A limit law for the favorite points of random walks on regular trees

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based on joint work with

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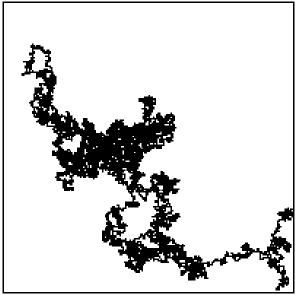
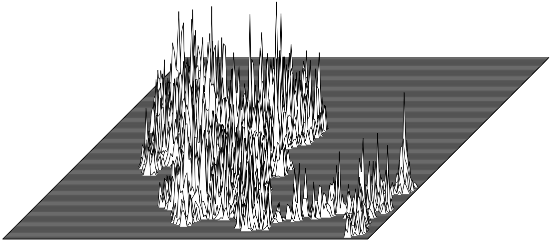
A **favorite** point of a walk of finite time-length is that visited most frequently.

Erdős and Taylor (1960) asked: How much time does the simple random walk on $\mathbb{Z}^d$ of time-length n spend at its favorite point(s)?





Let $X = \{X_n : n \geqslant 0\}$:= **simple random walk** on $\mathbb{Z}^d$, initial value $X_0 := 0$

Time spent at x (a.k.a. **local time**):

$$\ell_n(x) := \sum_{j=0}^{n} 1_{\{X_j = x\}}$$

Erdős and Taylor's question: Find asymptotic behavior (as $n \to \infty$) of

$$M_n := \max_{x \in \mathbb{Z}^d} \ell_n(x)$$

In $d \geqslant 3$ [ET60]: $\frac{1}{\log n} M_n \to c_d \in (0, \infty)$ Maximum of n i.i.d. Geometric(c_d)

In $d = 1$: $M_n/\sqrt{n}$ admits a weak limit. Lévy's theory of Brownian local time

In $d = 2$ [ET60]: With high probability,

$$\frac{1}{\pi}(\log n)^2 \lesssim M_n \lesssim \frac{4}{\pi}(\log n)^2$$

Maximum of n i.i.d. Geometric($c \log n$)

Theorem (Dembo, Peres, Rosen and Zeitouni (2001))

In $d = 2$: $$\frac{1}{(\log n)^2} \max_{x \in \mathbb{Z}^2} \ell_n(x) \xrightarrow[n\to\infty]{P} \frac{4}{\pi}$$

Q: Beyond leading order? Limit law? Other local maxima?

Q: What makes $d = 2$ special? Excursions reach points at scale of domain

A: Local time is **logarithmically correlated**

Setting:

- $G = (V \cup \{\varrho\}, E)$ finite connected graph with $\varrho :=$ **distinguished vertex**
- $X = \{X_t : t \geqslant 0\} :=$ **continuous-time** simple random walk on $V \cup \{\varrho\}$ with unit jump rate across each edge. $P^x :=$ law of X started from x.
- **Local time** (parametrized by actual time)

$$\ell_t(x) := \int_0^t 1_{\{X_s = x\}} \mathrm{d}s$$

- The **first hitting time** of ϱ defined as

$$\tau_\varrho := \inf\{t \geqslant 0 : X_t \in \varrho\}$$

gives rise to the **Green function** via

$$G^V(x,y) := E^x\big(\ell_{\tau_\varrho}(y)\big)$$

Fact: G^V is symmetric & positive definite, so it is the covariance of a centered Gaussian process called **Gaussian Free Field** (zero b.c. at ϱ).

Key idea: Parametrize local time by the time spent at ϱ. Indeed, set

$$T_\varrho(t) := \inf\{s \geqslant 0 \colon \ell_s(\varrho) \geqslant t\} \quad \text{and} \quad L_t(x) := \ell_{T_\varrho(t)}(x)$$

Then for all $t \geqslant 0$ and $x, y \in V \cup \{\varrho\}$:

$$E^\varrho\big(L_t(x)\big) = t$$

and

$$\text{Cov}_{P^\varrho}\big(L_t(x), L_t(y)\big) = 2t\, G^V(x,y)$$

As $t \mapsto L_t(x)$ has **independent increments**, we get:

Corollary (of multivariate CLT)

Let h = GFF (zero b.c. at ϱ). Then

$$\frac{L_t(\cdot) - t}{\sqrt{2t}} \xrightarrow[t\to\infty]{\text{law}} h(\cdot)$$

Theorem (Bramson, Ding, Zeitouni (2016), B.-Louidor (2016))

