Extreme order statistics of logarithmically correlated processes: Part I

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Plan of the lectures

Part I:

- Extreme value theory: a primer
- Logarithmically correlated processes, examples
- Extremal behavior — randomly shifted Gumbel PPP

Part II:

- Some technical proofs: tightness of maximum
- Invariant point process via Liggett & Choquet-Deny
- Local time of random walks

Extreme value theory: a primer

Let $X_1, X_2, \ldots, X_n$ be i.i.d., $M_n := \max_{i=1,\ldots,n} X_i$.

Q: Is there a limit law for appropriately centered and scaled M_n?

Alternatively, let F = CDF of X. Then

$$\mathbb{P}(M_n \leqslant t) = F(t)^n$$

Q: For what F are there $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ such that $x \mapsto F^n(a_n x + b_n)$ converges to a non-constant limit as $n \to \infty$?

Q: What functions G are limits of $x \mapsto F(a_n x + b_n)^n$ as $n \to \infty$?

Theorem (Fisher-Tippet 1928, Gnedenko 1943)

The set of non-constant limits of $x \mapsto F(a_n x + b_n)^n$ *as* $n \to \infty$ *for some CDF* F *and real-valued* $\{a_n\}_{n \geqslant 1}$ *and general* $\{b_n\}_{n \geqslant 1}$ *takes the form*

$$\{x \mapsto G_\gamma(ax+b) \colon \gamma \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\}$$

where

$$G_\gamma(x) := \begin{cases} \exp\{-(1+\gamma x)^{-\gamma}\} & \text{if } 1+\gamma x > 0, \\ 0 & \text{else,} \end{cases}$$

with the interpretation

$$G_\gamma(x) := \exp\{-e^{-x}\}, \quad x \in \mathbb{R},$$

if $\gamma = 0$.

We call γ the **extreme value index**.

- **Fréchet** $\gamma < 0$. After a shift/scaling:

$$G(x) = \exp\{-(-x)^{-\gamma}\}, \quad x < 0 \quad (:= 0 \text{ else})$$

Domain of attraction includes e.g., Cauchy.

- **Weibull** $\gamma > 0$. After a shift/scaling:

$$G(x) = \exp\{-x^{-\alpha}\}, \quad x > 0 \quad (:= 0 \text{ else})$$

Domain of attraction includes e.g., uniform or lower tail exponential.

- **Gumbel** $\gamma = 0$:

$$G(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}$$

Domain of attraction includes e.g., Gaussian and upper tail exponential.

Unified form Jenkinson 1955, von Mises 1936.

Let $X_1, \ldots, X_n$ and let

$$\widehat{X}_n \geqslant \widehat{X}_{n-1} \geqslant \cdots \geqslant \widehat{X}_1$$

be its decreasing reordering. We already explored $M_n = \widehat{X}_n$.

Q: Given $k \geqslant 0$, is there a joint limit law of appropriately scaled and centered random vector

$$(\widehat{X}_n, \ldots, \widehat{X}_{n-k+1})$$

as $n \to \infty$?

Theorem (Order statistics)

For $F :=$ CDF of X_1, assume $F(a_n \cdot + b_n)^n \to G_\gamma$ for some $\gamma \in \mathbb{R}$. Then for all $k \geqslant 0$,

$$\left(\frac{\widehat{X}_n - b_n}{a_n}, \dots, \frac{\widehat{X}_{n-k+1} - b_n}{a_n}\right) \xrightarrow[n\to\infty]{\text{law}} \left(\frac{Y_1^\gamma - 1}{\gamma}, \frac{(Y_1 + Y_2)^\gamma - 1}{\gamma}, \dots, \frac{(Y_1 + \dots Y_k)^\gamma - 1}{\gamma}\right)$$

where $Y_1, \dots Y_k$ are i.i.d. Exponential(1). (For $\gamma = 0$ replace $u \mapsto \frac{u^\gamma - 1}{\gamma}$ by $\log(u)$.)

Renyi ($\gamma := 1$): $X_1, \dots, X_n$ = Exponential(1)

General case: Analysis of inverse CDFs (see book by de Haan and Ferreira).

