# Eigenvalue order statistics for random Schrödinger operators

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# Random Schrödinger operator

## Transport theory in disordered quantum systems

(Anderson tight-binding) **Hamiltonian** on lattice ($x \in \mathbb{Z}^d$):

$$Hf(x) := \sum_{y:\, |y-x|=1} \left[f(y) - f(x)\right] + \xi(x) f(x)$$

Objects:

$$f(x) = \text{wave function at site } x$$
$$\xi(x) = \text{potential energy at } x$$

**Disordered system:** $\xi(x)$ random, i.i.d.

NOTE: Different sign convention from physics.

# Time evolution

## “To i or not to i”

**Quantum evolution:**

$$f(t,x) = \langle \delta_x, \mathrm{e}^{\mathrm{i}tH} f(0,\cdot) \rangle_{\ell^2(\mathbb{Z}^d)}$$

$f(t,x)$ = wave function of an electron in a disordered metal

**Diffusive dynamics:**

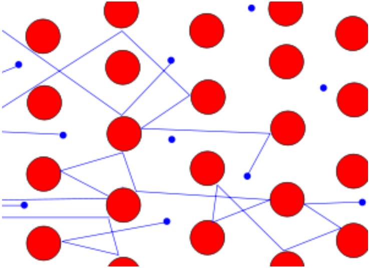
$$f(t,x) = \langle \delta_x, \mathrm{e}^{tH} f(0,\cdot) \rangle_{\ell^2(\mathbb{Z}^d)}$$

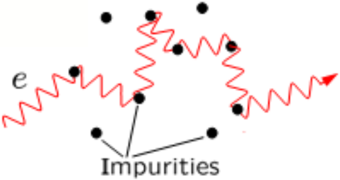
$f(t,x)$ = density profile in landscape of sources & sinks

# Quantum evolution

**Transport theory of metals 101:**

- Paul Drude (1900): mean-free path of an electron
- Felix Bloch (1928): ideal crystals are perfect conductors
- Conclusion: resistivity must come from crystal imperfections
- Phil Anderson (1958): strong disorder —> transport stops



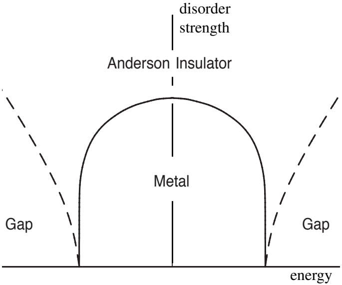


# Anderson localization

Nobel prize for physics 1977: Anderson, van Vleck, Mott

A. Lagendijk, B. van Tiggelen, D.S. Wiersma (2009): Fifty years of Anderson localization, *Physics Today* **62**, no. 8

- **Conductor** $\Leftrightarrow$ continuous spectrum a.k.a. **metal** or **extended** state
- **Insulator** $\Leftrightarrow$ discrete spectrum a.k.a. **localized** state
- **Mott transition** = line in-between also called **mobility edge**




**Rigorous work:** Fröhlich & Spencer (1983), Dreifus & Klein (1989), Aizenman & Molchanov (1993), ...

Delocalization on Cayley tree:

Abou-Chacra & Anderson & Thouless (1973), Klein (1998), Aizenman & Sims & Warzel (2006)

# Diffusive dynamics
## Back to probability

**Parabolic Anderson model (PAM):**

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \Delta u(t,x) + \xi(x) u(t,x) \\ u(0,x) = \delta_0(x). \end{cases}$$

**Applications:**

- chemical kinetics (Zel'dovich *et al*)
- hydrodynamics (Carmona and Molchanov)
- magnetic phenomena (Molchanov and Ruzmaikin)

**In probability:**

- population dynamics w/ inhomogeneous rates
- Brownian motion among obstacles
- interacting random polymers

# Heuristic picture

## Feynman-Kac + spectral analysis

**Probabilistic solution:** $(X_s)_{s\geq 0}$ continuous-time SRW

$$u(t,x) = E^x\left(\exp\left\{\int_0^t \xi(X_s)\mathrm{d}s\right\} 1_{\{X_t=x\}}\right)$$

$\Rightarrow$ path wants to be in regions of large potential

**Functional-analytic solution:** $\{\lambda^{(k)}(\xi)\}_{k\geq 1}$ ordered eigenvalues

$$u(t,x) = \sum_k \mathrm{e}^{t\lambda^{(k)}(\xi)} v_k(x)^\star v_k(0)$$

$\Rightarrow$ nearly maximal eigenvalues are those most desired

**Trade-off:** Benefit of nice region against cost of getting there

# Literature

Gärtner & Molchanov (1990, 1998)
Carmona & Molchanov (1994)
Gärtner & den Hollander (1999)

Donsker & Varadhan, Bolthausen, ..., Fukushima
Sznitman (1998), Antal (1995), Povel, ...

