Extremal points of random walks on planar and tree graphs

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based on joint papers with

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Erdős and Taylor (1960): How much time does the simple random walk on $\mathbb{Z}^d$ of time-length n spend at its most favorite point?

Let $X = \{X_n : n \geqslant 0\} :=$ simple random walk on $\mathbb{Z}^d$, initial value $X_0 := 0$
Time spent at x (a.k.a. local time):

$$\ell_n(x) := \sum_{j=0}^{n} 1_{\{X_j = x\}}$$

Question: Asymptotic behavior (as $n \to \infty$) of

$$M_n := \max_{x \in \mathbb{Z}^d} \ell_n(x)$$

In $d \geqslant 3$ [ET60]: $\frac{1}{\log n} M_n \to c_d \in (0, \infty)$
In $d = 2$ [ET60]: With high probability,

$$\frac{1}{\pi}(\log n)^2 \lesssim M_n \lesssim \frac{4}{\pi}(\log n)^2$$

In $d = 1$: $M_n/\sqrt{n}$ admits weak limit. At most 3 maximizers! (Tóth 2001)

Dembo, Peres, Rosen and Zeitouni (2001): In $d = 2$,

$$\frac{1}{(\log n)^2} \max_{x \in \mathbb{Z}^2} \ell_n(x) \xrightarrow[n \to \infty]{P} \frac{4}{\pi}$$

Q: Beyond leading order? Limit law?

Wilf's (1989) "white screen problem": How long does it take for a random walk on a finite graph to visit every vertex?

Confine the SRW to finite $\Lambda \subseteq \mathbb{Z}^d$, e.g., torus $\Lambda_n := [0,n)^d \cap \mathbb{Z}^d$

Cover time:

$$\tau_{\text{cov}}^{(n)} := \inf\Big\{k \geqslant 0\colon \min_{x\in\Lambda_n} \ell_k(x) > 0\Big\}$$

Question: Asymptotic behavior of $\tau_{\text{cov}}^{(n)}$ as $n \to \infty$?

In $d \geqslant 3$ Aldous and Fill book: $\frac{1}{n^2(\log n)^2}\tau_{\text{cov}}^{(n)} \to \tilde{c}_d \in (0,\infty)$

In $d = 2$ Aldous-Fill and Zuckerman (1992):

$$\frac{1}{\pi}n^2(\log n)^2 \lesssim \tau_{\text{cov}}^{(n)} \lesssim \frac{4}{\pi}n^2(\log n)^2$$

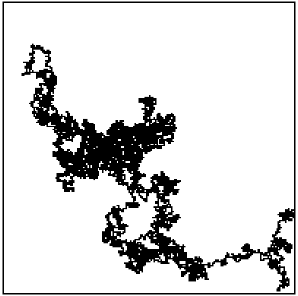
Lawler (1992): $\frac{1}{\pi} \to \frac{2}{\pi}$

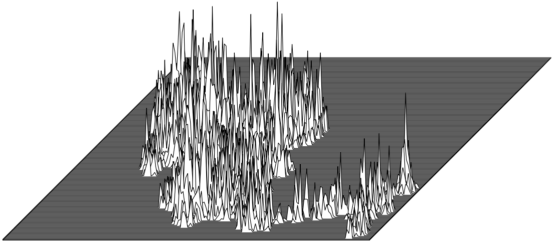
In $d = 1$: $\frac{1}{n^2}\tau_{\text{cov}}^{(n)}$ tight but no deterministic limit

Dembo, Peres, Rosen and Zeitouni (2004): In $d = 2$,

$$\frac{1}{n^2(\log n)^2}\,\tau_{\text{cov}}^{(n)} \xrightarrow[n\to\infty]{P} \frac{4}{\pi}$$

Q: Beyond leading order? Limit law?



