Proofs of phase transitions by comparison to mean-field theory

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joint work with

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Nearest-neighbor ferromagnets

$\mathbf{S}_x$ = spin at $x \in \mathbb{Z}^d$, $\mathbf{S}_x \in \Omega \subset \mathbb{R}^\nu$

Hamiltonian: $\Lambda \subset \mathbb{Z}^d$

$$H_\Lambda(\mathbf{S}) = -\frac{1}{2d} \sum_{\substack{\langle x,y \rangle \\ x,y \in \Lambda}} \mathbf{S}_x \cdot \mathbf{S}_y$$

Minus sign = ferromagnet

Gibbs measure:

$$\mu_\Lambda(\mathrm{d}\mathbf{S}) = \frac{\mathrm{e}^{-\beta H_\Lambda(\mathbf{S})}}{Z_\Lambda} \prod_{x \in \Lambda} \mu_0(\mathrm{d}\mathbf{S}_x)$$

μ_0 = *a priori* measure on spins (i.i.d.)

Simple examples

(1) *Ising model*: $\sigma_x \in \Omega = \{-1, +1\} \subset \mathbb{R}$

$$H(\sigma) = -\sum_{\langle x,y\rangle} \sigma_x \sigma_y$$

(2) *The $O(N)$-model*: $\mathbf{S}_x \in \Omega = \{z \in \mathbb{R}^N : |z| = 1\} \subset \mathbb{R}^N$

$$H(\mathbf{S}) = -\sum_{\langle x,y\rangle} \mathbf{S}_x \cdot \mathbf{S}_y$$

Both cases: μ_0 = uniform on Ω

Another example

(3) *Potts model* :

standard form: $\sigma_x \in \{1, \dots, q\}$

$$H(\sigma) = -\sum_{\langle x,y \rangle} \delta_{\sigma_x \sigma_y}$$

vector representation:

$\Omega = \{\hat{v}_1, \dots, \hat{v}_q\} \subset \mathbb{R}^{q-1}$

$\hat{v}_i$ = vertices of a simplex in $\mathbb{R}^{q-1}$

$$\hat{v}_i \cdot \hat{v}_j = \frac{q-1}{q} \times \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{q-1} & \text{o/w} \end{cases}$$

μ_0 = uniform

Yet another example

(4) *Nematic liquid crystal*:

standard form: $\mathbf{u}_x \in \{z \in \mathbb{R}^N : |z| = 1\}$

$$H(\mathbf{u}) = -\sum_{\langle x,y\rangle} (\mathbf{u}_x \cdot \mathbf{u}_y)^2$$

matrix representation: $Q = (Q_{ij}) \leftrightarrow \mathbf{u} \in \mathbb{R}^N$ ($|\mathbf{u}| = 1$)

$$Q_{ij} = u_i u_j - \frac{1}{N}\delta_{ij} \qquad 1 \leq i,j \leq N$$

If $Q \leftrightarrow \mathbf{u}$ and $\tilde{Q} \leftrightarrow \mathbf{v}$ then

$$(\mathbf{u} \cdot \mathbf{v})^2 = \mathrm{Tr}(Q\tilde{Q}) + \frac{1}{N}$$

Ω = set of such matrices, $\Omega \subset \mathbb{R}^{N^2}$
inner product $Q \cdot \tilde{Q} = \mathrm{Tr}(Q\tilde{Q}^{\mathsf{T}})$

Mean-field theory

MFE for magnetization

Magnetization: $\mathbf{m} = E_\mu(\mathbf{S}_0)$

DLR equation:

$$E_\mu(\mathbf{S}_0) = E_\mu\left(\frac{E_{\mu_0}(\mathbf{S}\,\mathrm{e}^{\beta \mathbf{M}_0\cdot\mathbf{S}})}{E_{\mu_0}(\mathrm{e}^{\beta \mathbf{M}_0\cdot\mathbf{S}})}\right)$$

where $\mathbf{M}_0 = \frac{1}{2d}\sum_{x\sim 0}\mathbf{S}_x$

Assume that $\mathbf{M}_0$ is concentrated: $\mathbf{M}_0 \approx \mathbf{m}$

Mean-field equation for magnetization:

$$\mathbf{m} = \frac{E_{\mu_0}(\mathbf{S}\,\mathrm{e}^{\beta \mathbf{m}\cdot\mathbf{S}})}{E_{\mu_0}(\mathrm{e}^{\beta \mathbf{m}\cdot\mathbf{S}})}$$

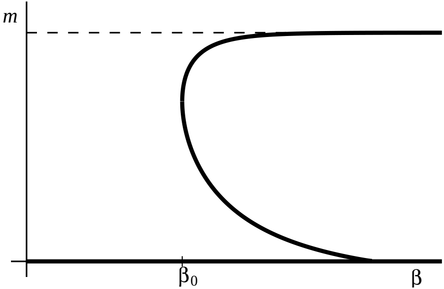
Typical situation
q-state Potts model ($q \geq 3$)