An equivalent formulation looks at convergence of empirical counts

$$\left(\sum_{i=1}^{n} 1_{\{X_i > a_n t_1 + b_n\}}, \dots, \sum_{i=1}^{n} 1_{\{X_i > a_n t_k + b_n\}}\right)$$

for all possible and $t_1 > t_2 > \cdots > t_k$. These are encoded via the **empirical (extremal) point process**

$$\eta_n := \sum_{i=1}^{n} \delta_{\frac{X_i - b_n}{a_n}}$$

resp., its integrals

$$\langle \eta_n, f \rangle := \int f \, \mathrm{d}\eta_n = \sum_{i=1}^{n} f\left(\frac{X_i - b_n}{a_n}\right)$$

Key advantage of linearity: multivariate questions handled by

$$(\lambda_1, \dots, \lambda_k) \cdot \left(\sum_{i=1}^{n} 1_{\{X_i > a_n t_1 + b_n\}}, \dots, \sum_{i=1}^{n} 1_{\{X_i > a_n t_k + b_n\}}\right) = \left\langle \eta_n, \sum_{j=1}^{k} \lambda_j 1_{(t_j, \infty)} \right\rangle$$

and the Cramér-Wold device. (Upshot: Law of $\langle \eta_n, f \rangle$ for each f suffices.)

Point process limit

Need to show that $\langle \eta_n, f \rangle$ admits a weak limit for a sufficiently large class of f's. Non-negative continuous f's with compact support suffice.

Theorem (Poisson convergence)

Suppose F is a CDF such that $F(a_n \cdot + b_n) \to G$ for some $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ and some non-constant G. Given i.i.d. r.v.'s $X_1, X_2, \dots$ with CDF F, let

$$\eta_n := \sum_{i=1}^{n} \delta_{\frac{X_i - b_n}{a_n}}$$

Then for all $f : \mathbb{R} \to [0, \infty)$ continuous with $\operatorname{supp}(f)$ bounded from below,

$$\mathbb{E}\big(e^{-\langle \eta_n, f\rangle}\big) \underset{n\to\infty}{\longrightarrow} \exp\left\{-\int (1 - e^{-f})\, d(\log G)\right\}$$

Proof of Theorem: Poissonization

Suppose $\operatorname{supp}(f) \subseteq [t, \infty)$. By i.i.d. property of $X_1, \ldots, X_n$,

$$\begin{aligned}
\mathbb{E}\left(e^{-\langle \eta_n, f\rangle}\right) &= \mathbb{E}\prod_{i=1}^{n} e^{-f((X_i-b_n)/a_n)} = \left[\mathbb{E}\left(e^{-f((X_1-b_n)/a_n)}\right)\right]^n \\
&= \Big[1 - \mathbb{E}\big(\underbrace{1 - e^{-f((X_1-b_n)/a_n)}}_{=\,0 \text{ unless } X_1 \geqslant a_n t + b_n}\big)\Big]^n \\
&= \exp\left\{-\left(1 + o(1)\right) n\mathbb{E}\left(1 - e^{-f((X_1-b_n)/a_n)}\right)\right\}
\end{aligned}$$

By a similar argument,

$$\mathbb{P}(X_1 \leqslant a_n t + b_n)^n \underset{n\to\infty}{\longrightarrow} G(t) \quad \Leftrightarrow \quad n\mathbb{P}(X_1 > a_n t + b_n) \underset{n\to\infty}{\longrightarrow} -\log G(t)$$

Using continuity of f, hence we get

$$n\mathbb{E}\left(1 - e^{-f((X_1-b_n)/a_n)}\right) \underset{n\to\infty}{\longrightarrow} \int \left(1 - e^{-f}\right) d(\log G) \quad \square$$

Consider independent random variables $\{N_j\}_{j\geqslant 1}$ and $\{Y_{j,i}\}_{j,i\geqslant 1}$ with

- $N_j = \text{Poisson}(\lambda_j)$
- $Y_{j,i}$ having distribution μ_j

A naturally associated point process $\xi := \sum_{j\geqslant 1}\sum_{i=1}^{N_j} \delta_{Y_{j,i}}$