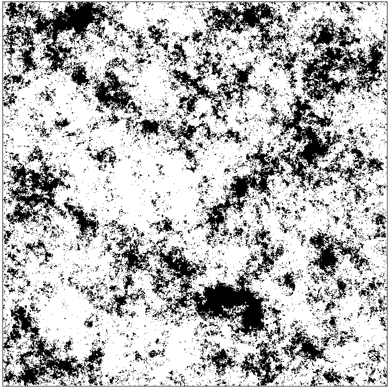


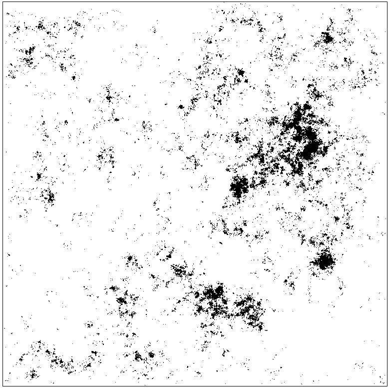
Set of visited sites

Local time profile

Note: Fractal support, local time fractal structure on the support

Points avoided by SRW run up to θ-multiple of the cover time:





$n := 2000$, $\theta := 0.1$ (left) and $\theta := 0.3$ (right)

Dembo, Peres, Rosen and Zeitouni (2006): What does the set of unvisited points look like for random walk on torus?

The set of θ, n-late (to be called "avoided") points:

$$\mathcal{L}_n(\theta) := \Big\{x \in \Lambda_n \colon \ell_{k(n)}(x) = 0 \text{ for } k(n) := \theta\frac{4}{\pi}n^2(\log n)^2\Big\}$$

Then w.h.p. $\mathcal{L}_n(\theta) = \emptyset$ for $\theta > 1$ and, for $\theta \in (0,1)$,

$$\big|\mathcal{L}_n(\theta)\big| = n^{2(1-\theta)+o(1)}$$

where $o(1) \to 0$ in probability

Q: Beyond leading order? Limit law?

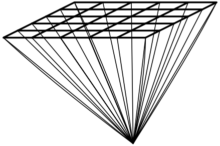
In $d \geqslant 3$: strong coupling of $\mathcal{L}_n(\theta)$ to Bernoulli($n^{-\theta d}$) (Miller and Souzi 2017)

Planar domain: $D \subseteq \mathbb{R}^2$ bounded, open, "nice"

Discretization:

$$D_N := \left\{x \in \mathbb{Z}^2 \colon d_\infty\left(\tfrac{x}{N}, D^{\mathrm{c}}\right) > \tfrac{1}{N}\right\}$$

Return mechanism: "boundary vertex" ϱ



$P^x :=$ law of SRW on $D_N \cup \{\varrho\}$ started from x

Reparametrized local time:

$$\forall t \geqslant 0\colon \quad L_t(x) := \frac{1}{\deg(x)} \ell_{\lfloor t \deg(\overline{D}_N)\rfloor}(x)$$

where $\deg(\overline{D}_N) := \sum_{x \in D_N \cup \{\varrho\}} \deg(x)$.

Note: Random walk is run for actual time $t_N \deg(\overline{D}_N) \asymp N^2 (\log N)^2$

Theorem (Abe-B.-Lee 2019)

There exist random a.s.-finite diffuse Borel measures $\{\mathcal{Z}_\lambda^D : \lambda \in (0,1)\}$ *on* D *charging all non-empty open sets a.s. such that for all* $\theta \in (0,1)$ *and any* $\{t_N\}_{N \geqslant 1}$ *with*

$$t_N \sim 2g\theta(\log N)^2$$

we have

$$\frac{1}{\widehat{W}_N} \sum_{x \in D_N} 1_{\{L_{t_N}(x)=0\}} \, \delta_{x/N} \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \mathcal{Z}_{\sqrt{\theta}}^D$$

where $\widehat{W}_N := N^2 \mathrm{e}^{-\frac{t_N}{g \log N}}$*. Here* $g := \frac{2}{\pi}$*.*

Note: Similar statements true for thick and thin points (with same $\mathcal{Z}_\lambda^D$'s!)

Q: What are the $\mathcal{Z}_\lambda^D$'s?

A: Versions of LQG measures.

Let h = GFF on D with Dirichlet boundary conditions, i.e., mean-zero generalized Gaussian process on D with covariance kernel

$$C(x,y) = -g\log\|x-y\| + g\int_D \log\|z-y\|\,\Pi^D(x,\mathrm{d}z)$$

where $\Pi^D(x,\cdot)$ is harmonic measure on D and $\|\cdot\|$ = Euclidean norm

Need to approximate h by continuous fields: Let $C_n\colon D\times D\to\mathbb{R}$ be s.t.

- C_n symmetric and positive definite
- $\forall x,y\in D\colon\ C_n(x,y)\geqslant 0$
- $x,y\mapsto C_n(x,y)$ continuous
- $\forall x,y\in D\colon\ C(x,y)=\sum_{n\geqslant 0} C_n(x,y)$

Such C_n's exist because C is the Green function.

