Large-scale metric properties of long-range percolation

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based on joint papers with
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In memory of Dima Ioffe

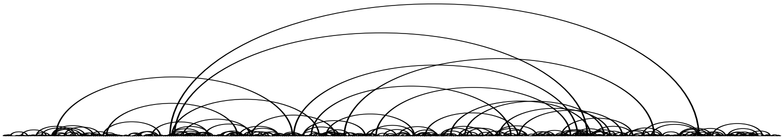
Graph with vertices $\mathbb{Z}^d$ and edge between x and y with probability

$$\mathfrak{p}(x,y) := 1 - \exp\{-\mathfrak{q}(x-y)\}$$

where $\mathfrak{q}\colon \mathbb{Z}^d \to [0,\infty]$. Usually, power-law decay assumed:

$$\mathfrak{q}(x) \underset{|x|\to\infty}{\sim} \beta|x|^{-s}$$

for $\beta, s > 0$ parameters and a norm $|\cdot|$ on $\mathbb{R}^d$.



Q: For what powers does a phase transition occur in Ising model on $\mathbb{Z}$?

A: $1/r^2$-decay critical

Kac-Thompson (1968), Dyson (1971), . . . , Fröhlich-Spencer (1982)

Thouless phenomenon (Thouless 1971, Anderson-Yuval 1971): Discontinuity of magnetization (expected spin at origin) at β_c

Proofs:

Aizenman-Newman (1986) for percolation

Aizenman-Chayes^2-Newman (1986) for random cluster model

Note: $1/r^2$-percolation model percolates at criticality!

Define the graph-theoretical distance by

$$D(x,y) := \inf\{n \geqslant 0 \colon \exists \text{ path } x \to y \text{ with } n \text{ edges}\}$$

Key questions:

- Scaling of $D(x,y)$ with $|x-y|$?
- Asymptotic shape of metric balls

$$\{x \in \mathbb{Z}^d \colon D(0,x) \leqslant n\}$$

  as $n \to \infty$?
- Asymptotic shape of isoperimetric sets?

Abbreviate $N := |x-y|$. Then, as $N \to \infty$,

$$D(x,y) \begin{cases} \to \lceil \frac{d}{d-s} \rceil & \text{if } s < d \\ = (d+o(1))\frac{\log N}{\log\log N} & \text{if } s = d \\ \asymp (\log N)^{\Delta(s,d)} & \text{if } d < s < 2d \\ \asymp N^{\theta(\beta)} & \text{if } s = 2d \\ \sim N & \text{if } s > 2d \end{cases}$$

- $s < d$: Benjamini, Peres, Kesten & Schramm (2004)
- $s = d$: Coppersmith, Gamarnik & Siridenko (2002), T. Wu
- $d < s < 2d$: B. (2004), B.& Lin (2018)
- $s = 2d$: Ding & Sly (2014), Bäumler (2022)
- $s > 2d$: Berger (2007)

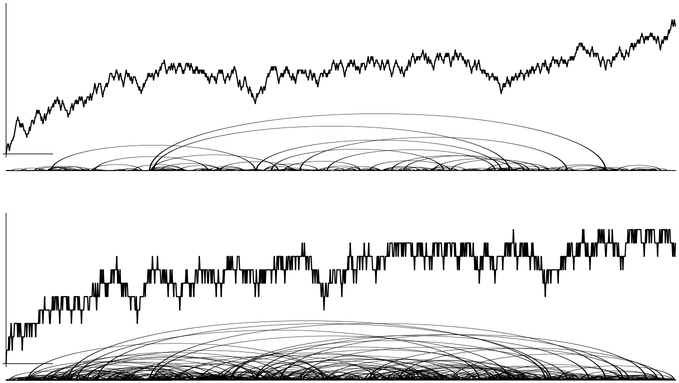
Here $\Delta(s,d) := \frac{1}{\log_2(2d/s)}$ and $\theta(\beta) \in (0,1)$ for each $\beta \in (0,\infty)$

No percolation at criticality for $d < s < 2d$!!! Berger (2002), Hutchcroft (2020), Bäumler-Berger (2022)

Quenched invariance principle for random walk for $s > 2d$ B.-Chen-Kumagai-Wang (2021)

Simulations of $x \mapsto D(0,x)$ in $d = 1$

$s = 1.5$



$\beta = 1$

$\beta = 5$

Theorem (B.-Lin 2019, B.-Krieger 2021)