Let $V_N := (0,N)^2 \cap \mathbb{Z}^2$, $\varrho := \partial V_N$ *(wired boundary),* h = *GFF (zero b.c. at* ϱ*). Set*

$$m_N := 2\sqrt{g}\,\log N - \frac{3}{4}\sqrt{g}\,\log\log N$$

where $g := \frac{2}{\pi}$. *Then for all* $u \in \mathbb{R}$,

$$P\Big(\max_{x\in V_N} h_x \leqslant m_N + u\Big) \underset{N\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\mathcal{Z}\mathrm{e}^{-\alpha u}}\big)$$

In short, $\max_{x\in V_N} h_x - m_N$ *tend in law to* $\alpha^{-1}(\log \mathcal{Z} + G)$ *where* $G :=$ *Gumbel independent of* $\mathcal{Z}$.

Caution(!): Approximation by GFF only good for times $t \gg$ cover time of V.

In transient dimensions ($d \geqslant 3$): power-law decay

$$G^V(x,y) \propto |x-y|^{-a}$$

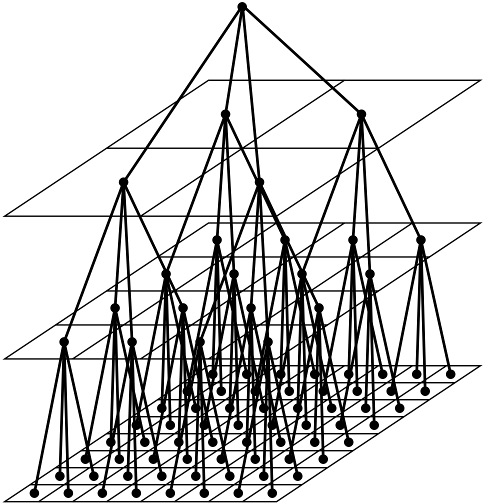
In $d = 2$, we get **logarithmic correlations**

$$G^V(x,y) = g \log \frac{N}{|x-y|+1} + O(1)$$

where $N := \mathrm{diam}(V)$ and $g := \frac{2}{\pi}$

Logarithmic correlations appear for the Green function on **regular trees** where, it turns out, we can answer the above questions.

Random walk on a tree a.k.a. hierarchical lattice



$\mathbb{T}_n$:= **regular rooted tree** of depth n with forward degree $b \geqslant 2$

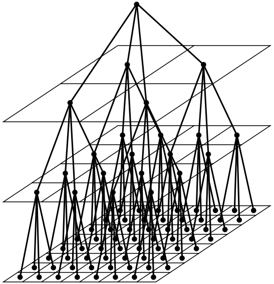
$\mathbb{L}_n$:= **leaf vertices** at depth n

As before: $\{X_t : t \geqslant 0\}$:= continuous-time SRW on $\mathbb{T}_n$,

$$\ell_t(x) := \int_0^t 1_{\{X_s = x\}} \mathrm{d}s$$

and, for ϱ := **root** of $\mathbb{T}_n$,

$$\tau_\varrho := \inf\{t \geqslant 0 : X_t = \varrho\}$$



"Connection" with $\mathbb{Z}^2$:

The random walk started from $x \in \mathbb{L}_n$ and run until τ_ϱ is similar to random walk started at 0 and run until first exit from $(-\frac{1}{2}4^n, \frac{1}{2}4^n) \cap \mathbb{Z}^2$.

Theorem (B.-Louidor 2021)

For any $x_n \in \mathbb{L}_n$ and all $u \in \mathbb{R}$,

$$P^{x_n}\left(\max_{x\in\mathbb{L}_n}\sqrt{\ell_{\tau_\varrho}(x)} \leqslant n\sqrt{\log b} - \frac{1}{\sqrt{\log b}}\log n + u\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left(e^{-\mathcal{Z}e^{-2u\sqrt{\log b}}}\right)$$

where $\mathcal{Z}$ is an a.s.-positive and finite random variable. Thus

$$\frac{1}{n}\left(\max_{x\in\mathbb{L}_n}\ell_{\tau_\varrho}(x) - \left(n^2\log b - 2n\log n\right)\right) \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \log\mathcal{Z} + G$$

where G is a normalized Gumbel random variable independent of $\mathcal{Z}$.

