Extremal process associated with 2D discrete Gaussian Free Field

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Based on joint work with O. Louidor

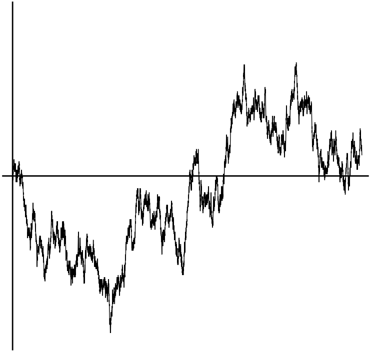
Plan

- Prelude about random fields   blame Eviatar!
- DGFF: definitions, level sets, maximum
- Extremal point process
- Some proofs   all roads lead back to UCLA
- Conformal invariance rediscovered
- Connection to Liouville Quantum Gravity

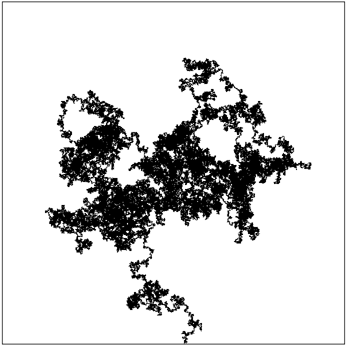
Prelude
Stochastic processes vs random fields

A **stochastic process:** $\{X_t\}$ where t = time ($t \in \mathbb{N}$ or $t \in \mathbb{R}_+$)

Simple symmetric random walk in $d = 1$ and $d = 2$



10,000 steps



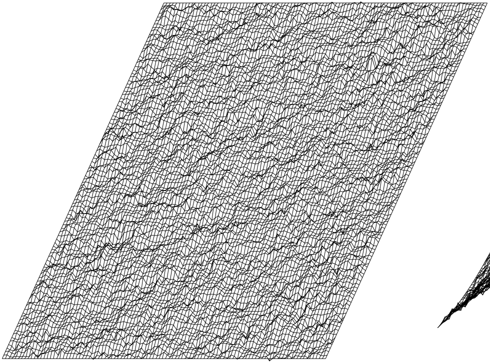
682,613 steps, 1000×1000 box

Questions: Scaling limit? Occupation time measure?

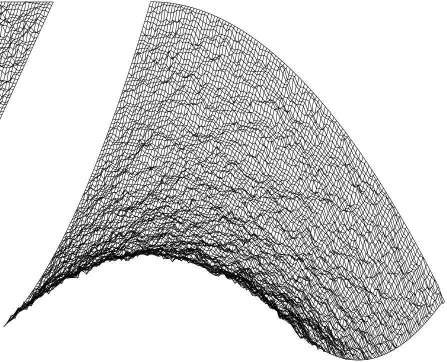
Prelude

Stochastic processes vs random fields

A **random field:** $\{\phi_x\}$ where x = space ($x \in \mathbb{Z}^d$ or $x \in \mathbb{R}^d$)



Linear b.c.



Parabolic b.c.

Questions: Asymptotic shape, correlations, fluctuations?
Refined questions: Local structure, level sets, extreme values?

Random fields in physics
Euclidean theory only

General form of law:

$$Z^{-1} \mathrm{e}^{-H(\phi)} \mathscr{D}\phi$$

where

- $Z = \int \mathrm{e}^{-H(\phi)} \mathscr{D}\phi$ = normalization
- $H(\phi)$ = Hamiltonian (energy of $\phi : \mathbb{R}^d \to \mathbb{R}$)
- $\mathscr{D}\phi = \prod_x \nu(\mathrm{d}\phi_x)$

Examples:

- **White noise:** $H = 0$, $\nu \sim$ standard normal
- **GFF:** $H(\phi) = \frac{1}{2}\int |\nabla\phi(x)|^2 \mathrm{d}x$, ν = Lebesgue
- **ϕ^4-theory:** ν = Lebesgue

$$H(\phi) = \frac{1}{2}\int |\nabla\phi(x)|^2 \mathrm{d}x + \lambda \int \phi(x)^4 \mathrm{d}x$$

Note: (most of the time) **ill defined!** (no product Lebesgue measure)