Suppose $\mathfrak{q}_\beta(x) = +\infty$ *for* $x \sim 0$. *Then there exists* $\phi_\beta\colon (0,\infty) \to (0,\infty)$ *continuous and* $\Sigma \subseteq (0,\infty)$ *at most countable such that, for* $\gamma := \frac{s}{2d}$,

$$\forall r > 1\colon\ \phi_\beta(r) = \phi_\beta(r^\gamma)$$

and such that for all $\beta \in (0,\infty) \smallsetminus \Sigma$ *and all* $\epsilon > 0$,

$$\frac{1}{r^d}\#\left(\left\{x \in \mathbb{Z}^d \colon |x| \leqslant r \ \wedge\ \left|\frac{D(0,x)}{\phi_\beta(r)(\log r)^\Delta} - 1\right| > \epsilon\right\}\right) \underset{r\to\infty}{\overset{P}{\longrightarrow}} 0$$

where $\Delta := \frac{1}{\log_2(2d/s)}$.

In short:

$$D(0,x) \underset{|x|\to\infty}{=} \big[\phi_\beta(|x|) + o(1)\big](\log|x|)^\Delta$$

for typical (large) x with high probability.

Classical proof in FPP context

Subadditivity relation

$$D(0,x+y) \leqslant D(0,x) + D(x,x+y)$$

with

$$D(x,x+y) \overset{\text{law}}{=} D(0,y)$$

Kingman's Subadditive Ergodic Theorem: For each $x \in \mathbb{Z}^d$, the limit

$$\varrho(x) := \lim_{n\to\infty} \frac{D(0,nx)}{n}$$

exists a.s. and (if non-zero for $x \neq 0$) is a restriction of a norm on $\mathbb{R}^d$.

Problem: The limit vanishes when $s \leqslant 2d$ so no information gained here!!!

Given any $\gamma_1, \gamma_2 \in (0,1)$ and a unit vector $e \in \mathbb{R}^d$,

$$P\Big([-N^{\gamma_1}, N^{\gamma_1}]^d \;\longleftrightarrow\; Ne + [-N^{\gamma_2}, N^{\gamma_2}]^d\Big) \approx 1 - \exp\Big\{-\beta c \frac{N^{d\gamma_1} N^{d\gamma_2}}{N^s}\Big\}$$

So an edge like this

- will exist w.h.p. if $d(\gamma_1 + \gamma_2) \gtrsim s$
- will NOT exist w.h.p. if $d(\gamma_1 + \gamma_2) \lesssim s$

Boundary case: $\gamma_1 + \gamma_2 = 2\gamma$ with $\gamma := \frac{s}{2d}$.

Can be built into a proof of an upper bound B. (2004)
Lower bound now proved using a different argument Trapman (2010)

Restricted "distance:"

$$\widetilde{D}(x,y) := \inf\left\{ n \geqslant 0\colon \begin{array}{l} \{(x_{k-1},x_k)\colon k=1,\dots,n\} \text{ edges }, x_0 = x, \\ x_n = y, \forall k = 1,\dots,n\colon\ |x_k - x| < 2|x-y| \end{array} \right\}.$$

Proposition (Subadditivity inequality)

Fix $\eta \in (0,1)$ *and* $\overline{\gamma} \in (\gamma, 1)$ *and let* Z, Z' *be i.i.d.* $\mathbb{R}^d$*-valued with common law*

$$P(Z \in B) = \sqrt{\eta\beta} \int_B \mathrm{e}^{-\eta\beta c_0 |z|^{2d}} \mathrm{d}z,$$

where c_0 *is a normalization constant. There are* $c_1, c_2 \in (0,\infty)$ *and, for each* $x \in \mathbb{Z}^d$*, an event* $A(x) \in \sigma(Z,Z')$ *such that*

$$\widetilde{D}(0,x) \overset{\text{law}}{\leqslant} \widetilde{D}\big(0, |x|^\gamma Z\big) + \widetilde{D}'\big(0, |x|^\gamma Z'\big) + 1 + |x|_1 1_{A(x)}$$

where $\widetilde{D}'$ *be an independent copy of* $\widetilde{D}$*, independent of* Z *and* Z'*, and*

$$P\big(A(x)\big) \leqslant c_1 \mathrm{e}^{-c_2 |x|^{\overline{\gamma}-\gamma}}$$

Identity to prove:

$$\widetilde{D}(0,x) \overset{\text{law}}{\leqslant} \widetilde{D}\big(0,|x|^\gamma Z\big) + \widetilde{D}'\big(0,|x|^\gamma Z'\big) + 1 + |x|_1 1_{A(x)}$$

Key steps:

- Use $\eta < 1$ to couple most long edges to a Poisson point process.
- Z and Z' are then the scaled distances to endpoints of the (long) edge chosen so that (Z,Z') minimizes

$$z,z' \mapsto |z|^{2d} + |z'|^{2d}$$

  This choice ensures (by way of a calculation) that $Z \perp\!\!\!\perp Z'$
- Thanks to working with modified distance, $Z,Z' \perp\!\!\!\perp \widetilde{D}$ and also that $\widetilde{D}(0,Z|x|^\gamma)$ and $\widetilde{D}(x,x+Z'|x|^\gamma) \overset{\text{law}}{=} \widetilde{D}'(0,Z'|x|^\gamma)$ are independent.
- If anything in the above description fails — which is what defines event $A(x)$ — then instead bound $\widetilde{D}(0,x) \leqslant |x_1|1_{A(x)}$.