Then for $f : \mathbb{R} \to [0,\infty)$ continuous with compact support,

$$\mathbb{E}\big(e^{-\langle \xi, f\rangle}\big) = \mathbb{E}\exp\Big\{-\sum_{j\geqslant 1}\sum_{i=1}^{N_j} f(Y_{j,i})\Big\} = \prod_{j\geqslant 1}\sum_{n\geqslant 0}\frac{\lambda_j^n}{n!}e^{-\lambda_j}\big[\mathbb{E}(e^{-f(Y_{j,1})})\big]^n$$

$$= \prod_{j\geqslant 1}\exp\Big\{-\lambda_j\int(1-e^{-f})d\mu_j\Big\} = \exp\Big\{-\int(1-e^{-f})d\mu\Big\}$$

provided $\mu := \sum_{j\geqslant 1}\lambda_j\mu_j$ is a Radon measure. We call

- μ the **intensity measure**
- ζ a **Poisson point process** with intensity measure μ, or **PPP(μ)** for short.

Since integrals w.r.t. functions in $C_c^+(\mathbb{R})$ determine the random measure, the above Theorem reads

$$\eta_n \xrightarrow[n\to\infty]{\text{law}} \mathrm{PPP}\big(\mathrm{d}(\log G)\big)$$

We have thus shown that, for many random variables X:

- A properly centered and scaled maximum $M_n := \max_{i\leqslant n} X_i$ of n i.i.d. copies $X_1, \ldots, X_n$ of X tends in law to a non-degenerate random variable with CDF G.
- If this happens, the empirical extremal point process $\sum_{i=1}^{n} \delta_{X_i}$ tends in law to Poisson point process with intensity measure $\mathrm{d}(\log G)$.

Q: How much does this generalize to dependent random variables?

Logarithmically correlated processes

We need:

- A metric space $(\mathscr{X}, \varrho)$ of finite $\operatorname{diam}(\mathscr{X}) := \sup_{x,y\in\mathscr{X}} \varrho(x,y)$
- A **random field** on $\mathscr{X}$; i.e., a random function $\varphi\colon \mathscr{X} \to \mathbb{R}$

Definition

We say that φ is **logarithmically correlated** if (assuming $\varphi_x \in L^2$)

$$\operatorname{Cov}(\varphi_x, \varphi_y) = O(1) + g \log\left(\frac{\operatorname{diam}(\mathscr{X})}{1+\varrho(x,y)}\right)$$

where $g :=$ constant and $O(1)$ is bounded locally uniformly on $\mathscr{X} \times \mathscr{X}$

Key features:

- Non-trivial correlations up to the system size
- Differences are tight, $\mathbb{E}((\varphi_x - \varphi_y)^2) = O(1) + g\log(1+\varrho(x,y))$
- If ρ is norm-metric, then φ is approximately **scale invariant**

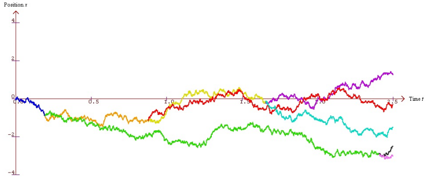
Examples to be discussed:

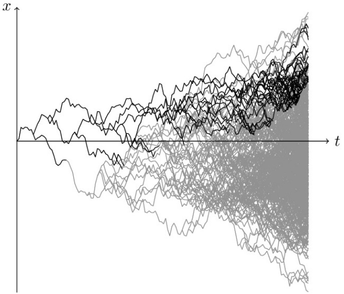
- Branching Brownian Motion
- Branching Random Walk
- Two-dimensional Gaussian Free Field
- Local time of two-dimensional simple random walk (maybe?)

Examples to be ignored:

- Other Gaussian processes (e.g., the membrane model)
- Non-Gaussian measures (“Integer-valued” Gaussian, $P(\phi)$-field theory)
- Characteristic polynomial of a random matrix ensemble
- Riemann’s zeta function on the critical line

The BBM is a process that "moves" as the standard Brownian motion while spawning a copy of itself at exponential rate 1.