Discrete Gaussian Free Field
DGFF for short

Field $\phi\colon \mathbb{Z}^d \to \mathbb{R}$

Gibbs measure: for $\Lambda \subset \mathbb{Z}^d$ finite, $\bar{\phi}$ = **boundary condition**

$$\mu_\Lambda^{\bar{\phi}}(\mathrm{d}\phi) := \frac{1}{Z_\Lambda^{\bar{\phi}}} \mathrm{e}^{-H_\Lambda(\phi)} \prod_{x\in\Lambda} \mathrm{d}\phi_x \prod_{x\notin\Lambda} \delta_{\bar{\phi}_x}(\mathrm{d}\phi_x)$$

Hamiltonian ($\mathbb{B}(\Lambda) :=$ n.n. edges incident with Λ)

$$H_\Lambda(\phi) := \frac{\kappa}{2} \sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda)} (\phi_x - \phi_y)^2$$

We will choose $\kappa = \frac{1}{2d}$ and $\bar{\phi}_x = 0$ for all x (zero b.c.)

Model of: crystal deformations, random interface, . . .

Green function: For X = SRW,

$$G_\Lambda(x,y) := E^x\left(\sum_{k=0}^{\tau-1} 1_{\{X_k=y\}}\right)$$

where $\tau :=$ first exit time from Λ

DGFF on Λ = Gaussian process $\{\phi_x : x \in \mathbb{Z}^d\}$ with

$$E(\phi_x) = 0 \quad \text{and} \quad E(\phi_x\phi_y) = G_\Lambda(x,y)$$

Note: Zero values outside Λ

Alternative def's: Gaussian Hilbert spaces, Langevin dynamics, . . .

Why 2D?

$V_N = N \times N$ **box,** $G_N = G_{V_N}$

For x with $\mathrm{dist}(x, V_N^{\mathrm{c}}) > \delta N$,

$$\mathrm{Var}(\phi_x) = G_N(x,x) \asymp \begin{cases} N & \text{if } d = 1 \\ \log N & \text{if } d = 2 \\ 1 & \text{if } d \geq 3 \end{cases}$$

In $d = 2$:

$$\begin{aligned} G_N(x,y) &= g \log N - a(x,y) + o(1) \qquad & N \gg 1 \\ a(x,y) &= g \log|x-y| + c_0 + o(1) \qquad & |x-y| \gg 1 \end{aligned}$$

where

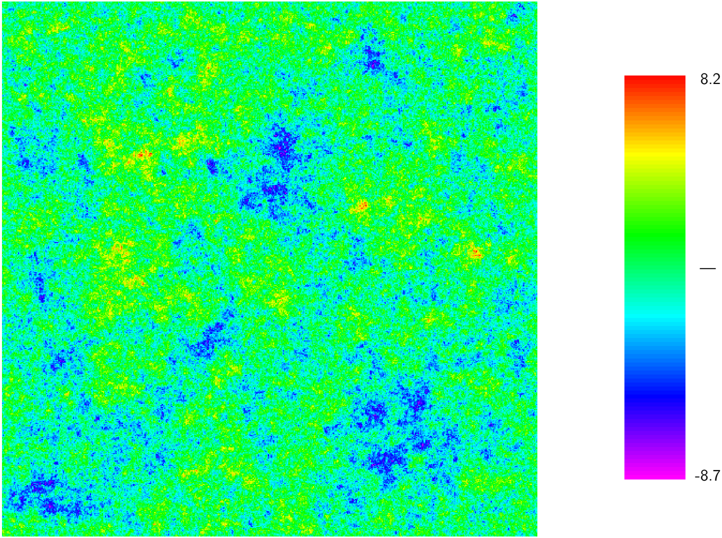
$$g := \frac{2}{\pi} \qquad \left(\text{In physics, } g := \tfrac{1}{2\pi}\right)$$

