Chasing a blind snail through a random maze

Random walk on random graphs

Marek Biskup

Based on joint work with

Noam Berger

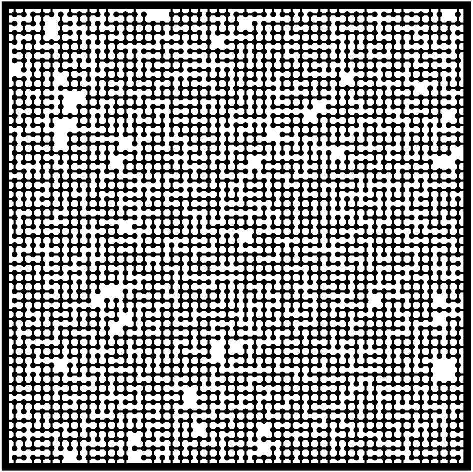
Random maze

Bond percolation on $\mathbb{Z}^2$

Bond percolation:

- Keep edge with probability p.
- Remove it with probability $1 - p$.

Think of $1 - p \ll 1$.



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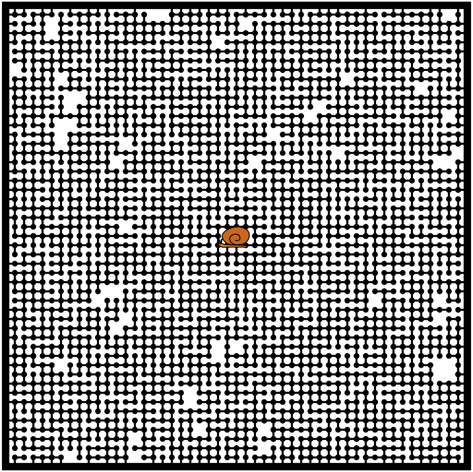