Note: $|\mathbb{L}_n| = b^n$ and so $n^2\log b = \frac{1}{\log b}(\log|\mathbb{L}_n|)^2$ in accord with [ET60]

Theorem (B.-Louidor (2021))

There is $c_\star \in (0,\infty)$ such that $\mathcal{Z}$ from previous theorem obeys

$$c_\star\, b^{-2n} \sum_{x\in\mathbb{L}_n} \Bigl(n\sqrt{\log b} - \sqrt{\ell_{\tau_\varrho}(x)}\,\Bigr)^{+} \ell_{\tau_\varrho}(x)^{1/4}\, \mathrm{e}^{2\sqrt{\log b}\,\sqrt{\ell_{\tau_\varrho}(x)}} \xrightarrow[n\to\infty]{\text{law}} \mathcal{Z}$$

Take-home message:

- $\max_{x\in\mathbb{L}_n} \sqrt{\ell_{\tau_\varrho}(x)}$ centered & scaled tends to a **randomly shifted Gumbel**
- The shift takes a **"derivative martingale"** form

Confirms **emergent universality** among log-correlated fields.

Branching Brownian motion (Bramson 1978, Lalley and Selke 1987, . . .), Branching random walk (Aïdekon 2013, . . .), discrete GFF in $d = 2$ (Bramson, Ding, Zeitouni 2015, . . .), membrane model in $d = 4$ (Schweiger 2020), . . .

Note for experts: 2nd order term in centering sequence **different** from BRW

Recall that $L_t(x) := \ell_{T_\varrho(t)}(x)$ where $T_\varrho(t) := \inf\{s \geqslant 0\colon \ell_s(\varrho) \geqslant t\}$

Tree geometry implies:

Lemma

Set

$$\mathcal{H}_{n,t} := \Big\{\max_{x\in\mathbb{L}_n} L_t(x) > 0\Big\} \overset{\text{a.s.}}{=} \Big\{\mathbb{L}_n \text{ hit before } T_\varrho(t)\Big\}$$

Then for any $x_n \in \mathbb{L}_n$,

$$\max_{x\in\mathbb{L}_n} L_t(x) \text{ under } P^\varrho(\,\cdot\,|\,\mathcal{H}_{n,t}) \;\xrightarrow[t\downarrow 0]{\text{law}}\; \max_{x\in\mathbb{L}_n} \ell_{\tau_\varrho}(x) \text{ under } P^{x_n}$$

So we first control $\max_{x\in\mathbb{L}_n} L_t(x)$ under $P^\varrho(\,\cdot\,|\,\mathcal{H}_{n,t})$

Note: Need a version of Lemma that is uniform in n.

For any $t \geqslant 0$ set

$$Z_n(t) := b^{-2n} \sum_{x \in \mathbb{L}_n} \Big(n\sqrt{\log b} - \sqrt{L_t(x)} \Big)^+ L_t(x)^{1/4} \, \mathrm{e}^{2\sqrt{\log b}\sqrt{L_t(x)}},$$

Note: if leaves not hit by $T_\varrho(t)$, then $Z_n(t) = 0$.

Theorem

For all $t > 0$, we have $Z_n(t) \xrightarrow[n\to\infty]{\text{law}} Z(t)$ where $Z(t)$ is a.s.-finite and non-negative. Moreover,

$$\lim_{n\to\infty} P^\varrho\Big(\max_{x\in\mathbb{L}_n} L_t(x) > 0 \Big) = \mathbb{P}\big(Z(t) > 0\big) \in (0,1)$$

and there is $c_\star \in (0,\infty)$ such that for all $u \in \mathbb{R}$,

$$P^\varrho\Bigg(\max_{x\in\mathbb{L}_n} \sqrt{L_t(x)} \leqslant n\sqrt{\log b} - \frac{1}{\sqrt{\log b}} \log n + u \Bigg) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\Big(\mathrm{e}^{-c_\star Z(t) \mathrm{e}^{-2u\sqrt{\log b}}} \Big)$$

The local time field $\{L_t(x)\colon x \in \mathbb{T}_n\}$ enjoys a **spatial Markov property**.

Notation: Given $x \in \mathbb{L}_k$, let $\mathbb{T}_{n-k}(x) :=$ **subtree** rooted at x.

Lemma (Spatial Markov property)

For any $k = 1, \dots, n-1$, $x \in \mathbb{L}_k$ *and* $u\colon \mathbb{T}_n \to \mathbb{R}$,

$$\{L_t(z)\colon z \in \mathbb{T}_{n-k}(x)\} \text{ under } P^{\varrho}\Big(\cdot \,\Big|\, L_t(\cdot) = u(\cdot) \text{ on } \mathbb{T}_n \smallsetminus \mathbb{T}_{n-k}(x)\Big)$$

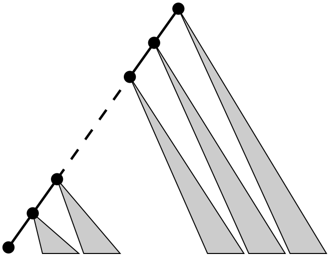
is equidistributed to

$$\{L_{u(x)}(z)\colon z \in \mathbb{T}_{n-k}\} \text{ under } P^{\varrho}$$

This makes local time L_t similar to **Branching random walk** albeit with a Markovian (rather than i.i.d.) step distribution.