“Solving” subadditivity via a random argument

Iterating the subadditivity bound leads to arguments of the form $|x|^{\gamma^{n+1}}$ times $Z_0 \prod_{k=1}^n |Z_k|^{\gamma^k}$, where $Z_0, Z_1, \ldots$ are i.i.d. copies of Z. In order to close the expression, we insert a random argument from the start:

Lemma

Let $Z_0, Z_1, \ldots$ be i.i.d. copies of Z. The random variable

$$W := Z_0 \prod_{k=1}^{\infty} |Z_k|^{\gamma^k}$$

is well defined and has moments of all orders. Moreover, for each $r > 1$ the limit

$$L_\beta(r) := \lim_{n\to\infty} \frac{1}{2^n} \mathbb{E}\big[\widetilde{D}(0, Wr^{\gamma^{-n}})\big]$$

where $W \perp\!\!\!\perp \widetilde{D}(\cdot,\cdot)$, exists and obeys

$$\forall r > 1: \quad L_\beta(r) = 2L_\beta(r^\gamma)$$

where $\gamma := \frac{s}{2d}$.

The existence of W goes via Z having at least Gaussian tails. Note that then

$$Z \perp\!\!\!\perp W \;\Rightarrow\; Z|W|^\gamma \overset{\text{law}}{=} W$$

Subadditivity gives

$$\widetilde{D}(0, Wr) \overset{\text{law}}{\leqslant} \widetilde{D}\big(0, |W|^\gamma Z\big) + \widetilde{D}'\big(0, |W|^\gamma Z'\big) + 1 + |Wr|_1 1_{A(Wr)}$$

which under expectation translates into

$$\mathbb{E}\big[D(0, Wr)\big] \leqslant 2\mathbb{E}\big[\widetilde{D}(0, |W|^\gamma Z r^\gamma)\big] + 1 + \mathbb{E}\big(|Wr|_1 1_{A(Wr)}\big)$$

As $\sup_{r>1} \mathbb{E}\big(|Wr|_1 1_{A(Wr)}\big) < \infty$, we have

$$\mathbb{E}\big[D(0, Wr)\big] \leqslant 2\mathbb{E}\big[\widetilde{D}(0, Wr^\gamma)\big] + c$$

This shows that $n \mapsto 2^{-n}[\mathbb{E}D(0, Wr^{\gamma^{-n}}) + c]$ is non-increasing and so

$$L_\beta(r) := \lim_{n\to\infty} \frac{1}{2^n} E\big[\widetilde{D}(0, Wr^{\gamma^{-n}})\big]$$

exists as claimed. The identity $L_\beta(r) = 2L(r^\gamma)$ follows.

From $L_\beta(r) = 2L(r^\gamma)$ and $\gamma^{-\Delta} = 2$ we get $L_\beta(r) \approx (\log r)^\Delta$. Define ϕ_β so that

$$L_\beta(r) = \phi_\beta(r)(\log r)^\Delta$$

This gives log-log-periodicity relation

$$\forall r > 1: \quad \phi_\beta(r) = \phi_\beta(r^\gamma)$$

Remaining steps (details omitted):

- Prove concentration (to improve convergence in the mean)
- Replace W by a deterministic point
- Interpolate doubly exponential sequences
- Replace restricted "distance" by actual distance

Warning: Limit does NOT exist a.s., at best in probability

Summarizing the above, we have derived

$$D(0,x) \underset{|x|\to\infty}{=} \big[\phi_\beta(|x|) + o(1)\big](\log|x|)^\Delta$$

As LRP model is scale-covariant, a natural conjecture is that ϕ_β is a constant. This would mean that also

$$\psi_\beta(t) := \phi_\beta\big(e^{\gamma^{-t}}\big)$$

which is a 1-periodic function on the reals, $\psi_\beta(t+1) = \psi_\beta(t)$, would be constant. However, this turns out to be false ...