Scalar magnetization: $\mathbf{m} = m\hat{v}_j$

Mean-field equation:

$$m = \frac{e^{\beta m} - 1}{e^{\beta m} + q - 1}$$

Solutions:



Ambiguity if more than one solution

Mean-field free energy
Model on complete graph

Cumulant generating function:

$$G(\mathbf{h}) = \log \int_{\Omega} \mu_0(\mathrm{d}\mathbf{S})\, \mathrm{e}^{\mathbf{S}\cdot\mathbf{h}}$$

Entropy:

$$S(\mathbf{m}) = \inf_{\mathbf{h}} \big[G(\mathbf{h}) - \mathbf{h}\cdot\mathbf{m} \big]$$

Free energy function:

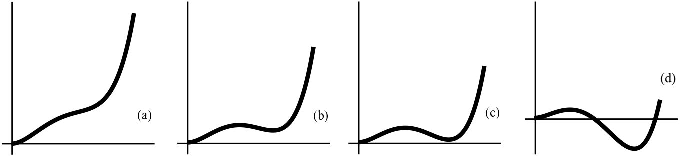
$$\Phi_\beta(\mathbf{m}) = -\frac{\beta}{2}|\mathbf{m}|^2 - S(\mathbf{m})$$

Lemma 1

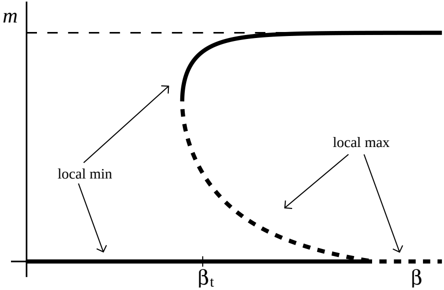
$$\nabla\Phi_\beta(\mathbf{m}) = 0 \;\Leftrightarrow\; \mathbf{m} = \nabla G(\beta\mathbf{m}) \;\Leftrightarrow\; \mathbf{m} \textit{ solves MFE}$$

Typical situation continued

Free energy $m \mapsto \Phi_\beta(m\hat{v}_1)$ as β increases:



Magnetization picture revised: $(\beta_t \neq \beta_0)$



Main result

Fourier transform of lattice Laplacian:

$$\hat{D}(k) = 1 - \tfrac{1}{d}\sum_{j=1}^{d} \cos(k_j)$$

Theorem 2 (Physical magnetization nearly minimizes Φ_β)

Let $\mathbf{m}_\star$ be a value of magnetization in an extremal translation invariant Gibbs state. Then

$$\Phi_\beta(\mathbf{m}_\star) \le \inf_{\mathbf{m}} \Phi_\beta(\mathbf{m}) + c\beta\mathcal{I}_d$$

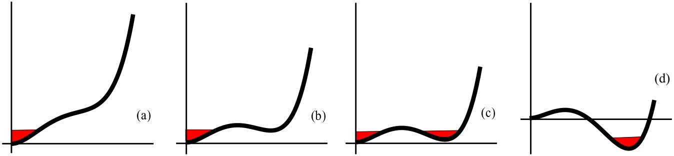
where c is a constant (depending only on Ω) and

$$\mathcal{I}_d = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{[1-\hat{D}(k)]^2}{\hat{D}(k)}$$

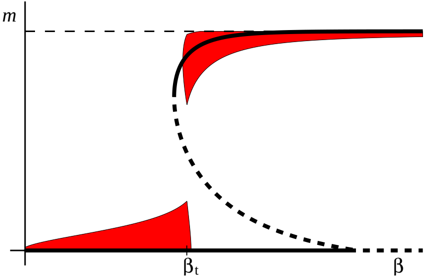
Note: $\mathcal{I}_d < \infty$ for $d \ge 3$ and $\mathcal{I}_d \sim \frac{1}{2d}$ as $d \to \infty$

Potts model revisited

Free energy “landscape” as β increases:



Allowed values of *physical* magnetization:



Implies *discontinuous* phase transition!

Proof: Convexity inequality

Lemma 3

Let μ be a translation & rotation invariant Gibbs state. Let $\mathbf{m}_\star = E_\mu(\mathbf{S}_0)$. Then:

$$\Phi_\beta(\mathbf{m}_\star) \le \inf_{\mathbf{m}} \Phi_\beta(\mathbf{m}) + \frac{\beta}{2}\left[E_\mu\left(\mathbf{S}_0 \cdot \mathbf{S}_x\right) - |\mathbf{m}_\star|^2\right]$$

Sketch of proof.

