Extreme order statistics of logarithmically correlated processes: Part II

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Stochastics Meeting Lunteren

November 14, 2023

Plan of the lecture

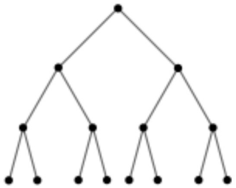
- Some technical proofs: tightness of maximum
- Extremal process of 2D GFF
- Invariant point process via Liggett & Choquet-Deny

Tightness of the maximum

We focus on Branching Random Walk indexed by b-ary tree $\mathbb{T}_n$. Recall:

- $\Lambda_n :=$ leaves of $\mathbb{T}_n$ at depth n
- For $x \in \Lambda_n$, the family history is the root-to-x path $(x_0, \dots, x_n)$
- Given i.i.d. copies $\{Y_x : x \in \mathbb{T}_n\}$ of Y, we have

$$\xi_n = \sum_{x \in \Lambda_n} \delta_{\varphi_x} \quad \text{for} \quad \varphi_x := \sum_{i=1}^{n} Y_{x_i}$$



- Denote $M_n := \max_{x \in \Lambda_n} \varphi_x$

Note: Additivity/scaling allows us to reduce to the case

$$\mathbb{E}(Y) = 0 \quad \text{and} \quad \mathbb{E}(Y^2) = 1$$

We will also assume that Y is continuously distributed with exponential tails.

Tightness of maximum, statement

Theorem

Assuming above setting with $b \geqslant 2$ and Y continuously distributed with exponential tails and subject to $\mathbb{E}(Y) = 0$ and $\mathbb{E}(Y^2) = 1$, let

$$m_n := \sqrt{2 \log b}\, n - \frac{3}{2} \frac{1}{\sqrt{2 \log b}} \log n$$

Then

$$\{M_n - m_n : n \geqslant 1\} \quad \text{is tight}$$

A bad upper bound

A natural instinct is to try to prove an upper by way of union bound:

$$\mathbb{P}(M_n > m_n) = \mathbb{P}\big(\exists x \in \Lambda_n \colon \varphi_x > m_n\big) \leqslant \sum_{x\in\Lambda_n} \mathbb{P}(\varphi_x > m_n)$$

Since φ_x is a random walk with $\mathrm{Var}(\varphi_x) = n$, the Local CLT gives

$$\mathbb{P}(\varphi_x > m_n) \lesssim \frac{c}{\sqrt{n}} \mathrm{e}^{-\frac{1}{2}\frac{1}{n}m_n^2}$$

But

$$\frac{1}{n}m_n^2 = (2\log b)n - 3\log n + o(1)$$

and so

$$\mathbb{P}(\varphi_x > m_n) \leqslant \frac{c'}{\sqrt{n}} \mathrm{e}^{-\frac{1}{2}(2\log b)n + \frac{3}{2}\log n} = c' n b^{-n}$$

As $|\Lambda_n| = b^n$, this will NOT be small when summed over $x \in \Lambda_n$!!!

A better idea:

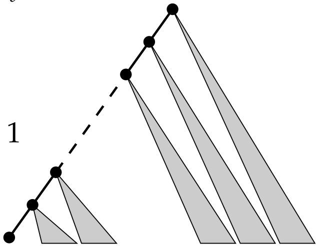
- Condition on the position of the maximum, say x. This x is unique by continuity of the law of Y.
- $\mathbb{T}_n$ then partitions into the path $(x_0, \dots, x_n)$ and a collection of mutilated subtrees $\mathbb{T}'_0, \dots, \mathbb{T}'_{n-1}$ with $\mathbb{T}'_k$ rooted at x_k.
- Introduce the abbreviations

$$S_k := \varphi_{x_k} \quad \text{and} \quad M'_k := \max_{y \in \Lambda_n \cap \mathbb{T}'_k} \varphi_y$$

The uniqueness of the maximum then requires

$$S_k + M'_{n-k} < S_n, \quad k = 0, \dots, n-1$$

Note that S is a random walk.