Let $h_n :=$ mean zero Gaussian process with covariance $\sum_{k=0}^n C_k$ and set

$$Z_\lambda^{D,(n)}(\mathrm{d}x) := \mathrm{e}^{\lambda\alpha h_n(x)-\frac{1}{2}(\lambda\alpha)^2\mathrm{Var}(h_n(x))}\mathrm{d}x$$

where $\alpha := 2/\sqrt{g}$. WLOG: All h_n's are realized on same probability space.

Kahane (1985): For each $\lambda \in (0,1)$ there is an a.s. non-vanishing finite diffuse random Borel measure Z_λ^D on D such that for all Borel $A \subseteq D$,

$$Z_\lambda^{D,(n)}(A) \underset{n\to\infty}{\longrightarrow} Z_\lambda^D(A) \quad \text{a.s.}$$

Moreover, the law of Z_λ^D does not depend on approximation scheme and

$$EZ_\lambda^D(A) = \int_A r^D(x)^{2\lambda^2}\mathrm{d}x$$

for $r^D(x) :=$ conformal radius of D from x.

Duplantier-Sheffield (2011): $Z_\lambda^D =$ LQG measure. Uniqueness: Shamov (2016)
B.-Louidor (2019): Z_λ^D spatial "distribution" of GFF λ-thick points

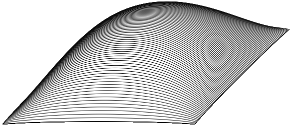
Denote

$$Z_\lambda^{D,0}(\cdot) \overset{\text{law}}{=} \left(Z_\lambda^D(\cdot) \,\Big|\, \int_D h(x)\mathrm{d}x = 0 \right)$$

$$\sigma_D^2 := \int_{D\times D} \mathrm{d}x\mathrm{d}y\, C(x,y)$$

and

$$\mathfrak{d}(x) := \mathrm{Leb}(D) \frac{\int_D \mathrm{d}y\, C(x,y)}{\int_{D\times D} \mathrm{d}z\, \mathrm{d}y\, C(z,y)}$$



Theorem

For each $\lambda \in (0,1)$ *there is* $\mathfrak{c}(\lambda) \in (0,\infty)$ *such that*

$$\mathcal{Z}_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathfrak{c}(\lambda)\mathrm{e}^{\lambda\alpha(\mathfrak{d}(x)-1)Y}\, Z_\lambda^{D,0}(\mathrm{d}x)$$

where $Y \perp\!\!\!\perp Z_\lambda^{D,0}$ *with* $Y = \mathcal{N}(0,\sigma_D^2)$

Jego (2019) proved an analogous result for random walk thick points.

Let $D \subseteq \mathbb{R}^2$ be "nice" bounded domain, D_N its discretization. Assume $0 \in D_N$.

$$\mu_\theta^{D,(N)} := \frac{\log N}{N^{2(1-\theta)}} \sum_{x \in \mathbb{Z}^d} 1_{\{\ell_{\tau^{D_N}}(x) \geqslant 2\theta g (\log N)^2\}} \delta_{x/N}$$

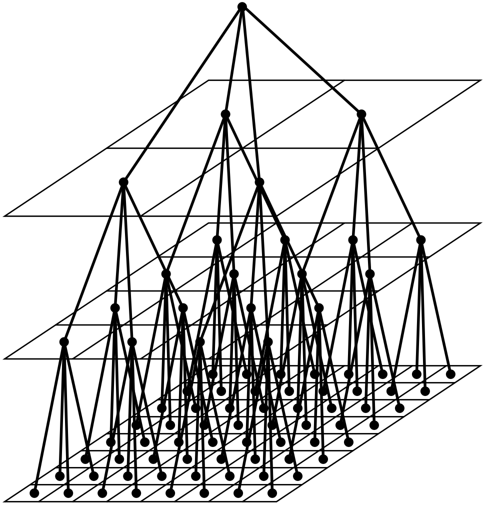
where $\tau^{D_N} := \inf\{k \geqslant 0 \colon X_k \notin D_N\}$. Then, under P^0, for all $\theta \in (0,1)$,

$$\mu_\theta^{D,(N)} \xrightarrow[N\to\infty]{\text{law}} \mu_\theta^D$$

where μ_θ^D is constructed similarly as Z_λ^D albeit with GFF replaced by square-root of Brownian local time. Called: Brownian multiplicative chaos.

