centers of eigenfunctions

$\psi^{(k)}_{D,\xi}$:= k**-th eigenfunction** (real-valued)

$$H_\xi \psi^{(k)}_{D,\xi} = \lambda^{(k)}_D(\xi)\psi^{(k)}_{D,\xi}$$

Localization center:

$$X_k := \operatorname{argmax}_{x\in D} \left|\psi^{(k)}_{D,\xi}(x)\right|$$

Ties resolved using lexicographic order

Ball: $B_R(x) := \{z \in \mathbb{Z}^d \colon |x-z| \le R\}$

Main theorem
Eigenvalue order statistics

Suppose ξ is in doubly-exponential class with parameter ρ. There are $\chi > 0$ and a_L with $a_L = \max_{x \in B_L} \xi(x) - \chi + o(1)$ as $L \to \infty$ such that for any bounded open $D \subset \mathbb{R}^d$:

(1) *for any $R_L \to \infty$ and each $k \geq 1$,*

$$\sum_{z\colon |z - X_k| \leq R_L} \left| \psi^{(k)}_{D_L, \xi}(z) \right|^2 \xrightarrow[L \to \infty]{P} 1$$

(2) *The law of*

$$\left\{ \left(\frac{X_k}{L}, \frac{\log |D_L|}{\rho} \big(\lambda^{(k)}_{D_L}(\xi) - a_L \big) \right) \colon k = 1, \dots, |D_L| \right\}$$

converges weakly to the ranking of a Poisson point process on $D \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda} \mathrm{d}\lambda$.

Uniformized version
Unfolded eigenvalues

Corollary

For each $k \in \mathbb{N}$,

$$\left(e^{-\frac{1}{\rho}(\lambda_{D_L}^{(1)}(\xi)-a_L)\log|D_L|},\dots,e^{-\frac{1}{\rho}(\lambda_{D_L}^{(k)}(\xi)-a_L)\log|D_L|}\right) \xrightarrow[L\to\infty]{\text{law}} (Z_1, Z_1+Z_2,\dots,Z_1+\cdots+Z_k),$$

where $Z_1, Z_2, \dots$ are i.i.d. exponential with parameter one.

Falls under **Gumbel Extreme-Order Class**

Ideas from proof

Setting scales

Note: For purely doubly exponential case:

$$\max_{x\in B_L} \xi(x) = \rho \log\log L^d + o(1), \qquad L \to \infty.$$

Gärtner and Molchanov:

$$\lambda_D^{(1)}(\xi) = \rho \log\log L^d - \chi + o(1)$$

where

$$\chi := -\sup\Big\{\lambda_{\mathbb{Z}^d}^{(1)}(\varphi)\colon \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho} \le 1\Big\}$$

Minimizer(s) "spread" all over $\mathbb{Z}^d$ — doubly-exponential class!

In general, $\rho \log\log L^d$ term gets lower-order corrections.

Ideas from proof
Percolation arguments

Standard ideology:

- Top of spectrum "carried" by spatial regions where ξ large and properly shaped (maximizer of χ)
- Eigenfunctions decay exponentially away from these regions

These ideas underly all proofs of localization at spectral edges, but Green-function/averaging techniques obscure this

Our contribution: We use these ideas throughout all proofs.

Extreme islands

Percolation in action

Fix $D \subset \mathbb{Z}^d$ and for $A > 0$ set

$$U := \bigcup_{\substack{z \in D \\ \xi(z) \geq \lambda_D^{(1)}(\xi) - 2A}} B_R(z) \cap D$$

Proposition

For all $k = 1, 2, \ldots, |U|$ *such that*

$$\lambda_D^{(k)}(\xi) \geq \lambda_D^{(1)}(\xi) - \frac{A}{2},$$

we have

$$\left|\lambda_D^{(k)}(\xi) - \lambda_U^{(k)}(\xi)\right| \leq 2d\left(1 + \frac{A}{2d}\right)^{1-2R}$$

Proof of Proposition

Martingale approximation argument

Lemma

Let (λ, ψ) be eigenvalue pair in $D \subset \mathbb{Z}^d$ and $Y = (Y_0, Y_1, \dots)$ a path of a (discrete-time) SRW. Set

$$\tau := \inf\{k \geq 0\colon \xi(Y_k) \geq \lambda \text{ or } Y_k \notin D\}.$$

Then $M_{\tau\wedge n}$, where $M_0 := \psi(Y_0)$ and

$$M_n := \psi(Y_n) \prod_{k=0}^{n-1} \frac{2d}{2d + \lambda - \xi(Y_k)}, \qquad n \geq 1,$$

is a martingale for the filtration $\mathscr{F}_n := \sigma(Y_0, \dots, Y_n)$.