Relying on the symmetries of $\mathbb{T}_n$, we have proved:

Lemma

Let $\{S_k\}_{k=0}^n$ be a random walk with step distribution Y and let $M'_0, \dots, M'_{n-1}$ be independent with above laws. Then for all Borel $A \subseteq \mathbb{R}$,

$$\mathbb{P}(M_n \in A) = b^n \, \mathbb{P}\left(\{S_n \in A\} \cap \bigcap_{k=0}^{n-1} \{S_k + M'_{n-k} < S_n\} \right)$$

Note: We know that

$$b^n \mathbb{P}(S_n > m_n) = O(n)$$

so we need to extract **additional** $1/n$ **term** from the large intersection.

Want to prove that

$$\mathbb{P}\Big(\{S_n > m_n\} \cap \bigcap_{k=0}^{n-1}\{S_k + M'_{n-k} < S_n\}\Big)$$

is order $1/n$. On the event in question, we will have $S_n \approx m_n$. Without the giant intersection, S_k wants to be $\approx \frac{k}{n}S_n \pm \sqrt{n}$

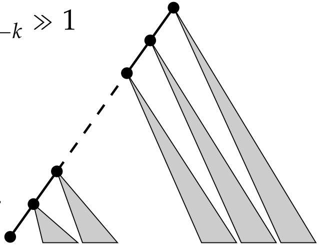
Assuming $M'_{n-k} - m_{n-k}$ tight, the positive fluctuation is out:

$$S_k - \frac{k}{n}S_n \gg \log(n-k) \quad \Rightarrow \quad m_{n-k} - M'_{n-k} \gg 1$$

We thus have to show that, on the event in question,

$$S_k - \frac{k}{n}S_n \lesssim \log(n-k)$$

As tails of M'_{n-k} needed, we need to proceed inductively.



To demonstrate this, we will show:

Lemma

For each $c, \alpha \in (0, \infty)$ *there exists* $C \in (0, \infty)$ *such that the following holds for all* $n \geqslant 1$*: Suppose that*

$$\mathbb{P}(M_k > m_k + u) \leqslant ce^{-\alpha u}$$

hold for all $k = 1, \ldots, n-1$ *and all* $u \geqslant 0$*. Then*

$$\mathbb{P}(M_n > m_n) \geqslant C$$

Note that

$$\mathbb{P}(M'_k > u) \leqslant \mathbb{P}(M_k > u) \leqslant 2\mathbb{P}(M'_k > u)$$

so we may assume the same about M'_k.

Note: Since Y has exponential tails, the assumption holds of $n := 1$.

Abbreviate

$$\theta_n(k) := \min\{k^{1/3}, (n-k)^{1/3}\}$$

Then

$$\mathbb{P}\Bigg(\{S_n > m_n\} \cap \bigcap_{k=0}^{n-1}\{S_k + M'_{n-k} < S_n\}\Bigg)$$
$$\geqslant \mathbb{P}\Bigg(\{S_n > m_n\} \cap \bigcap_{k=0}^{n-1}\{S_k + M'_{n-k} < S_n\} \cap \bigcap_{k=1}^{n-1}\{S_k \leqslant \tfrac{k}{n}S_n - 2\theta_n(k)\}\Bigg)$$

Now $S_k \leqslant \frac{k}{n}S_n - 2\theta_n(k)$ gives $S_n - S_k \geqslant \frac{k}{n}m_n + 2\theta_n(k) \geqslant m_{n-k} + \theta_n(k)$. So

$$M'_{n-k} \leqslant m_{n-k} + \theta_n(k) \quad \Rightarrow \quad S_k + M'_{n-k} < S_n$$

whenever $k = 1, \ldots, n-1$. (The case $k = 0$ must be treated separately.)