Jensen's inequality: $Z_\Lambda \ge \exp\left\{-|\Lambda| \inf_{\mathbf{m}} \Phi_\beta(\mathbf{m}) + O(\partial\Lambda)\right\}$. From

$$e^{|\Lambda| G(\mathbf{h})} = E_\mu\left(e^{\mathbf{h}\cdot M_\Lambda + \beta H_\Lambda} Z_\Lambda\right)$$

we get

$$|\Lambda| G(\mathbf{h}) \ge |\Lambda|(\mathbf{h}\cdot\mathbf{m}_\star) + \beta E_\mu(H_\Lambda) - |\Lambda| \inf_{\mathbf{m}} \Phi_\beta(\mathbf{m}) + O(\partial\Lambda)$$

Now divide by $|\Lambda|$, let $\Lambda \uparrow \mathbb{Z}^d$ and optimize over $\mathbf{h}$. $\square$

Proof: Principal ingredient

Infrared bound: (Dyson, Fröhlich, Lieb, Simon, Spencer)

$$\sum_{x,y\in\mathbb{Z}^d} v_x v_y E_\mu\big((\mathbf{S}_x - \mathbf{m}_\star)\cdot(\mathbf{S}_y - \mathbf{m}_\star)\big) \le \frac{\nu}{\beta}\sum_{x,y\in\mathbb{Z}^d} v_x v_y D^{-1}(x-y)$$

Choosing $v_x = \frac{1}{2d}$ for $x \sim 0$ and $v_x = 0$ o/w yields

Lemma 4 (Key estimate)

$$E_\mu\left|\frac{1}{2d}\sum_{x\sim 0}\mathbf{S}_x - \mathbf{m}_\star\right|^2 \le \frac{\nu}{\beta}\mathcal{I}_d$$

This proves the main “assumption” of mean-field theory!

Problem: Only torus states obey IRB—need to approximate

Proof: Energy concentration

Lemma 5

Let μ be an extremal translation & rotation invariant Gibbs state. Let $\mathbf{m}_\star = E_\mu(\mathbf{S}_0)$. Then

$$0 \le E_\mu\big(\mathbf{S}_0 \cdot \mathbf{S}_x\big) - |\mathbf{m}_\star|^2 \le c\,\nu\mathcal{I}_d$$

Sketch of proof.

Symmetrize & apply DLR:

$$E_\mu\big(\mathbf{S}_x \cdot \mathbf{S}_0\big) = E_\mu\big(\mathbf{M}_0 \cdot \mathbf{S}_0\big) = E_\mu\big(\mathbf{M}_0 \cdot \nabla G(\beta\mathbf{M}_0)\big)$$

$$E_\mu\big(\mathbf{S}_0 \cdot \mathbf{S}_x\big) - |\mathbf{m}_\star|^2 = E_\mu\Big[(\mathbf{M}_0 - \mathbf{m}_\star) \cdot \big(\nabla G(\beta\mathbf{M}_0) - \nabla G(\beta\mathbf{m}_\star)\big)\Big]$$

RHS is less than $c\beta\mathrm{Var}_\mu(\mathbf{M}_0) \le c\nu\mathcal{I}_d$ where $c = \|\nabla\nabla G\|_\infty$ □

Results for specific models

For class of models considered above, to prove first order phase transitions it suffices to understand mean-field theory.

This yields first-order phase transitions in e.g.

- q-state Potts model for $q \geq 3$
- $O(N)$-nematic for $N \geq 3$

provided $d \gg 1$

Note: Both were open problems for $\gtrsim 20$ years

Details: CMP **238** (2003) 53-93

Extensions to other interactions
Joint work w/ Nick Crawford

Principal requirement: reflection positivity

$$H(\mathbf{S}) = -\sum_{x,y\in\mathbb{Z}^d} J_{x,y}\,\mathbf{S}_x\cdot\mathbf{S}_y$$

Satisfied by e.g.

- n.n. & n.n.n. interactions
- Yukawa potentials: $J_{x,y} = \mathrm{e}^{-\lambda|x-y|_1}$
- Power-law decaying potentials:
  $$J_{x,y} = |x-y|_1^{-s} \qquad d < s < \min\{2d, d+2\}$$

Mean-field philosophy

Theorem 6
In the class of RP models, as interaction range diverges and/or $d \to \infty$, the following holds at any uniqueness point of MFT:

- *Magnetization converges to mean-field value*
- *Energy density converges to mean-field value*
- *Gibbs measure converges weakly to product measure*

This yields an alternative approach to "Kac limit"

Details: JSP **122** (2006) 1139-1193

Some open problems

- Extensions to other regular lattices
- Gauge theories, non-linear σ-models
- Continuous phase transitions
- Quantum models

THE END

Slides available from:
http://www.math.ucla.edu/~biskup/talks.html