Earlier/concurrent constructions of μ_θ^D: Bass, Burdzy and Khoshnevisan (1994), Aïdekon, Hu and Shi (2020)

Key idea of Jego: Characterization of μ_θ^D by a list of natural properties.



Fix $b \geqslant 2$ and let $\mathbb{T}_n :=$ regular rooted tree of depth n with forward degree b. Denote $\mathbb{L}_n :=$ leaf vertices.

Let $\{X_t \colon t \geqslant 0\}$ be continuous-time SRW on $\mathbb{T}_n$ and set

$$\ell_t(x) := \int_0^t 1_{\{X_s = x\}} \mathrm{d}s$$

For $\varrho :=$ root of $\mathbb{T}_n$, let

$$\tau_\varrho := \inf\{t \geqslant 0 \colon X_t = \varrho\}$$

Then: random walk started from $x \in \mathbb{L}_n$ and run until τ_ϱ is similar to random walk started at 0 and run until first exit from D_N.

Cortines, Louidor and Saglietti (2019) and Dembo, Rosen and Zeitouni (2019): scaling limit of cover time of $\mathbb{T}_n$.

Theorem (B.-Louidor 2021)

For any $x_n \in \mathbb{L}_n$ *and all* $u \in \mathbb{R}$,

$$P^{x_n}\left(\max_{x\in\mathbb{L}_n}\sqrt{\ell_{\tau_\varrho}(x)} \leqslant \sqrt{\log b}\,n - \frac{1}{\sqrt{\log b}}\log n + u\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big(\mathrm{e}^{-\mathcal{Z}\mathrm{e}^{-2u\sqrt{\log b}}}\big)$$

where $\mathcal{Z}$ *is an a.s.-positive and finite random variable. Thus*

$$\frac{1}{n}\left(\max_{x\in\mathbb{L}_n}\ell_{\tau_\varrho}(x) - \big(n^2\log b - 2n\log n\big)\right) \xrightarrow[n\to\infty]{\text{law}} \log\mathcal{Z} + G$$

where G *is a normalized Gumbel random variable independent of* $\mathcal{Z}$.

Note: $|\mathbb{L}_n| = b^n$ and so $n^2\log b = \frac{1}{\log b}(\log|\mathbb{L}_n|)^2$ in accord with [ET60]

Theorem (B.-Louidor (2021))

There is $c_\star \in (0,\infty)$ *such that* $\mathcal{Z}$ *from previous theorem obeys*

$$c_\star\, b^{-2n} \sum_{x\in\mathbb{L}_n} \Bigl(n\sqrt{\log b} - \sqrt{\ell_{\tau_\varrho}(x)}\,\Bigr)^+ \ell_{\tau_\varrho}(x)^{1/4}\, \mathrm{e}^{2\sqrt{\log b}\,\sqrt{\ell_{\tau_\varrho}(x)}} \ \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}}\ \mathcal{Z}$$

For random walk started from the root, let

$$\widetilde{\tau}_{\varrho}(t) := \inf\{s \geqslant 0 \colon \ell_s(\varrho) > t\}.$$

and

$$L_t(x) := \ell_{\widetilde{\tau}_{\varrho}(t)}(x)$$

Key facts:

- $\{L_t(x)\}_{x\in\mathbb{T}_n}$ has a Markovian structure
- Law of L_t on a path from leaf to a root explicit

So we first control the scaling limit of $\max_{x\in\mathbb{L}_n} L_t(x)$

For any $t \geqslant 0$ set

$$Z_n(t) := b^{-2n} \sum_{x \in \mathbb{L}_n} \left(n\sqrt{\log b} - \sqrt{L_t(x)} \right)^+ L_t(x)^{1/4} \, \mathrm{e}^{2\sqrt{\log b}\sqrt{L_t(x)}},$$

Note: if leaves not hit by $\widetilde{\tau}_\varrho(t)$, then $Z_n(t) = 0$.