The $d = 2$ model is **asymptotically scale invariant:**

$$G_{2N}(2x, 2y) = G_N(x,y) + o(1)$$

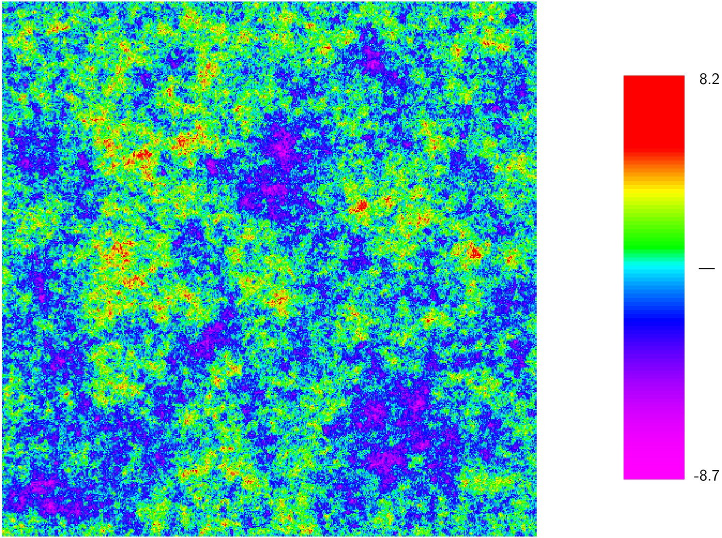
In fact: **conformally invariant**

DGFF on 500×500 square
Uniform color system



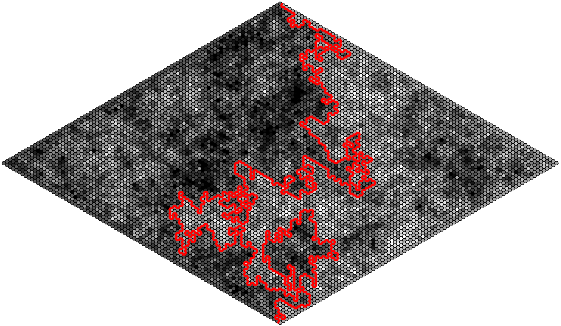
DGFF on 500×500 square

Emphasizing the extreme values



Level set (fractal) geometry
Some known facts

- $O(1)$-**level sets:** SLE_4 curves (Schramm & Sheffield)



- $O(\log N)$-**level sets:** Hausdorff dimension (Daviaud)

$$\left\{x \in D_N \colon \phi_x \geq 2\sqrt{g}\,\gamma \log N\right\}, \qquad 0 < \gamma < 1$$

has $N^{2(1-\gamma^2)+o(1)}$ vertices. Note: Maximum order $\log N$!

Main goal: Describe statistics of extreme values of ϕ as $N \to \infty$

Question of interest: What's the role of conformal invariance?

N.B.: Continuum GFF **not** a function!

Surprise connections:

- Invariant measures for independent particle systems
- Liouville Quantum Gravity, multiplicative cascades, ...

Further known facts

Absolute maximum

Setting and notation: $V_N := (0,N)^2 \cap \mathbb{Z}^2$

$$M_N := \max_{x \in D_N} \phi_x \quad \text{and} \quad m_N := E M_N$$

Leading scale (Bolthausen, Deuschel & Giacomin):

$$m_N \sim 2\sqrt{g} \log N$$

(hard to simulate)

Tightness for a subsequence (Bolthausen, Deuschel & Zeitouni):

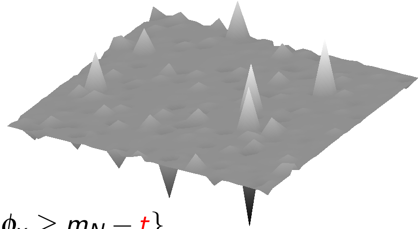
$$2E\big|M_N - m_N\big| \le m_{2N} - m_N$$

Full tightness (Bramson & Zeitouni):

$$EM_N = 2\sqrt{g}\log N - \frac{3}{4}\sqrt{g}\log\log N + O(1)$$

Convergence in law (Bramson, Ding & Zeitouni)

Some known facts
Extremal process tightness



Ding & Zeitouni, Ding established
extremal process tightness:

Extreme level set:

$$\Gamma_N(t) := \{x \in V_N : \phi_x \geq m_N - t\}$$

$$\exists c, C \in (0,\infty): \qquad \lim_{\lambda\to\infty} \liminf_{N\to\infty} P\left(\mathrm{e}^{c\lambda} \leq |\Gamma_N(\lambda)| \leq \mathrm{e}^{C\lambda}\right) = 1$$

and $\exists c > 0$ s.t.

$$\lim_{r\to\infty} \limsup_{N\to\infty} P\Big(\exists x, y \in \Gamma_N(c \log\log r) : r \leq |x-y| \leq N/r\Big) = 0$$

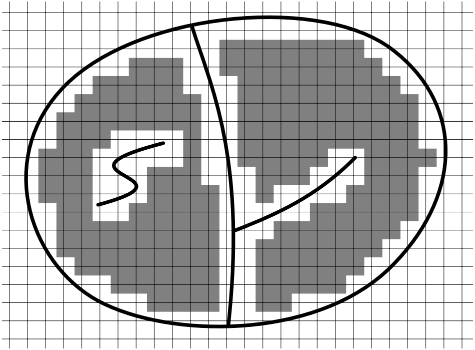
Generalizes to "bounded and open" domains

Domains we will consider

Class of domains:

$$\mathfrak{D} = \{D \subset \mathbb{C}\colon \text{bounded, open, } \partial D = \text{finite \# of pieces}\}$$

Discretized as $D_N := \{x \in \mathbb{Z}^2\colon \operatorname{dist}(x/N, D^{\mathrm{c}}) > {}^1\!/_N\}$.