Figs: courtesy of M. Roberts and M. Pain

Given a law on (locally finite) point process η, the **BRW with step** η is a sequence $\{\xi_n\}_{n\geqslant 0}$ of random point measures on $\mathbb{R}$ evolving as follows: At each time, we replace each existing point by an independent copy of η.

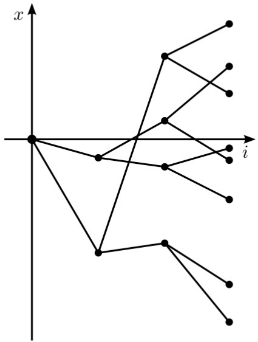


Fig: courtesy Wikipedia

Typically $\xi_0 := \delta_x$ (and typically $x := 0$).

Definition

A **BRW with step** η is a sequence $\{\xi_n\}_{n\geqslant 0}$ of random $\mathbb{N}$-valued Radon measures on $\mathbb{R}$ such that for each $n \geqslant 0$ and each $f \in C_c^+(\mathbb{R})$,

$$\mathbb{E}\big(e^{-\langle \xi_{n+1}, f\rangle} \,\big|\, \mathcal{F}_n\big) = e^{-\langle \xi_n, \tilde f\rangle} \quad \text{where} \quad e^{-\tilde f(x)} := \mathbb{E}\big(e^{-\langle \eta, f(x+\cdot)\rangle}\big)$$

Here $\mathcal{F}_n := \sigma(\xi_0, \ldots, \xi_n)$.

Explanation: Write $\{x_i\}_{i\geqslant 1}$ for the points of ξ_n. Then ξ_{n+1} has points $\{x_i + y_j^{(i)} : i, j \geqslant 1\}$ where $\{y_j^{(i)}\}_{j\geqslant 1}$ enumerates the points of $\eta^{(i)}$ for $\{\eta^{(i)}\}_{i\geqslant 1}$ i.i.d. copies of η. A computation shows

$$\begin{aligned}\mathbb{E}\big(e^{-\langle \xi_{n+1}, f\rangle} \,\big|\, \mathcal{F}_n\big) &= \mathbb{E}\Bigg(\prod_{i\geqslant 1}\prod_{j\geqslant 1} e^{-f(x_i + y_j^{(i)})} \,\Bigg|\, \mathcal{F}_n\Bigg) \\ &= \prod_{i\geqslant 1} \mathbb{E}\big(e^{-\langle \eta, f(x_i+\cdot)\rangle}\big) = \prod_{i\geqslant 1} e^{-\tilde f(x_i)} = e^{-\langle \xi_n, \tilde f\rangle}\end{aligned}$$

(Note: The marginal laws of $\langle \xi_{n+1}, f\rangle$ for f as above determine the law of ξ_{n+1}.)

Suppose for simplicity:

- branching is always into b "children"
- positions are chosen independently

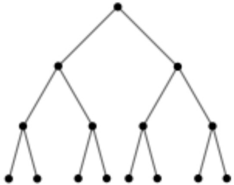
i.e., $\eta = \sum_{i=1}^{b} \delta_{Y_i}$ where $Y_1, \dots, Y_b$ are i.i.d.

The total progeny at "time" n is indexed by vertices of b**-ary rooted tree** $\mathbb{T}_n$ of depth n. Write $\Lambda_n :=$ **leaves at depth** n.

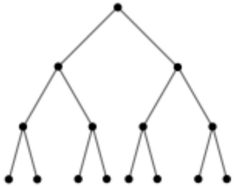
A **family history** of $x \in \Lambda_n$ is the sequence $(x_0, x_1, \dots, x_n)$ of neighbors in $\mathbb{T}_n$ where $x_0 :=$ root and $x_n = x$.

A natural **"scale" invariant metric** on Λ_n is

$$\varrho(x,y) := b^{1-\min\{i=1,\dots,n\colon x_i \neq y_i\}}, \quad x \neq y$$



Corresponds, roughly, to natural embedding of Λ_n to $[0,1]$.