Lower bound continued

Using the independence of S and M_k's, this bounds the probability by

$$\mathbb{P}\left(\{S_n > m_n\} \cap \bigcap_{k=1}^{n-1} \{S_k \leqslant \tfrac{k}{n} S_n - 2\theta_n(k)\}\right)$$

times

$$\mathbb{P}(M_n' < m_n) \prod_{k=1}^{n-1} \mathbb{P}\big(M_{n-k}' \leqslant m_{n-k} + \theta_n(k)\big)$$

from below. Induction assumption (upper bound) tells us that the product is positive uniformly in $n \geqslant 1$. In conjunction with Lemma, this shows

$$\frac{\mathbb{P}(M_n > m_n)}{\mathbb{P}(M_n < m_n)} \geqslant c b^n \, \mathbb{P}\left(\{S_n > m_n\} \cap \bigcap_{k=1}^{n-1} \{S_k \leqslant \tfrac{k}{n} S_n - 2\theta_n(k)\}\right)$$

We have reduced the problem to a question about the random walk!

Lemma (Bolthausen)

Let $X_1, \ldots, X_n$ be i.i.d. with positive continuous density. Then

$$\mathbb{P}\left(\bigcap_{k=1}^{n-1}\{X_1 + \cdots + X_k < 0\} \,\middle|\, X_1 + \cdots + X_n = 0\right) = \frac{1}{n}$$

Proof idea: Interpret this as the even that the cyclic path has maximum at "time" zero. Note that the law of X_i does not matter. A more elaborate argument shows

$$\mathbb{P}\left(\{S_n > m_n\} \cap \bigcap_{k=1}^{n-1}\{S_k \leqslant \tfrac{k}{n}S_n - 2\theta_n(k)\}\right) \geqslant \frac{c}{n}\mathbb{P}(S_n > m_n)$$

E.g., for Gaussian random walk lower bound by Brownian motion and calculate (Bramson). Since $\mathbb{P}(S_n > m_n) \geqslant c' b^{-n} n$, we are done. □

The proof proceeds by the following steps:

- Bootstrap $\mathbb{P}(M_n > m_n)$ to exponential lower tail (percolation argument).
- Use another "ballot theorem" argument to treat the upper tail.

One can actually prove a lower bound $\mathbb{P}(M_n > m_n)$ without induction.

For GFF the proof is more complicated due to non-hierarchical structure of covariances. The above representation via path-&-subtrees representation still exist, called **concentric decomposition**.

Further details: M.Biskup. Extrema of the two-dimensional Discrete Gaussian Free Field. 2017 PIMS summer school lecture notes. Springer volume (edited by M. Barlow, G. Slade).

Extrema of GFF

Recall that GFF in U_N is a Gaussian process $\{\varphi_x \colon x \in U_N\}$ with

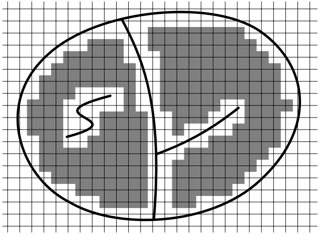
$$\mathbb{E}(\varphi_x) = 0 \quad \text{and} \quad \text{Cov}(\varphi_x, \varphi_y) = G^{U_N}(x,y)/\deg(y)$$

Here U_N is a "good" discretization of U, e.g.,

$$U_N := \left\{x \in \mathbb{Z}^2 \colon \text{dist}_\infty(x/N, U^c) > 1/N\right\}$$

This is needed so that harmonic functions on U_N converge to those in U.

Change of labeling: N is the linear scale of U_N



Tightness of maximum (Bramson & Zeitouni): Denote

$$m_N := 2\sqrt{g}\log N - \frac{3}{4}\sqrt{g}\log\log N$$

where $g := \frac{1}{2\pi}$. Then $\{\max_{x\in U_N}\varphi_x - m_N\}_{N\geqslant 1}$ is tight.