Really need: Weak convergence of harmonic measures (discrete to continuous)

Setup for extreme order theory

Extremal point process

Full process: Measure η_N on $\overline{D} \times \mathbb{R}$

$$\eta_N := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{\phi_x - m_N}$$

Problem: Values in each "peak" strongly correlated

Local maxima only: $\Lambda_r(x) := \{z \in \mathbb{Z}^2 \colon |z - x| \le r\}$

$$\eta_{N,r} := \sum_{x \in D_N} 1_{\{\phi_x = \max_{z \in \Lambda_r(x)} \phi_z\}} \delta_{x/N} \otimes \delta_{\phi_x - m_N}$$

Including clusters: (work in progress)

$$\widetilde{\eta}_{N,r} := \sum_{x \in D_N} 1_{\{\phi_x = \max_{z \in \Lambda_r(x)} \phi_z\}} \delta_{x/N} \otimes \delta_{\phi_x - m_N} \otimes \delta_{\{\phi_x - \phi_z \colon z \in \mathbb{Z}^d\}}$$

Main result

Convergence to Cox process

Recall $G_N(x,x) = g \log N + O(1)$

Theorem (B.-Louidor 2013, 2014)

There is a random Borel measure Z^D on $\overline{D}$ with $0 < Z^D(\overline{D}) < \infty$ a.s. such that for any $r_N \to \infty$ and $N/r_N \to \infty$,

$$\eta_{N,r_N} \xrightarrow[N\to\infty]{\text{law}} \mathrm{PPP}\Big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\phi}\mathrm{d}\phi\Big)$$

where $\alpha := 2/\sqrt{g} = \sqrt{2\pi}$.

Corollaries

Asymptotic law of maximum: Setting $Z := Z^D(\overline{D})$,

$$P\big(M_N \le m_N + t\big) \underset{N\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

N.B.: Laplace transform of Z

Joint law position/value: $A \subset D$ open, $\widehat{Z}(A) = Z^D(A)/Z^D(\overline{D})$

$$P\Big(M_N \le m_N + t,\, N^{-1}\operatorname{argmax}\phi \in A\Big) \underset{N\to\infty}{\longrightarrow} E\big(\widehat{Z}(A)\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

In fact: Key steps of the proof

Proof of Theorem

Invariance under "Dysonization"

Idea: As $\{\eta_{N,r_N} : N \geq 1\}$ tight, extract a converging subsequence and characterize its distribution uniquely.

Abbreviate:
$$\langle \eta, f \rangle := \int \eta(\mathrm{d}x, \mathrm{d}\phi) f(x,\phi)$$

Proposition (Distributional invariance)

Suppose $\eta :=$ a weak-limit point of some $\{\eta_{N_k, r_{N_k}}\}$. Then for any $f\colon D \times \mathbb{R} \to [0,\infty)$ continuous, compact support,

$$E\left(\mathrm{e}^{-\langle \eta, f\rangle}\right) = E\left(\mathrm{e}^{-\langle \eta, f_t\rangle}\right), \qquad t > 0,$$

where

$$f_t(x,\phi) := -\log E\mathrm{e}^{-f(x,\phi + B_t - \frac{\alpha}{2}t)}$$

with $B_t :=$ standard Brownian motion.