Proof of Lemma
Key calculation

On $\{\tau > n\}$,

$$E(\psi(Y_{n+1})|Y_1, \dots, Y_n) = \psi(Y_n) + \frac{1}{2d}(\Delta\psi)(Y_n).$$

But $(\Delta + \xi)\psi = \lambda\psi$ and so

$$\begin{aligned} &E(M_{n+1}|Y_1, \dots, Y_n) \\ &\quad = \left[\psi(Y_n) + \frac{1}{2d}(\Delta\psi)(Y_n)\right] \prod_{k=0}^{n} \frac{2d}{2d + \lambda - \xi(Y_k)} \\ &\quad = \psi(Y_n)\left[1 + \frac{1}{2d}(\lambda - \xi(Y_n))\right] \prod_{k=0}^{n} \frac{2d}{2d + \lambda - \xi(Y_k)} = M_n, \end{aligned}$$

Hence, $M_{\tau \wedge n}$ is a martingale. □

Proof of Proposition
Mass outside islands

$$U := \bigcup_{\substack{z \in D \\ \xi(z) \geq \lambda_D^{(1)}(\xi) - 2A}} B_R(z) \cap D$$

Corollary

$$\sum_{x \notin U} |\psi(x)|^2 \leq \left(1 + \frac{A}{2d}\right)^{-2R} \|\psi\|_2^2.$$

Proof: $|M_{\tau \wedge n}|^2$ is submartingale and so

$$|\psi(x)|^2 = E^x |M_0|^2 \leq E^x |M_{\tau \wedge n}|^2$$

But $|M_{\tau \wedge R}| \leq (1 + \frac{A}{2d})^{-R} |\psi(X_{\tau \wedge R})|$ pointwise and

$$\sum_{x \notin U} E^x |\psi(X_{\tau \wedge R})|^2 \leq \|\psi\|_2^2$$

by reversibility of the SRW. □

Proof of Proposition

Deforming potential landscape

$$U := \bigcup_{\substack{z \in D \\ \xi(z) \geq \lambda_D^{(1)}(\xi) - 2A}} B_R(z) \cap D$$

Set $\xi_s(x) := \xi(x) - s\,1_{\{x \notin U\}}$. Then

$$\lambda_D^{(k)}(\xi_\infty) = \lambda_U^{(k)}(\xi), \qquad k = 1, \dots, |U|$$

Now, for a.e. s,

$$\frac{\mathrm{d}}{\mathrm{d}s} \lambda_D^{(k)}(\xi_s) = \sum_{x \notin U} \left| \psi_{D,\xi_s}^{(k)}(x) \right|^2$$

From Corollary:

$$\text{R.H.S.} \leq \left(1 + \frac{A + s}{2d} \right)^{-2R} \|\psi\|_2^2$$

Integrate over s from 0 to ∞ to get the result. □

Coupling to i.i.d. process

Theorem

For each $L \geq 1$ there are random variables $\lambda_1(\xi), \dots, \lambda_{m_L}(\xi)$, with $m_L \to \infty$, and a number $A > 0$ such that

(1) $\lambda_1(\xi), \dots, \lambda_{m_L}(\xi)$ *are i.i.d.*

(2) *If $\hat{\lambda}_1(\xi), \dots, \hat{\lambda}_{m_L}(\xi)$ is a decreasing re-ordering of $\lambda_1(\xi), \dots, \lambda_{m_L}(\xi)$, then for all $k = 1, \dots, m_L$,*

$$\hat{\lambda}_1(\xi) - \hat{\lambda}_k(\xi) < A \quad \Rightarrow \quad |\lambda_{D_L}^{(k)}(\xi) - \hat{\lambda}_k(\xi)| < 2d\left(1 + \frac{A}{2d}\right)^{1-2R_L}$$

In fact, $\lambda_1(\xi), \dots, \lambda_{m_L}(\xi)$ are principal Dirichlet eigenvalues in (disjoint) boxes of intermediate scale

Why principal eigenvalues enough?

Enforced spectral gap

- To get large $\lambda_D^{(1)}(\xi)$, the shape of the potential needs to be close to that of a maximizer of χ
- But then $\lambda_D^{(2)}(\xi)$ is smaller than $\lambda_D^{(1)}(\xi)$ by (essentially) $\rho \log 2$.

Precise version links gap size to **large-deviation** rate function for potential profiles

Bottom line: Only principal (local) eigenvalues can contribute to top of spectrum

Consequences of coupling

Coupling to i.i.d. variables implies:

- order statistics can only be one of three **max-order classes**
- it remains to find a_L and b_L such that, roughly,

$$\mathbb{P}\big(\lambda_{B_R}^{(1)} \geq a_L + sb_L\big) = L^{-d}\mathrm{e}^{-s}\big(1 + o(1)\big)$$

Here $s = 0$ is definition of a_L. For $s \neq 0$, this is a Lemma.

Proof of this Lemma is the only point where double-exp needed

Eigenvector localization
Exponential decay

Deterministic claim: Whenever $|\lambda^{(k)} - \lambda^{(k\pm1)}| > \epsilon_R$

$$|\psi^{(k)}(z)| \le c_1 \mathrm{e}^{-c_2 a_L |z - X_k|}$$

Gap ensured *a posteriori* by scaling limit

Proof based on controlling deformation $\xi \mapsto \xi_s$ outside U.

Bottom line: Minami estimate replaced by existential argument.

THE END