Gärtner & König & Molchanov (2000)
B. & König (2001), van der Hofstad & König & Mörters (2006)

# Localization in PAM

## From Feynman-Kac to eigenvalues

Technically falls in the realm of **large deviation analysis**

**Key issue:** scale at which dominant "islands" distinguished

For this we need:

- Eigenvalue scaling limit
- Eigenvector localization
  - location of "localization center"
  - decay rate of eigenfunction from the "center"

# Heavy tailed case

van der Hofstad & Mörters & Sidorova (2008)
König & Lacoin & Mörters & Sidorova (2009)
Astrauskas (2008, 2009)

Answers are "easy" —> extremes of $\xi$

NOTE: Gaussian or even exponential tails are heavy!

# Doubly exponential tails

Gärtner & Molchanov (1998), Gärtner & den Hollander (1999)

$$\mathrm{Prob}\big(\xi(0) > r\big) = \exp\big\{-\mathrm{e}^{r/\rho}\big\}$$

**Definition:** We say that $\mathbb{P}$ is in **doubly-exponential class** if $\xi(0)$ is continuously distributed and the limit

$$\frac{1}{\rho} := \lim_{r\to\infty} \frac{\log\log\big[\mathbb{P}(\xi(0) > r)^{-1}\big]}{r}$$

exists in $(0, \infty)$.

One out of **4 universality classes**

# Results
## Notation and setup

Background box: $B_L := [-L/2, L/2]^d \cap \mathbb{Z}^d$

Open set $D \subset [-1/2, 1/2]^d$ —> scaled up lattice version

$$D_L := \{x \in \mathbb{Z}^d : x/L \in D\}$$

Ordered eigenvalues of $H$ in $D_L$:

$$\lambda^{(1)}_{D_L}(\xi) \geq \lambda^{(2)}_{D_L}(\xi) \geq \dots$$

Eigenfunctions:

$$v^{(k)}_{D_L,\xi}(x), \qquad k = 1, 2, \dots$$

Continuous distribution: Well-defined almost surely

**Theorem (Eigenvalue order statistics)**

*Suppose $\mathbb{P}$ is in doubly-exponential class with parameter $\rho$. Then for each $L \geq 1$ there are $X_1, X_2, \dots \in B_L(0)$ and $(a_L)$ with*

$$a_L = \max_{x \in B_L} \xi(x) - \chi + o(1), \qquad L \to \infty,$$

*such that: For any open $D \subset [-1/2, 1/2]^d$ and any $R_L \to \infty$,*

$$\sum_{z\colon |z - X_k| \leq R_L} \left| v^{(k)}_{D_L,\xi}(z) \right|^2 \underset{L \to \infty}{\longrightarrow} 1$$

*in probability, for each $k \geq 1$. Moreover, the law of*

$$\left\{ \left( \frac{X_k}{L}, (\lambda^{(k)}_{D_L}(\xi) - a_L) \log L \right) : X_k \in D_L \right\}_{k \geq 1} \tag{1}$$

*converges weakly to the ranking, by the value of the last coordinate, of a Poisson point process on $D \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \kappa \mathrm{e}^{-\kappa\lambda} \mathrm{d}\lambda$ where $\kappa := d/\rho$.*

# Compare with bulk of spectrum

**Localization regime:**
Homogeneous Poisson point process
Killip & Nakano (2007)

**Delocalization regime:** (random matrices)
Dyson's Brownian motions, etc
**Edge:** Tracy-Widom law

# Ideas from proof

## Scales

Note: For purely doubly exponential case:

$$\max_{x\in B_L} \xi(x) = \rho \log\log L^d + o(1), \qquad L \to \infty.$$

Gärtner and Molchanov:

$$\lambda_D^{(1)}(\xi) = \rho \log\log L^d - \chi + o(1)$$

where

$$\chi := -\sup\Big\{\lambda_{\mathbb{Z}^d}^{(1)}(\varphi) \colon \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho} \le 1\Big\}$$

# Extreme islands

Fix $D \subset \mathbb{Z}^d$ and set

$$U := \bigcup_{\substack{z \in D \\ \xi(z) \geq \lambda_D^{(1)}(\xi) - 2A}} B_R(z) \cap D$$