Extreme level set tightness (Ding & Zeitouni): Denote

$$\Gamma_N(t) := \{x \in U_N \colon \varphi_x \geqslant m_N - t\}$$

Then

$$\exists c, C \in (0,\infty)\colon \qquad \lim_{\lambda\to\infty}\liminf_{N\to\infty}\mathbb{P}\bigl(\mathrm{e}^{c\lambda} \leqslant |\Gamma_N(\lambda)| \leqslant \mathrm{e}^{C\lambda}\bigr) = 1$$

and $\exists c > 0$ s.t.

$$\lim_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}\Bigl(\exists x,y\in\Gamma_N(c\log\log r)\colon r\leqslant |x-y|\leqslant N/r\Bigr) = 0$$

Upshot: $\Gamma_N(t)$ consists of islands of $O(1)$ vertices separated by distance $O(N)$.

Full process: Measure η_N on $\overline{U} \times \mathbb{R}$

$$\eta_N := \sum_{x \in U_N} \delta_{x/N} \otimes \delta_{\varphi_x - m_N}$$

Problem: Values in one peak strongly correlated

Local maxima only: Set $B_r(x) := \{z \in \mathbb{Z}^2 \colon |z - x| \leqslant r\}$ and let

$$\eta_{N,r} := \sum_{x \in U_N} 1_{\{\varphi_x = \max_{z \in B_r(x)} \varphi_z\}} \, \delta_{x/N} \otimes \delta_{\varphi_x - m_N}$$

Note: We record both position and value of the field.

Theorem (Convergence to Cox process)

There exists a random Borel measure Z^U *on* $\overline{U}$ *with* $0 < Z^U(\overline{U}) < \infty$ *a.s. such that for any* $r_N \to \infty$ *and* $N/r_N \to \infty$,

$$\eta_{N,r_N} \xrightarrow[N\to\infty]{\text{law}} \mathrm{PPP}\Big(Z^U(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\Big)$$

where $\alpha := 2/\sqrt{g}$ *for* $g := \frac{1}{2\pi}$.

Asymptotic law of maximum: Setting $Z := Z^U(\overline{U})$,

$$\mathbb{P}\big(M_N \leqslant m_N + t\big) \underset{N\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

Joint law position/value: $A \subseteq U$ open, $\widehat{Z}(A) = Z^U(A)/Z^U(\overline{U})$

$$\mathbb{P}\Big(M_N \leqslant m_N + t,\, N^{-1}\mathrm{argmax}\,(\varphi) \in A\Big) \underset{N\to\infty}{\longrightarrow} E\big(\widehat{Z}(A)\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

Recall that $\langle \eta, f \rangle := \int \eta(\mathrm{d}x, \mathrm{d}h) f(x,h)$

Thanks to tightness, $\{\eta_{N,r_N} : N \geqslant 1\}$ tight as sequence of Radon measures. This means that we can extract converging subsequences.

Proposition (Distributional invariance)

Suppose $\eta :=$ a weak-limit point of some $\{\eta_{N_k, r_{N_k}}\}$. Then for any $f : U \times \mathbb{R} \to [0,\infty)$ continuous, compact support,

$$E\big(\mathrm{e}^{-\langle \eta, f \rangle}\big) = E\big(\mathrm{e}^{-\langle \eta, f_t \rangle}\big), \qquad t > 0,$$

where

$$f_t(x,h) := -\log E\big(\mathrm{e}^{-f(x, h + B_t - \frac{\alpha}{2} t)}\big)$$

with $B_t :=$ standard Brownian motion.

We may write

$$\eta = \sum_{i\geqslant 1} \delta_{x_i,h_i}$$

Letting $\{B_t^{(i)}\}$ be independent standard Brownian motions, set

$$\eta_t := \sum_{i\geqslant 1} \delta_{x_i,h_i+B_t^{(i)}-\frac{\alpha}{2}t}$$

Well defined as $t \mapsto |\Gamma_N(t)|$ grows only exponentially. Then

$$E\big(e^{-\langle \eta_t,f\rangle}\big) = E\big(e^{-\langle \eta,f_t\rangle}\big)$$

and so Proposition says

$$\eta_t \overset{\text{law}}{=} \eta, \qquad t > 0$$

Note: x-values do not "move"!