Proposition explained

We may write

$$\eta = \sum_{i\geq 1} \delta_{(x_i,\phi_i)}$$

Let $\{B_t^{(i)}\}$:= i.i.d. standard Brownian motions. Set

$$\eta_t := \sum_{i\geq 1} \delta_{(x_i,\phi_i + B_t^{(i)} - \frac{\alpha}{2}t)}$$

Well defined as $t \mapsto |\Gamma_N(t)|$ grows only exponentially. Then

$$E\left(\mathrm{e}^{-\langle \eta_t, f\rangle}\right) = E\left(\mathrm{e}^{-\langle \eta, f_t\rangle}\right)$$

and so Proposition says

$$\eta_t \overset{\text{law}}{=} \eta, \qquad t > 0$$

i.e., η **invariant under a Dysonization of its points**

Proof of Proposition (part 1)

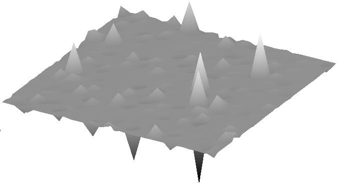
Gaussian interpolation: $\phi', \phi'' \overset{\text{law}}{=} \phi$, independent

$$\phi \overset{\text{law}}{=} \underbrace{\left(1 - \frac{t}{g \log N}\right)^{1/2} \phi'}_{\text{main term}} + \underbrace{\left(\frac{t}{g \log N}\right)^{1/2} \phi''}_{\text{perturbation}}$$

For x = large r-local maximum of ϕ':

$$\begin{aligned}\left(1 - \frac{t}{g \log N}\right)^{1/2} \phi'_x &= \phi'_x - \frac{1}{2} \frac{t}{g \log N} \phi'_x + o(1) \\ &= \phi'_x - \frac{t}{2} \frac{m_N}{g \log N} + o(1) \\ &= \phi'_x - \frac{\alpha}{2} t + o(1)\end{aligned}$$

Proof of Proposition (part 2)



Concerning ϕ'', abbreviate

$$\widetilde{\phi}''_x := \left(\frac{t}{g \log N}\right)^{1/2} \phi''_x$$

Properties of Green function:

$$\mathrm{Cov}\big(\widetilde{\phi}''_x, \widetilde{\phi}''_y\big) = \begin{cases} t + o(1), & \text{if } |x-y| \le r \quad \text{nearly constant} \\ o(1), & \text{if } |x-y| \ge N/r \quad \text{nearly independent} \end{cases}$$

So we conclude: The law of

$$\Big\{\widetilde{\phi}''_x \colon x = \text{large local min. of } \phi'\Big\}$$

is asymptotically that of independent B.M.'s $\square$

Proof of Theorem continued
Invariant laws for independent particle systems

The question of what limit process we get has been reduced to:

Problem: Characterize point processes on $\overline{D} \times \mathbb{R}$ are invariant under independent Dysonization

$$(x, \phi) \mapsto \left(x, \phi + B_t - \frac{\alpha}{2} t\right)$$

of (the "field coordinate" of) its points

Easy to check: $\mathrm{PPP}(\nu(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\phi}\mathrm{d}\phi)$ okay for any ν (even random)

Any other solutions?

Liggett's "folk" theorem

Setting:

- Markov chain on (nice space) $\mathscr{X}$ w/ transition kernel P
- System of particles evolving independently by P
- $\mathscr{I} :=$ loc. finite **invariant measures** on particle systems

Theorem (Liggett 1977)

Assume **uniform dispersivity property:**

$$\sup_{x\in\mathscr{X}} \mathrm{P}^n(x,C) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall C\subset\mathscr{X} \text{ compact}$$

Then each $\mu\in\mathscr{I}$ takes the form $\mathrm{PPP}(M(\mathrm{d}x))$, where M is a random measure satisfying

$$M\mathrm{P} \overset{\text{law}}{=} M$$

Liggett's 1977 derivation

For $t > 0$ define Markov kernel P on $\overline{D} \times \mathbb{R}$ by

$$(\mathrm{P}g)(x,\phi) := E^0 g\left(x, \phi + B_t - \tfrac{\alpha}{2} t\right)$$

Set $g(x,\phi) := \mathrm{e}^{-f(x,\phi)}$ for $f \geq 0$ continuous with compact support. Proposition implies

$$E\left(\mathrm{e}^{-\langle \eta, f \rangle}\right) = E\left(\mathrm{e}^{-\langle \eta, f^{(n)} \rangle}\right)$$

where

$$f^{(n)}(x,\phi) = -\log(\mathrm{P}^n \mathrm{e}^{-f})(x,\phi)$$

P has **uniform dispersivity property** and so $\mathrm{P}^n \mathrm{e}^{-f} \to 1$ uniformly on $\overline{D} \times \mathbb{R}$. Expanding the log,

$$f^{(n)} \sim 1 - \mathrm{P}^n \mathrm{e}^{-f} \qquad \text{as } n \to \infty$$

Liggett's 1977 derivation (continued)