**Proposition**

*For all* $k = 1, 2, \ldots, |U|$ *such that*

$$\lambda_D^{(k)}(\xi) \geq \lambda_D^{(1)}(\xi) - \frac{A}{2}, \tag{2}$$

*we have*

$$\left|\lambda_D^{(k)}(\xi) - \lambda_U^{(k)}(\xi)\right| \leq 2d\left(1 + \frac{A}{2d}\right)^{1-2R} \tag{3}$$

# Martingale approximation argument

**Lemma**

*Let* $(\lambda, v)$ *be eigenvalue pair in* $D \subset \mathbb{Z}^d$ *and* $Y = (Y_0, Y_1, \dots)$ *a path of a (discrete-time) SRW. Set*

$$\tau := \inf\{k \geq 0 \colon \xi(Y_k) \geq \lambda \text{ or } Y_k \notin D\}. \tag{4}$$

*Then* $M_{\tau\wedge n}$*, where* $M_0 = v(Y_0)$ *and*

$$M_n := v(Y_n) \prod_{k=0}^{n-1} \frac{2d}{2d + \lambda - \xi(Y_k)}, \qquad n \geq 1, \tag{5}$$

*is a martingale for the filtration* $\mathscr{F}_n = \sigma(Y_0, \dots, Y_n)$*.*

# Proof of Lemma
## Key calculation

On $\{\tau > n\}$,

$$E(v(Y_{n+1})|Y_1, \dots, Y_n) = v(Y_n) + \frac{1}{2d}(\Delta v)(Y_n).$$

But $(\Delta + \xi)v = \lambda v$ and so

$$\begin{aligned} &E(M_{n+1}|Y_1, \dots, Y_n) \\ &\quad = \left[v(Y_n) + \frac{1}{2d}(\Delta v)(Y_n)\right] \prod_{k=0}^{n} \frac{2d}{2d + \lambda - \xi(Y_k)} \\ &\quad = v(Y_n)\left[1 + \frac{1}{2d}\left(\lambda - \xi(Y_n)\right)\right] \prod_{k=0}^{n} \frac{2d}{2d + \lambda - \xi(Y_k)} = M_n, \end{aligned}$$

Hence, $M_{\tau \wedge n}$ is a martingale.

# Mass outside islands

**Corollary**

$$\sum_{x \notin U} \left|v(x)\right|^2 \leq \left(1 + \frac{A}{2d}\right)^{-2R} \|v\|_2^2.$$

**Proof.**
(Sketch)

$$\left|v(x)\right|^2 \leq E^x |M_{\tau \wedge n}|^2$$

But $|M_{\tau \wedge R}| \leq (1 + \frac{A}{2d})^{-R} |v(X_{\tau \wedge R})|$ pointwise. □

# Deforming the landscape

## Proof of Proposition

Set $\xi_s(x) := \xi(x) - s\,1_{\{x \notin U\}}$. Then

$$\lambda_D^{(k)}(\xi_\infty) = \lambda_U^{(k)}(\xi), \qquad k = 1, \dots, |U|$$

Now, for a.e. $s$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\lambda_D^{(k)}(\xi_s) = \sum_{x \notin U} \left| v_{D,\xi_s}^{(k)}(x) \right|^2$$

From Corollary:

$$\text{R.H.S.} \le \left(1 + \frac{A+s}{2d}\right)^{-2R} \|v\|_2^2$$

Integrate over $s$ from 0 to $\infty$ to get the result.

# Further items

Above ensures eigenvalues can be coupled to i.i.d. RVs
$\Rightarrow$ order statistics can only be one of three **max-order classes**
$\Rightarrow$ need to find $a_L$ and $b_L$ such that, roughly,

$$\mathbb{P}\big(\lambda_{B_R}^{(1)} \geq a_L + sb_L\big) = L^{-d}\mathrm{e}^{-s}\big(1+o(1)\big)$$

Here $s=0$ is definition of $a_L$. For $s \neq 0$, this is a Lemma.

Proof of this Lemma is the only point where double-exp needed

**Eigenvector localization:** Whenever $\big|\lambda^{(k)} - \lambda^{(k\pm1)}\big| > \epsilon_R$

$$\big|v^{(k)}(z)\big| \leq c_1\mathrm{e}^{-c_2 a_L|z-X_k|}$$

**Gap** ensured by scaling limit (replaces Minami estimate).

# Remarks

Spectral control so strong that we can avoid Feynman-Kac

Preliminary calculations suggest that

**Max-order class is always Gumbel**

THE END