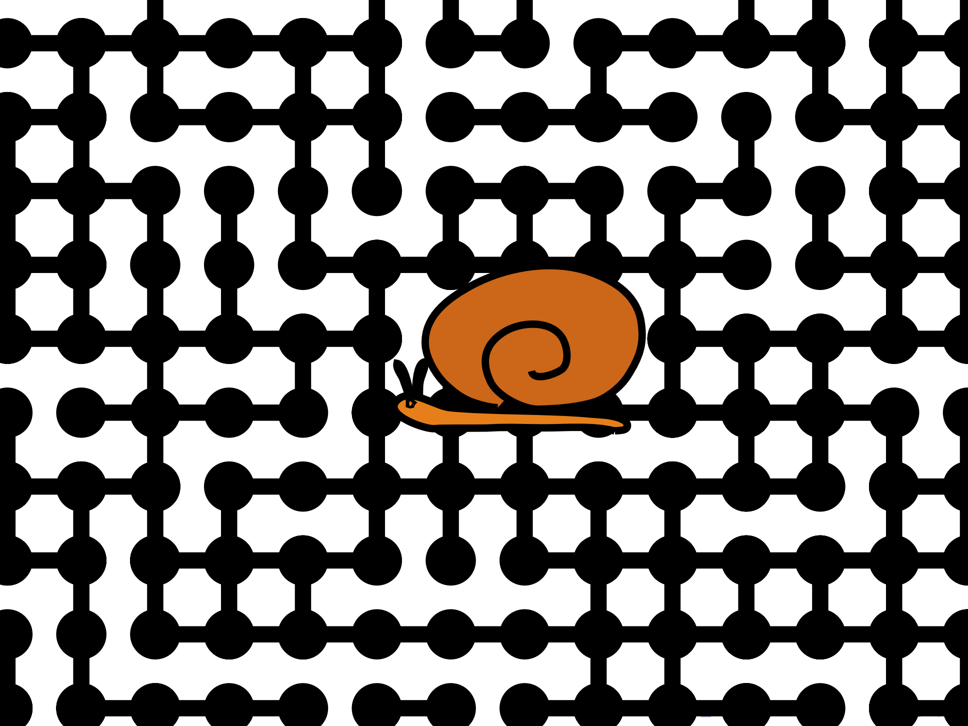

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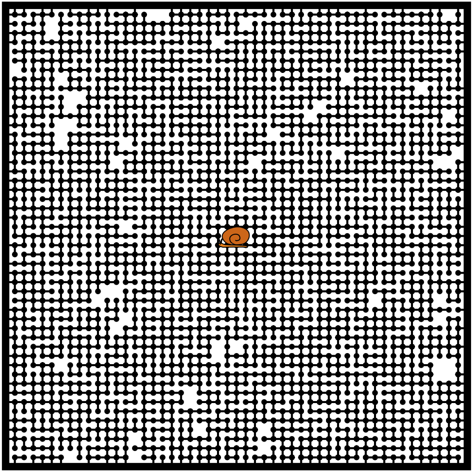
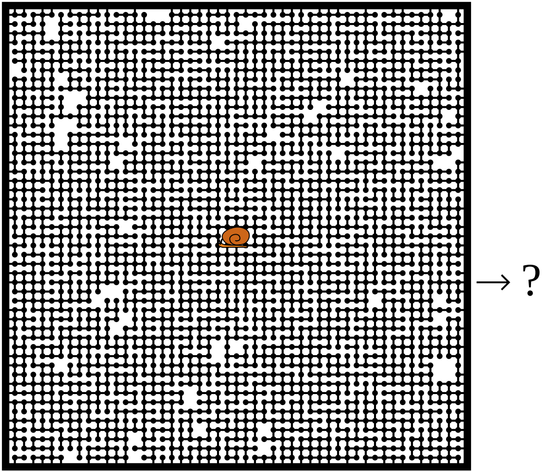


Random maze

Snail's random walk

Main questions:

- Exit point distribution



Random maze

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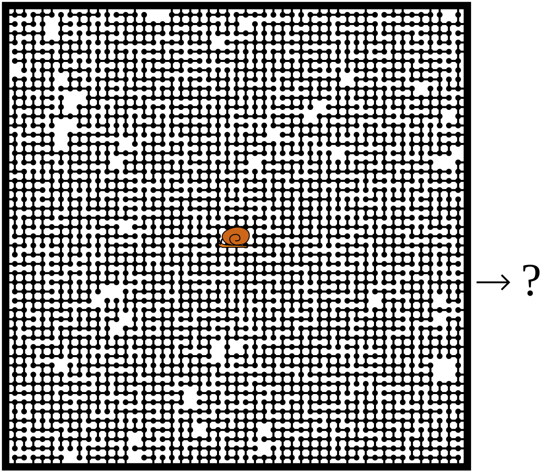
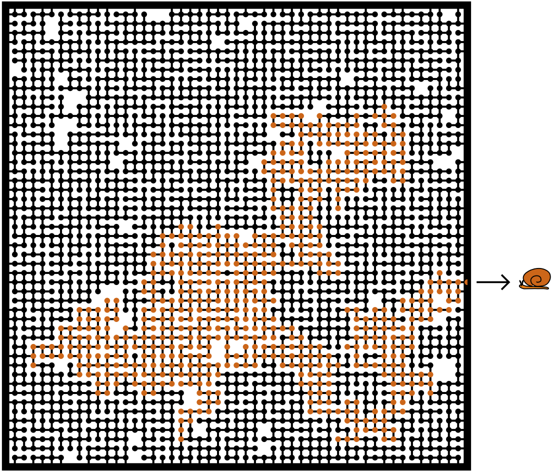


Random maze

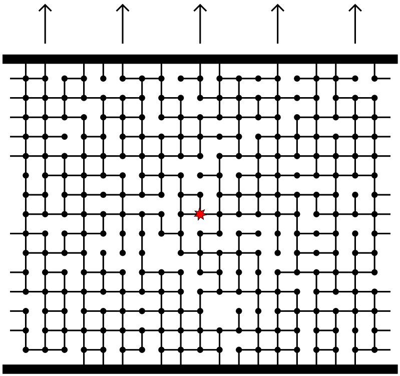
Snail's random walk

Main questions:

- Exit point distribution
- Time needed to exit
- Snail's path — is there a scaling limit?

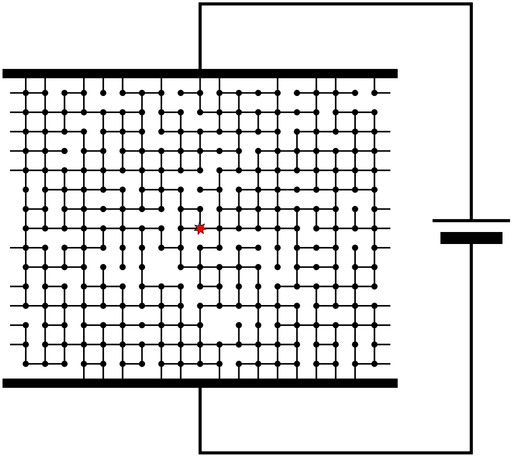


Hitting probability
Walk exits through the top side?