Hence

$$E\left(e^{-\langle \eta, f\rangle}\right) = \lim_{n\to\infty} E\left(e^{-\langle \eta, 1-\mathsf{P}^n e^{-f}\rangle}\right) \qquad (*)$$

But, as P is Markov,

$$\langle \eta, 1-\mathsf{P}^n e^{-f}\rangle = \langle \eta \mathsf{P}^n, 1-e^{-f}\rangle$$

$(*)$ shows that $\{\eta \mathsf{P}^n \colon n \geq 1\}$ is tight. Along a subsequence

$$\eta \mathsf{P}^{n_k}(\mathrm{d}x, \mathrm{d}\phi) \xrightarrow[k\to\infty]{\text{law}} M(\mathrm{d}x, \mathrm{d}\phi)$$

and so

$$E\left(e^{-\langle \eta, f\rangle}\right) = E\left(e^{-\langle M, 1-e^{-f}\rangle}\right)$$

i.e., $\eta = \mathrm{PPP}(M(\mathrm{d}x, \mathrm{d}\phi))$. Clearly,

$$M\mathsf{P} \overset{\text{law}}{=} M$$

Question: What M can we get in our case?

Theorem (Liggett 1977)

$MP \overset{\text{law}}{=} M$ implies $MP = M$ a.s. when P is a kernel of

(1) *an irreducible, recurrent Markov chain*

(2) *a random walk on a closed abelian group w/o proper closed invariant subset*

N.B.:**(2) covers our case** and

$$MP = M \text{ a.s.} \quad \Leftrightarrow \quad M \text{ random mixture of P-invariant laws}$$

For our chain Choquet-Deny (or $t \downarrow 0$) shows

$$M(\mathrm{d}x, \mathrm{d}h) = Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h} \mathrm{d}h + \widetilde{Z}^D(\mathrm{d}x) \otimes \mathrm{d}h$$

Tightness of maximum forces $\widetilde{Z}^D = 0$ a.s. □

Proof of Theorem completed

Uniqueness of the limit

We thus know $\eta_{N_k,r_{N_k}} \xrightarrow{\text{law}} \eta$ implies

$$\eta = \mathrm{PPP}\big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\big)$$

for some random Z^D — albeit **possibly depending** on $\{N_k\}$.

But for $Z := Z^D(\overline{D})$, this yields

$$P\big(M_{N_k} \le m_{N_k} + t\big) \underset{k\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-\alpha^{-1} Z \mathrm{e}^{-\alpha t}}\big)$$

Hence: **law of $Z^D(\overline{D})$ unique if limit law of maximum unique** (and we know this from Bramson & Ding & Zeitouni)

Existence of joint limit of maxima in finite number of disjoint subsets of $D \Rightarrow$ **uniqueness of law of $Z^D(\mathrm{d}x)$** $\square$

Details: B.-Louidor (arXiv:1306.2602)

Properties of Z^D-measure
B.-Louidor (arXiv:1410.4676)

Theorem

The measure Z^D satisfies:

(1) $Z^D(A) = 0$ *a.s. for any Borel* $A \subset \overline{D}$ *with* $\mathrm{Leb}(A) = 0$

(2) $\mathrm{supp}(Z^D) = \overline{D}$ *and* $Z^D(\partial D) = 0$ *a.s.*

(3) Z^D *is non-atomic a.s.*

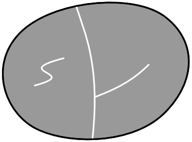
Property (3) is only barely true:

Conjecture

Z^D is supported on a set of zero Hausdorff dimension

Fancy properties

Gibbs-Markov for Z^D measure



Recall $\widetilde{D} \subseteq D$ yields $\phi^D \overset{\text{law}}{=} \phi^{\widetilde{D}} + \varphi^{D,\widetilde{D}}$

Fact: $\varphi^{D,\widetilde{D}} \xrightarrow{\text{law}} \Phi^{D,\widetilde{D}}$ on $\widetilde{D}$ where

(1) $\{\Phi^{D,\widetilde{D}}(x) \colon x \in \widetilde{D}\}$ mean-zero **Gaussian** field with

$$\operatorname{Cov}\bigl(\Phi^{D,\widetilde{D}}(x), \Phi^{D,\widetilde{D}}(y)\bigr) = G^D(x,y) - G^{\widetilde{D}}(x,y)$$

continuum Green functions

(2) $x \mapsto \Phi^{D,\widetilde{D}}(x)$ **harmonic** on $\widetilde{D}$ a.s.