For i.i.d. copies $\{Y_x : x \in \mathbb{T}_n\}$ of Y, we then have

$$\xi_n = \sum_{x \in \Lambda_n} \delta_{\varphi_x} \quad \text{for} \quad \varphi_x := \sum_{i=1}^{n} Y_{x_i}$$

and so $\mathbb{E}(\varphi_x) = n\mathbb{E}(Y)$ and

$$\mathrm{Cov}\big(\varphi_x, \varphi_y\big) = \mathrm{Var}(Y) \log_b\left(\frac{1}{\rho(x,y)}\right), \quad x \neq y$$

In short, the BRW is log-correlated.

Let $\mathcal{G} = (V, E)$ be connected, unoriented, locally-finite graph. Write $\deg(x)$ for degree of x in $\mathcal{G}$. The **simple random walk** on $\mathcal{G}$ is a Markov chain with state space V and transition probability

$$\mathsf{P}(x,y) := \frac{1}{\deg(x)} 1_{(x,y)\in E}$$

Notations:

- $X = \{X_k\}_{k\geqslant 0}$ is a sample path, P^x is the law of X with $P^x(X_0 = x) = 1$ and E^x is associated expectation
- Given a finite set $U \subsetneq V$, the **Green function in U** is

$$G^U(x,y) := E^x\left(\sum_{k=0}^{\tau_U - 1} 1_{\{X_k = y\}}\right)$$

  where $\tau_U := \inf\{k \geqslant 0 \colon X_k \notin U\}$

Definition (GFF)

The **Gaussian Free Field** on U is a Gaussian process $\{\varphi_x \colon x \in U\}$ with

$$\mathbb{E}(\varphi_x) = 0 \quad \text{and} \quad \mathrm{Cov}(\varphi_x, \varphi_y) = \frac{1}{\deg(y)} G^U(x,y), \quad x, y \in U$$

For $\mathcal{G} := \mathbb{Z}^d$ and $U_N := \{x \in \mathbb{Z}^d : x/N \in U\}$ for $U \subseteq \mathbb{R}^d$ a nice open set,

$$G^{U_N}(x,y) = \begin{cases} c_1|x-y| + O(1), & \text{if } d = 1 \\ c_d|x-y|^{2-d} + o(1), & \text{if } d \geqslant 3 \text{ and } |x-y| \gg 1 \\ \log\left(\frac{N}{1+|x-y|}\right) + O(1), & \text{if } d = 2 \end{cases}$$

provided x, y are "deep" inside U_N. (In $d \geqslant 3$ we can take $U_N = \mathbb{Z}^d$.)

Upshot: GFF on $\mathbb{Z}^2$ is logarithmically correlated

It gets event better ...

In $d = 2$ and with $U_N := \{x \in \mathbb{Z}^d \colon \text{dist}_\infty(x/N, \mathbb{R}^d \smallsetminus U) > 1/N\}$ we even get

$$G^{U_N}(x,x) = \frac{1}{2\pi}\log N + \frac{1}{2\pi}\log r^U(x/N) + c_0 + o(1)$$

where $r^U(x) :=$ **conformal radius** of U from x and c_0 is a known constant.

For off-diagonal values we get

$$G^{U_N}(x,y) = \widehat{G}^U\big(x/N, y/N\big) + o(1)$$

where

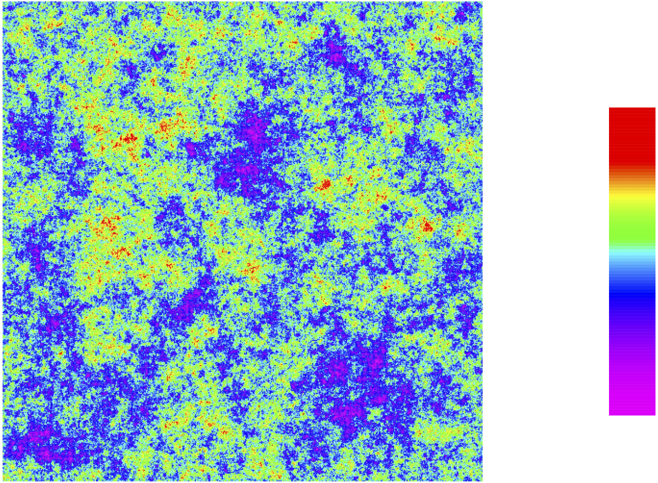
- $\widehat{G}^U$ is a **continuum Green function** in U — solving $\Delta\widehat{G}^U(x,\cdot) = \delta_x(\cdot)$
- $o(1) \to 0$ as $N \to \infty$ uniformly in $|x-y| \geqslant \epsilon N$, for each $\epsilon > 0$