Electrostatic version

Potential at the origin?



Discrete harmonic analysis

Definition 1

Let $G = (V, E)$ be a graph. A function $\varphi : V \to \mathbb{R}^d$ is called *discrete harmonic* if $\forall x \in V$,

$$(\Delta\varphi)(x) \stackrel{\text{def}}{=} \sum_{y:\ (x,y)\in E} [\varphi(y) - \varphi(x)] = 0.$$

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Automatic properties (no conditions):

(1) Maximum principle

Subtle properties (depend on the graph):

(2) Lieouville's theorem

(3) Harnack inequality

Connections with random walk

Let $X_1, X_2, \ldots$ = successive positions of the random walk on V
Walk started at x: Probability distribution P_x, expectation E_x

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Theorem 2 (Dirichlet problem)

Let $V_0 \subset V$ be finite. Let $\varphi : V_0 \to \mathbb{R}$ be harmonic on V_0 with boundary conditions ψ on ∂V_0. Then

$$\varphi(x) = E_x\big(\psi(X_T)\big), \qquad \forall x \in V_0,$$

where T = first time the walk leaves V_0.

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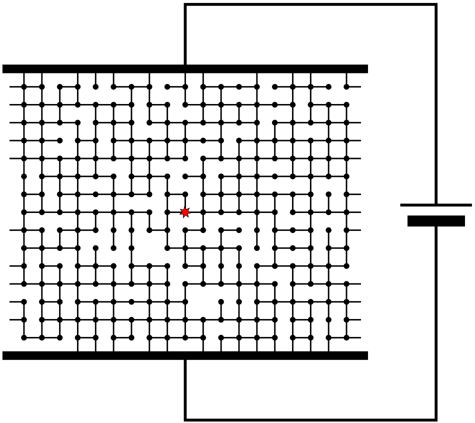
Proof.
$x \mapsto E_x\big(\psi(X_T)\big)$ is discrete harmonic on V_0 with b.c. ψ
Maximum principle $\Rightarrow \exists$ at most one such function □

Electrostatic problem revisited
Geometric embedding



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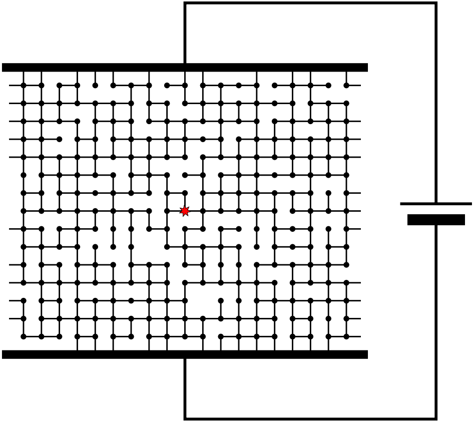
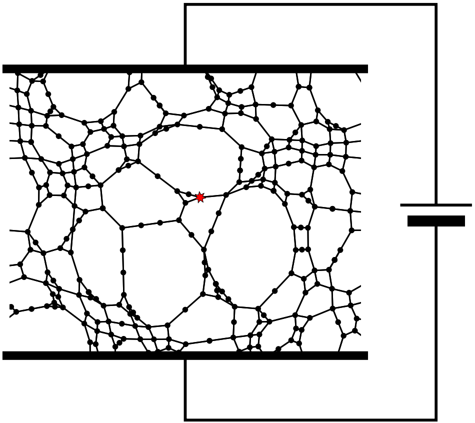


The position $\varphi(x) = x$ is *not* discrete-harmonic

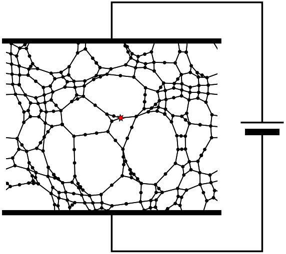
Electrostatic problem revisited

Harmonic embedding



The position $\varphi(x) = x + \chi(x)$ *is* discrete-harmonic

Electrostatic problem "solved"



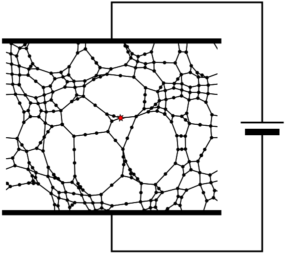
Notations:

(1) Infinite slab $\{(x, y) : |y| \leq N\}$

(2) Potential $+1$ on top bar, -1 on bottom bar

(3) $x + \chi(x)$ = "new" position of site x

Electrostatic problem “solved”



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Theorem 3

If $\varphi(x)$ is the potential at x, then

$$\varphi(x) = \frac{1}{N}\big[(x + \chi(x)) \cdot e_2\big].$$

Martingales

Definition 4

A sequence of random variables $M_0, M_1, \ldots$ is a *martingale* if

$$E(M_{n+1}|M_0, \ldots, M_n) = M_n, \qquad n \geq 0$$

E.g., game with zero expected profit

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Theorem 5 (Harmonic + RW = martingale)

Let $G = (V, E)$ be a graph and let $\varphi : V \to \mathbb{R}^d$ be harmonic. Let $X_0, X_1, \ldots$ be the random walk on V. Define

$$M_n = \varphi(X_n), \qquad n \geq 0.$$

Then $M_0, M_1, \ldots$ is a martingale.

Hitting probability
Martingale calculation

Let $M_n = X_n + \chi(X_n)$. Then M_n is

- random walk on deformed graph
- martingale

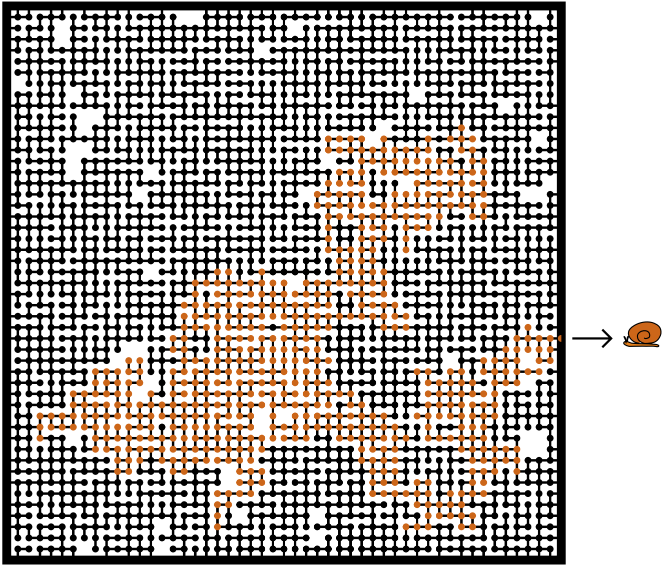
A classic martingale calculation:

$$e_2 \cdot \chi(0) = e_2 \cdot E_0(M_0) = e_2 \cdot E_0(M_T)$$
$$= 2P_0(\text{top hit before bottom}) - 1$$

From here we get

$$P_0(\text{walk exits thru top}) = \frac{1}{2}\left(1 + \frac{e_2 \cdot \chi(0)}{N}\right)$$

Snail’s slimy trail



Martingale functional CLT

Diffusive scaling: Scale space by $\sqrt{n}$ and time by n. Explicitly

$$B_n(t) = \frac{1}{\sqrt{n}}\left(M_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(M_{\lfloor tn \rfloor + 1} - M_{\lfloor tn \rfloor})\right)$$

Note: $t \mapsto B_n(t)$ is a continuous path.

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Theorem 6 (Martingale functional CLT—folk version)

If margingale (M_n) has stationary square-integrable increments, then as $n \to \infty$, the law of $(B_n(t)\colon t \ge 0)$ converges to that of Brownian motion.

Precise conditions of this theorem hold for $M_n = X_n + \chi(X_n)$ on almost-every percolation configuration.

Correction on deformation

All those thing under the rug. . .

Previous slide: Deformed walk $\longrightarrow$ Brownian motion

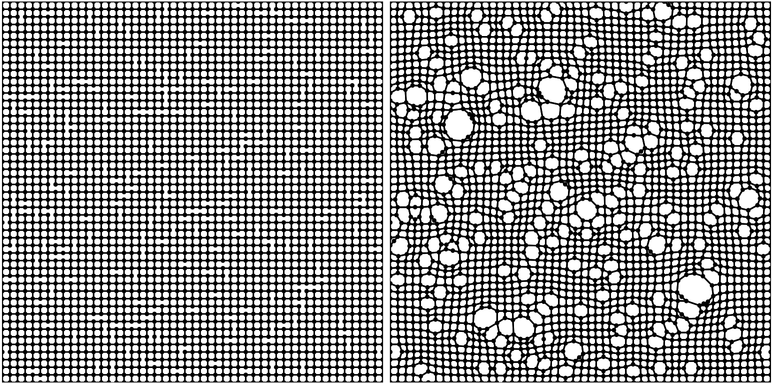
Need to correct on deformation. We show that

$$\chi(x) = o(|x|)$$

Proof quite nontrivial; see [BB05] for details.