Theorem

For all $t > 0$, we have $Z_n(t) \xrightarrow[n\to\infty]{\text{law}} Z(t)$ where $Z(t)$ is a.s.-finite and non-negative. Moreover,

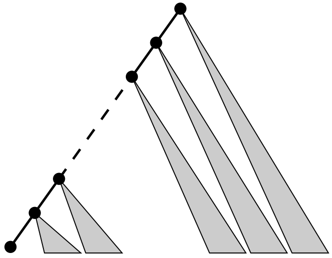
$$\lim_{n\to\infty} P^\varrho\Big(\max_{x \in \mathbb{L}_n} L_t(x) > 0 \Big) = \mathbb{P}\big(Z(t) > 0\big) \in (0,1)$$

and there is $c_\star \in (0,\infty)$ such that for all $u \in \mathbb{R}$,

$$P^\varrho\left(\max_{x \in \mathbb{L}_n} \sqrt{L_t(x)} \leqslant n\sqrt{\log b} - \frac{1}{\sqrt{\log b}} \log n + u \right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\Big(\mathrm{e}^{-c_\star Z(t) \mathrm{e}^{-2u\sqrt{\log b}}} \Big)$$

Main idea — drawn from work on maximum of GFF: Aïdekon 2013, B.-Louidor (2016, 2018), Bramson, Ding and Zeitouni (2016), etc:

- Condition on most favorite leaf to be $x \in \mathbb{L}_n$ and the local time along the path from ϱ to x.
- Local time in subtrees "hanging off" the path are bounded from above.
- This results in entropic repulsion for local time trajectory along the path.



In principle, we can use this as a *bootstrap method*: Assume decay of the upper tail of the maximum to get decay of lower tail and *vice versa*.

The argument is easier when the upper tail is known:

Lemma (Abe 2018)

For $t > 0$ *and* $n \geqslant 1$ *let*

$$a_n(t) := n\sqrt{\log b} - \frac{3}{4\sqrt{\log b}}\log n - \frac{1}{4\sqrt{\log b}}\log\Big(\frac{n+\sqrt{t}}{\sqrt{t}}\Big)$$

Then

$$P^{\varrho}\Big(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)} - \sqrt{t} - a_n(t) \geqslant u\Big) \leqslant c(1+u)\mathrm{e}^{-2u\sqrt{\log b}}$$

Proof: Calculations using "barrier estimate" and explicit law of L_t.

A "barrier calculation" then yields

$$\inf_{t\geqslant 1}\inf_{n\geqslant 1} P^{\varrho}\Big(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}\geqslant\sqrt{t}+a_n(t)\Big)>0$$

This gives tightness: Denote $\mathcal{H}_{n,t}:=\{\max_{x\in\mathbb{L}_n}L_t(x)>0\}$. Then

Theorem (Uniform tightness)

There are $c_1,c_2>0$ such that for all $n\geqslant 1$, all $t>0$ and all $u\in[0,n]$,

$$P^{\varrho}\bigg(\Big|\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}-\sqrt{t}-a_n(t\vee 1)\Big|>u\,\bigg|\,\mathcal{H}_{n,t}\bigg)\leqslant c_1\mathrm{e}^{-c_2u}$$

In particular, for each $t>0$, the family

$$\bigg\{\text{law of }\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)}-\Big(\sqrt{\log b}\,n-\frac{1}{\sqrt{\log b}}\log n\Big)\text{ under }P^{\varrho}(\,\cdot\,|\,\mathcal{H}_{n,t})\bigg\}_{n\geqslant 1}$$

of probability measures on $\mathbb{R}$ is tight.

Condition on $L_t(x)$ for $x \in \mathbb{L}_k$ and show that only x with $L_t(x)$ large can support a near-maximal leaf vertex. So it suffices:

Proposition (Sharp upper tail)

There is $c_\star \in (0,\infty)$ such that $o(1) = o_{n,t,u}(1)$ defined for integer $n \geqslant 1$ and real $t > 0$ and $u > 0$ by

$$P^\varrho\Big(\max_{x\in\mathbb{L}_n}\sqrt{L_t(x)} - \sqrt{t} - a_n(t) > u\Big) = c_\star u \mathrm{e}^{-2u\sqrt{\log b}}\big(1+o(1)\big)$$

obeys

$$\lim_{m\to\infty}\sup_{t,u\geqslant m}\limsup_{n\to\infty}\big|\,o_{n,t,u}(1)\big| = 0.$$

Note: Uniformity allows to study crossover to GFF asymptotic as $t \to \infty$

- Limit for the time spent at most favorite point by SRW on $\mathbb{Z}^2$. (Jego already constructed the limit random variable $\mathcal{Z}$.)
- Scaling limit of the cover time in “planar” domains and torus.

It appears that we are finally closing in on these …

THANK YOU!