Upshot: GFF is asymptotically (as $N \to \infty$) **conformally invariant**

Key issue: Pointwise limit **does not exist**! (Note that $\text{Var}(\varphi_x) \to \infty$.)

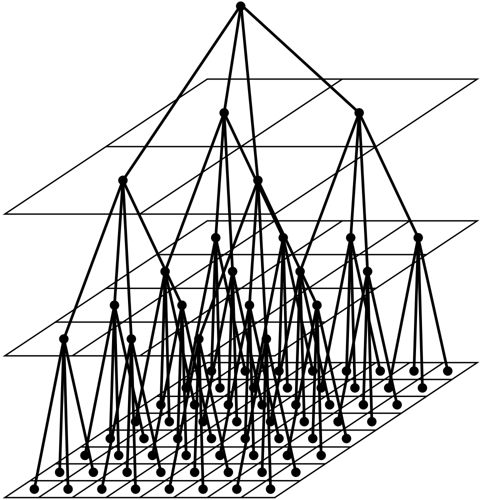
2D Gaussian Free Field — a conformally invariant random fractal

$U_N := (0, N)^d \cap \mathbb{Z}^d$ with $N := 500$



Level lines described by SLE_4-process: Schramm, Sheffield, Miller, etc

GFF is BRW in disguise

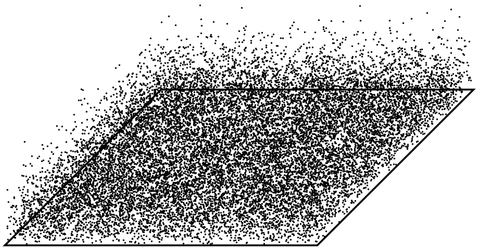


Extremal behavior of log-correlated processes

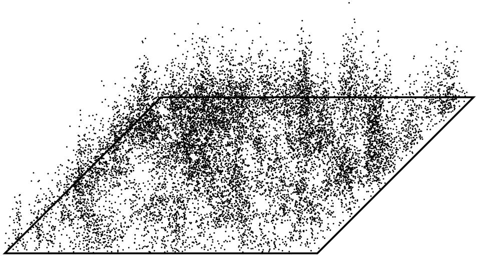
Key observations (based on above examples):

- Local correlations matter significantly — nearby values remain close so no "isolated" local maxima possible.
- Global correlation matter as well albeit by $O(1)$ effects

The PPP limit as for i.i.d. cannot hold but something similar maybe …



i.i.d. normals



GFF

Let $\{\varphi_x : x \in \Lambda_n\}$ be one of the processes:

- Branching Brownian Motion at time n
- Branching Random Walk at time n
- Gaussian Free Field in a box of side-length 2^n

For the BRW, we assume a.s. survival and certain moment conditions.

Denote

$$M_n := \max_{x \in \Lambda_n} \varphi_x$$

Then we have ...