Main idea of work on the maximum of BRW (Aïdekon 2013) and extremal process of GFF (B.-Louidor 2016, 2018, 2020):

- **Condition on favorite leaf** to be $x \in \mathbb{L}_n$ and the local time **along the path** from ϱ to x to grow **roughly linearly**.
- Local time in subtrees "hanging off" the path are curbed from above.
- This results in **entropic repulsion** for local time trajectory along the path.



In principle, we can use this as a **bootstrap method**: Assume decay of the upper tail of the maximum to control of lower tail and *vice versa*.

The argument is easier when the upper tail is known:

Lemma (Abe 2018)

For $t > 0$ and $n \geqslant 1$ let

$$a_n(t) := n\sqrt{\log b} - \frac{3}{4\sqrt{\log b}}\log n - \frac{1}{4\sqrt{\log b}}\log\left(\frac{n+\sqrt{t}}{\sqrt{t}}\right)$$

Then

$$P^{\varrho}\left(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)} - \sqrt{t} - a_n(t) \geqslant u\right) \leqslant c(1+u)\mathrm{e}^{-2u\sqrt{\log b}}$$

Proof: Calculations using “barrier estimate” and known facts about law of L_t.

Note: $a_n(t)$ captures the **crossover** between L_t and GFF/BRW

A "barrier calculation" then yields

$$\inf_{t\geqslant 1}\inf_{n\geqslant 1} P^{\varrho}\Big(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}\geqslant\sqrt{t}+a_n(t)\Big)>0$$

This gives **tightness:** Recall $\mathcal{H}_{n,t}:=\{\max_{x\in\mathbb{L}_n}L_t(x)>0\}$. Then

Theorem (Uniform tightness)

There are $c_1,c_2>0$ such that for all $n\geqslant 1$, all $t>0$ and all $u\in[0,n]$,

$$P^{\varrho}\Big(\Big|\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}-\sqrt{t}-a_n(t\vee 1)\Big|>u\,\Big|\,\mathcal{H}_{n,t}\Big)\leqslant c_1\mathrm{e}^{-c_2u}$$

In particular, for each $t>0$, the family

$$\Big\{\text{law of }\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}-\Big(\sqrt{\log b}\,n-\frac{1}{\sqrt{\log b}}\log n\Big)\text{ under }P^{\varrho}(\,\cdot\,|\,\mathcal{H}_{n,t})\Big\}_{n\geqslant 1}$$

of probability measures on $\mathbb{R}$ is tight.

Condition on $L_t(x)$ for $x \in \mathbb{L}_k$ and show that only x with $L_t(x)$ large can support a near-maximal leaf vertex. So it suffices:

Proposition (Sharp upper tail)

There is $c_\star \in (0,\infty)$ such that $o(1) = o_{n,t,u}(1)$ defined for integer $n \geqslant 1$ and real $t > 0$ and $u > 0$ by

$$P^{\varrho}\Big(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)} - \sqrt{t} - a_n(t) > u\Big) = c_\star u e^{-2u\sqrt{\log b}}\big(1+o(1)\big)$$

obeys

$$\lim_{m\to\infty}\sup_{t,u\geqslant m}\limsup_{n\to\infty}\big|o_{n,t,u}(1)\big| = 0.$$

Note: Uniformity allows to study **crossover to GFF asymptotic** as $t \to \infty$

- Location of maximum (for walk started on a leaf).
- Extremal process (governed by a measure $\mathcal{Z}(\mathrm{d}x)$) and cluster distribution (identical to that of GFF/BRW).
- Limit for the time spent at favorite point by SRW on $\mathbb{Z}^2$. (A. Jego already constructed a candidate for limit measure $\mathcal{Z}(\mathrm{d}x)$.)
- Broader perspective of universality. Recent work of
  1. Zeitouni and Schweiger on GFF in random environment
  2. Hofstetter et al on sine-Gordon model and $P(\phi)$ models
  3. my work with a student on DG model
  4. gradient fields by Belius and Wu
  5. etc
- Unified theory of extremal properties of log-correlated processes?