Theorem (Gibbs-Markov property)

Suppose $\widetilde{D} \subseteq D$ *be such that* $\operatorname{Leb}(D \smallsetminus \widetilde{D}) = 0$. *Then*

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha \Phi^{D,\widetilde{D}}(x)} Z^{\widetilde{D}}(\mathrm{d}x)$$

Theorem (Conformal symmetry)

Suppose $f : D \to f(D)$ analytic bijection. Then

$$Z^{f(D)} \circ f(\mathrm{d}x) \stackrel{\text{law}}{=} |f'(x)|^4 Z^D(\mathrm{d}x)$$

In particular, for D simply connected and $rad_D(x)$ conformal radius

$$\mathrm{rad}_D(x)^{-4} Z^D(\mathrm{d}x)$$

is invariant under conformal maps of D.

Note:

(1) $\mathrm{Leb} \circ f(\mathrm{d}x) = |f'(x)|^2 \mathrm{Leb}(\mathrm{d}x)$ and so $\mathrm{rad}_D(x)^{-2}\mathrm{Leb}(\mathrm{d}x)$ is invariant under conformal maps.

(2) By GM property it suffices to know law($Z^{\mathbb{D}}$) for $\mathbb{D} :=$ unit disc. So this is a **statement of universality**

Continuum GFF := Gaussian on $\mathrm{H}_0^1(D)$ w.r.t. norm $f \mapsto \pi\|\nabla f\|_2^2$

Formal expression: $\phi(x) = \sum_{n\geq 1} Z_n f_n(x)$ $\{f_n\}$ ONB

Exists only as a linear functional on $\mathrm{H}_0^1(D)$:

$$\phi(f) = \sqrt{\pi}\sum_{n\geq 1} Z_n \langle \nabla f, \nabla f_n \rangle_{L^2(D)}$$

Derivative martingale:

$$M'(\mathrm{d}x) = \big[2\mathrm{Var}(\phi(x)) - \phi(x)\big]\mathrm{e}^{2\phi(x) - 2\mathrm{Var}(\phi(x))}\mathrm{d}x$$

Defined by smooth approximations to h or expansion in ONB (Duplantier, Sheffield, Rhodes, Vargas)

KPZ relation links M'-measure of sets to Lebesgue measure

Unifying scheme?
Liouville Quantum Gravity/Multiplicative Chaos

Liouville Quantum Gravity (LQG):

$$M^D(\mathrm{d}x) := \mathrm{rad}_D(x)^2 M'(\mathrm{d}x)$$

Conjecture

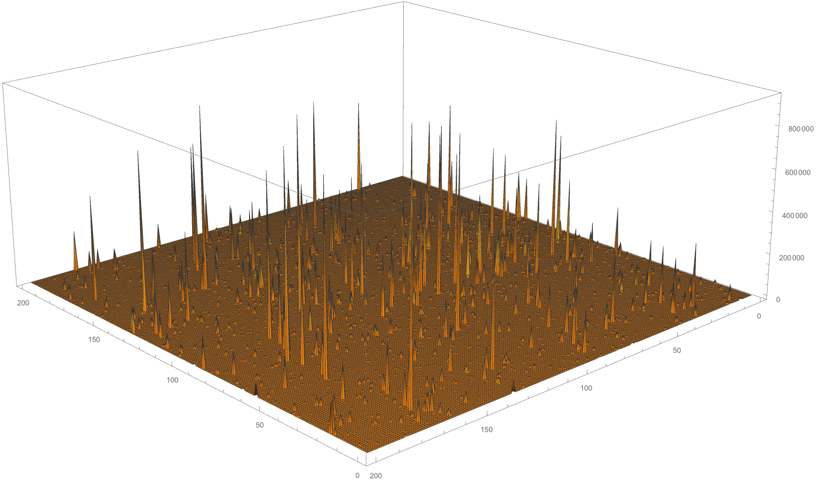
There is constant $c_\star \in (0,\infty)$ s.t. for all D

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} c_\star M^D(\mathrm{d}x)$$

Current status: Law of Z^D characterized by

- Gibbs-Markov property — okay for LQG
- shift and dilation symmetry — okay for LQG
- precise upper tails of $Z^D(A)$ — so far open

Liouville Quantum Gravity
200×200 box



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