Theorem (Randomly-shifted decorated PPP limit)

For each of the above processes there exist constants $c_1, c_2 \in (0,\infty)$ *such that, setting* $m_n := c_1 n - c_2 \log n$*, the family* $\{M_n - m_n\}_{n\geqslant 1}$ *is tight. Moreover, there exists*

- *an a.s.-positive random variable* Z*,*
- *a constant* $\alpha > 0$*, and*
- *a law* $\mathcal{D}$ *on locally finite upper bounded point processes on* $\mathbb{R}$*,*

such that for all $f \in C_c^+(\mathbb{R})$*,*

$$\mathbb{E}\Bigl(\exp\Bigl\{-\sum_{x\in\Lambda_n} f(\varphi_x - m_n)\Bigr\}\Bigr) \underset{n\to\infty}{\longrightarrow} E\Bigl(\exp\Bigl\{-Z\int \mathrm{e}^{-\alpha h}\mathrm{d}h\otimes\mathcal{D}(\mathrm{d}\chi)\bigl(1-\mathrm{e}^{-\langle\chi, f(h+\cdot)\rangle}\bigr)\Bigr\}\Bigr)$$

where expectation on the right is w.r.t. Z*.*

Credits go to many people, to be given later.

Unified setting — explanation of the limit

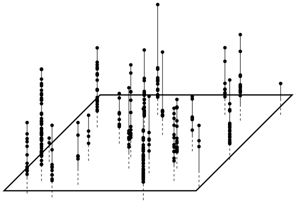
$$\mathbb{E}\big(\mathrm{e}^{-\sum_{x\in\Lambda_n} f(\varphi_x - m_n)}\big) \underset{n\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-Z\int \mathrm{e}^{-\alpha h}\mathrm{d}h\otimes\mathcal{D}(\mathrm{d}\chi)(1-\mathrm{e}^{-\langle\chi, f(h+\cdot)\rangle})}\big)$$

Sample independent objects:

- random variable Z,
- $\{h_i \colon i \geqslant 1\}$:= points of $\mathrm{PPP}(\mathrm{e}^{-\alpha h}\mathrm{d}h)$
- $\{d_j^{(i)} \colon j \geqslant 1\}_{i\geqslant 1}$:= i.i.d. samples from $\mathcal{D}$, called **decorations**

Theorem then says

$$\sum_{x\in\Lambda_n} \delta_{\varphi_x - m_n} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \sum_{i,j\geqslant 1} \delta_{\alpha^{-1}\log Z + h_i + d_j^{(i)}}$$



The quantity $\alpha^{-1}\log Z$ is a **random shift.**

Process on the right called:

Randomly-shifted decorated Poisson point process

As a consequence of above Theorem, we get:

Corollary

Let M_n be the maximum of any of the three processes above. Then there exists a constant $c_\star \in (0,\infty)$ such that

$$\mathbb{P}\big(M_n \leqslant m_n + u\big) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big(\mathrm{e}^{-c_\star Z \mathrm{e}^{-\alpha u}}\big), \quad u \in \mathbb{R}$$

In short, the centered maximum tends to a **randomly shifted Gumbel law**.

Proof: Approximate $\mathbb{P}(M_n \leqslant m_n + u)$ by $\mathbb{E}(\mathrm{e}^{-\langle \eta_n, f\rangle})$ for $f := \lambda 1_{(u,\infty)}$ and take $n \to \infty$ and $\lambda \to \infty$ to represent the limit as

$$E\left(\exp\left\{-Z\int \mathrm{e}^{-\alpha h}\mathrm{d}h \otimes \mathcal{D}(\mathrm{d}\chi) 1_{\{\langle \chi, 1_{(u-h,\infty)}\rangle = 0\}}\right\}\right)$$

As χ is upper bounded, the integral is computed to be $c_\star \mathrm{e}^{-\alpha u}$. □

Branching Brownian Motion:

- McKean (1975), Bramson (1978, 1983), Lalley and Selke (1987)
- Arguin, Bovier, Kistler (2011, 2012)
  Aïdekon, Berestycki, Brunnet, Shi (2013)

Branching Random Walk:

- Kingman (1975), Biggins (1976), Biggins and Kyprianou (2005)
- Addario-Berry, Reed (2009), Aïdekon, Shi (2010)
- Aïdekon (2013), Madule (2017), Maillard, Mallein (2021)

Gaussian Free Field:

- Bothausen, Deuschel, Zeitouni (2011), Bramson, Zeitouni (2012)
- Bramson, Ding, Zeitouni (2016)
- Biskup and Louidor (2016, 2018, 2020)

Ideas & proofs: 2nd lecture (tomorrow 9AM)