Some pictures
Percolation cluster and its deformation

50×50 box, $p = 0.95$



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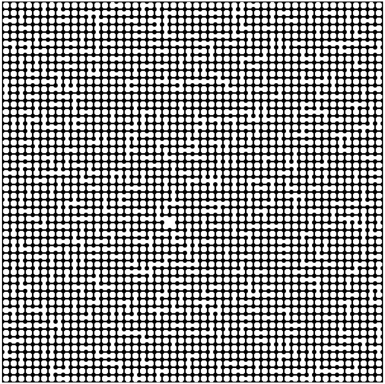
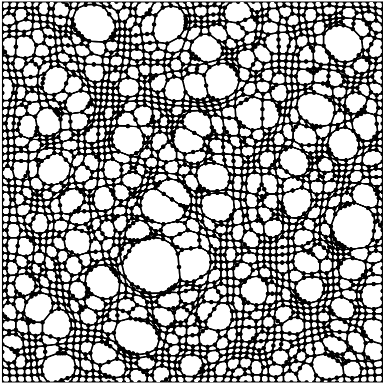
50×50 box, $p = 0.85$





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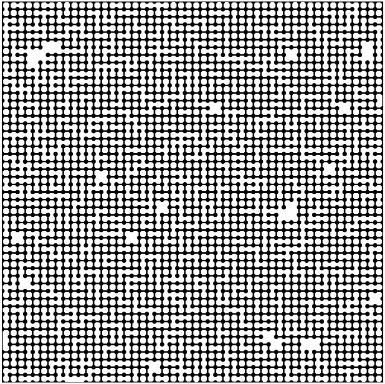
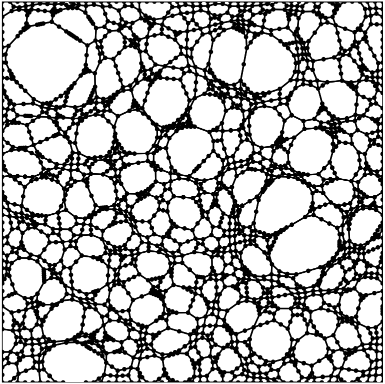
50×50 box, $p = 0.75$





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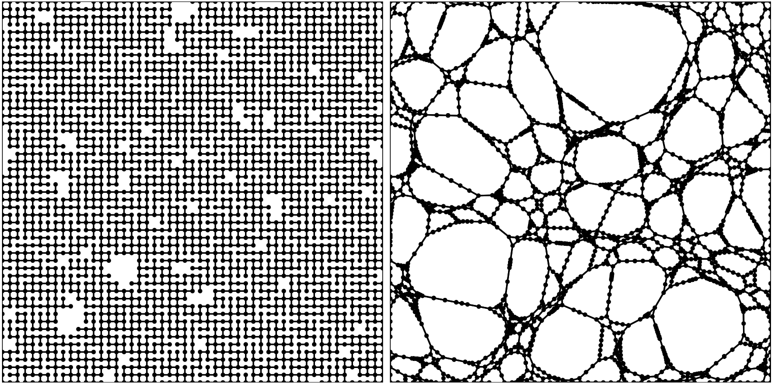

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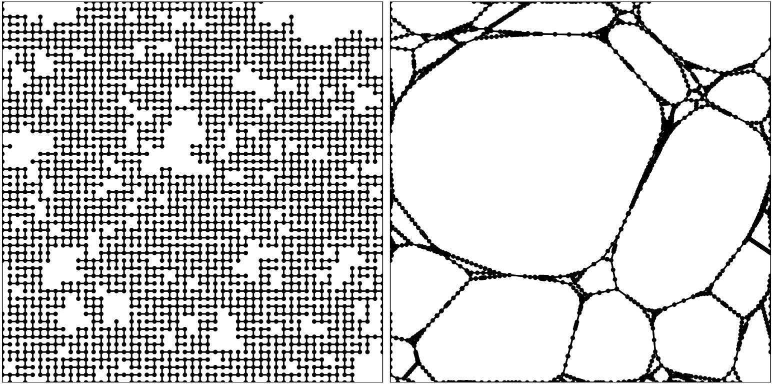

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