Gaussian interpolation: $\varphi', \varphi'' \overset{\text{law}}{=} h$, independent

$$\varphi \overset{\text{law}}{=} \left(1 - \frac{t}{g \log N}\right)^{1/2} \varphi' + \left(\frac{t}{g \log N}\right)^{1/2} \varphi''$$

Now let x be such that $\varphi'_x \geqslant m_N - \lambda$. Then

$$\begin{aligned} \left(1 - \frac{t}{g \log N}\right)^{1/2} \varphi'_x &= \varphi'_x - \frac{1}{2} \frac{t}{g \log N} \varphi'_x + o(1) \\ &= \varphi'_x - \frac{t}{2} \frac{m_N}{g \log N} + o(1) \\ &= \varphi'_x - \frac{\alpha}{2} t + o(1) \end{aligned}$$

Concerning φ'', denote

$$\tilde{\varphi}''_x := \left(\frac{t}{g\log N}\right)^{1/2} \varphi''_x$$

By properties of G^{U_N} we have

$$\mathrm{Cov}(\tilde{\varphi}''_x, \tilde{\varphi}''_y) = \begin{cases} t + o(1), & \text{if } |x-y| \leqslant r \\ o(1), & \text{if } |x-y| \geqslant N/r \end{cases}$$

So we conclude:

$$\left\{\tilde{\varphi}''_x \colon x \in U_N,\ \varphi'_x \geqslant m_N - \lambda,\ \varphi'_x = \max_{z \in B_r(x)} \varphi'_z\right\}$$

has asymptotically the law of $\{B_t^{(i)}\}$. □

Dysonization-invariant point processes

Key question: Characterize the point processes that are invariant under independent evolution via

$$x \mapsto x + B_t - \frac{\alpha}{2} t$$

of its points? (We may think of this as Dysonization.)

Easy to check: $\mathrm{PPP}(\nu(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h} \mathrm{d}h)$ okay for any ν (even random)

Any other solutions?

Liggett's 1977 derivation

For $t > 0$ define Markov kernel

$$(\mathsf{P}g)(x,h) := E^0 g\big(x, h + B_t - \tfrac{\alpha}{2}t\big)$$

Set $g(x,h) := \mathrm{e}^{-f(x,h)}$ for $f \in C_c^+(U \times \mathbb{R})$. Proposition implies

$$E\big(\mathrm{e}^{-\langle \eta, f\rangle}\big) = E\big(\mathrm{e}^{-\langle \eta, f^{(n)}\rangle}\big)$$

where

$$f^{(n)}(x,h) = -\log(\mathsf{P}^n \mathrm{e}^{-f})(x,h)$$

Kernel P has **uniform dispersivity property:** For $C \subset \overline{U} \times \mathbb{R}$ compact

$$\sup_{x,h} \mathsf{P}^n\big((x,h), C\big) \underset{n\to\infty}{\longrightarrow} 0$$

and thus $\mathsf{P}^n \mathrm{e}^{-f} \to 1$ uniformly. Expanding the log,

$$f^{(n)} \sim 1 - \mathsf{P}^n \mathrm{e}^{-f} \qquad \text{as } n \to \infty$$

Hence

$$E\big(e^{-\langle \eta, f\rangle}\big) = \lim_{n\to\infty} E\big(e^{-\langle \eta, 1-\mathsf{P}^n e^{-f}\rangle}\big) \qquad (*)$$

But, as P is Markov,

$$\langle \eta, 1-\mathsf{P}^n e^{-f}\rangle = \langle \eta\mathsf{P}^n, 1-e^{-f}\rangle$$

$(*)$ shows that $\{\eta\mathsf{P}^n\colon n\geqslant 1\}$ is tight. Along a subsequence

$$\eta\mathsf{P}^{n_k}(dx,dh) \xrightarrow[k\to\infty]{\text{law}} M(dx,dh)$$

and so

$$E\big(e^{-\langle \eta, f\rangle}\big) = E\big(e^{-\langle M, 1-e^{-f}\rangle}\big)$$

i.e., $\eta = \text{PPP}(M(dx,dh))$. Clearly,

$$M\mathsf{P} \overset{\text{law}}{=} M$$

Question: What M can we get?

Theorem (Liggett 1977)

$M\mathsf{P} \overset{\text{law}}{=} M$ implies $M\mathsf{P} = M$ a.s. when P is a kernel of

(1) *an irreducible, recurrent Markov chain*

(2) *a random walk on a closed abelian group w/o proper closed invariant subset*

(2) applies in our case.

Note: $M\mathsf{P} = M$ means M invariant for the chain. Choquet-Deny's Theorem says all positive harmonic functions are exponentials. By reversibility this gives invariant measures:

$$M(\mathrm{d}x, \mathrm{d}h) = Z^U(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h + \tilde{Z}^U(\mathrm{d}x) \otimes \mathrm{d}h$$

Tightness of maximum implies $\tilde{Z}^U = 0$ a.s. □

We thus know $\eta_{N_k,r_{N_k}} \xrightarrow{\text{law}} \eta$ implies

$$\eta = \mathrm{PPP}\big(Z^U(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\big)$$

for some random Z^U. But for $Z := Z^U(\overline{U})$, this reads

$$P\big(M_{N_k} \leqslant m_{N_k} + t\big) \underset{k\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1}Z\mathrm{e}^{-\alpha t}}\big)$$

Hence, the law of $Z^U(\overline{U})$ is unique if limit law of maximum unique. This we know thanks to Bramson, Ding and Zeitouni 2016.

An enhanced version implies uniqueness of law of $Z^U(\mathrm{d}x)$. □

Theorem (B-Louidor 2020)

The measure Z^U satisfies:

(1) $Z^U(A) = 0$ *a.s. for any Borel* $A \subset \overline{U}$ *with* $\mathrm{Leb}(A) = 0$

(2) $\mathrm{supp}(Z^U) = \overline{U}$ *and* $Z^U(\partial D) = 0$ *a.s.*

(3) Z^U *is non-atomic a.s.*

Property (3) is only barely true:

Theorem (B.-Gufler-Louidor 2023?)

Z^U is supported on a set of zero Hausdorff dimension a.s.

Theorem (B-Louidor 2020)

Under a conformal map $f\colon U \to \widetilde{U}$, the measure transforms as

$$Z^{\widetilde{U}} \circ f(\mathrm{d}x) \overset{\text{law}}{=} |f'(x)|^4 Z^U(\mathrm{d}x)$$

Recall we were interested in $\eta_N := \sum_{x\in U_N} \delta_{x/N} \otimes \delta_{\varphi_x - m_N}$

A better representation by **cluster process** on $\overline{U} \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$:

$$\widehat{\eta}_{N,r} := \sum_{x\in U_N} 1_{\{\varphi_x = \max_{z\in B_r(x)} \varphi_z\}}\, \delta_{x/N} \otimes \delta_{\varphi_x - m_N} \otimes \delta_{\{\varphi_x - \varphi_{x+z}\colon z\in\mathbb{Z}^2\}}$$

Theorem (B-Louidor 2018)

There is a measure μ on $\mathbb{Z}^2$ such that (for $r_N \to \infty$, $N/r_N \to \infty$)

$$\widehat{\eta}_{N,r_N} \xrightarrow[N\to\infty]{\text{law}} \text{PPP}\Big(Z^U(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h \otimes \mu(\mathrm{d}\phi)\Big)$$

where $\alpha := 2/\sqrt{g}$.

Capable of capturing **universality** w.r.t. short-range perturbations

THE END