Theorem (B.-Krieger 2021)

Denote $m(\beta) := \sup\{k \in \mathbb{Z} \colon \gamma^{-k} \leqslant \log\beta\}$ *for* $\gamma := \frac{s}{2d}$ *and set*

$$u(\beta) := \gamma^{m(\beta)} \log\beta \in [1, \gamma^{-1}).$$

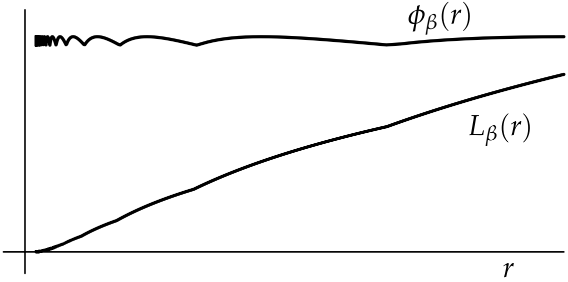
Then for all $t \in [0,1]$,

$$(\log\beta)^{\Delta}\, \psi_{\beta}\left(t - \frac{\log(\frac{u(\beta)}{2d-s})}{\log(1/\gamma)}\right) \xrightarrow[\beta\to\infty]{} \left[\frac{s}{2d-s}(2\gamma)^{-t} - 2\frac{s-d}{2d-s}2^{-t}\right](2d-s)^{\Delta}$$

with the limit uniform on $[0,1]$.

Corollary (Non-constancy)

For each $\mathfrak{q}$ *as above there is* $\beta_0 \in (0,\infty)$ *such that* ϕ_β *is not constant for* $\beta > \beta_0$.



Corollary (Scaling with β)

For each $\mathfrak{q}$ *as above there are* $c, C \in (0,\infty)$ *and* $\beta_1 > 1$ *such that*

$$\forall \beta > \beta_1 \, \forall r > 1 \colon \quad \frac{c}{(\log \beta)^\Delta} \leqslant \phi_\beta(r) \leqslant \frac{C}{(\log \beta)^\Delta}.$$

Idea: For $\beta \gg 1$ and each $n \geqslant 0$, we have

$$D(x,y) \begin{cases} \leqslant n, & \text{for } N := |x-y| \ll \beta^{\theta_n}, \\ \geqslant n+1, & \text{for } N \gg \beta^{\theta_n}, \end{cases}$$

for some exponent $\theta_n > 0$. This is checked inductively:

- $n = 0$: a direct edge between x and y occurs with probability $\sim \beta N^{-s}$, so we have $\theta_1 := 1/s$.
- $n = 1$: an edge between x and a ball of radius β^{θ_1} centered at y occurs with probability $\sim \beta \times \beta^{d\theta_1} N^{-s}$, so $\theta_2 := \frac{1}{s} + \frac{d}{s}\theta_1$.
- general n: an edge between a ball of radius β^{θ_k} centered at x and a ball of radius $\beta^{\theta_{n-k}}$ centered at y occurs with probability $\beta \times \beta^{d\theta_k} \times \beta^{d\theta_{n-k}} N^{-s}$ so

$$\theta_{n+1} := \frac{1}{s} + \frac{d}{s} \max_{0 \leqslant k \leqslant n} (\theta_k + \theta_{n-k})$$

Define $\theta_0 := 0$ and

$$\theta_{n+1} := \frac{1}{s} + \frac{d}{s} \max_{0 \leqslant k \leqslant n} (\theta_k + \theta_{n-k})$$

Lemma

Recall that $\gamma := \frac{s}{2d}$. *The following holds for all* $n \geqslant 0$*:*

$$\theta_{2^n-1} = \frac{1}{s} \frac{1-\gamma^n}{1-\gamma} \gamma^{-n+1}$$

and, for all integers k satisfying $2^n - 1 \leqslant k \leqslant 2^{n+1} - 1$,

$$\theta_k = \frac{2^{n+1} - 1 - k}{2^n} \theta_{2^n-1} + \frac{k - 2^n + 1}{2^n} \theta_{2^{n+1}-1}$$

In short, $k \mapsto \theta_k$ *is piecewise linear with explicit values for* $k \in \{2^n - 1 : n \geqslant 0\}$.

The above gives $D(0, \beta^{\theta_n} x) \sim n$. For $\lambda \in [0,1]$ with $2^n\lambda \in \mathbb{N}$ we get

$$D(0, \beta^{\theta_{\lambda 2^n + (1-\lambda)2^{n+1}}} x) \sim \lambda 2^n + (1-\lambda)2^{n+1} = [\lambda + (1-\lambda)2]2^n$$

Asymptotic analysis shows $\theta_{2^n} \sim \frac{1}{2d-s}\gamma^{-n}$ and so

$$\theta_{\lambda 2^n + (1-\lambda)2^{n+1}} \sim \lambda\theta_{2^n} + (1-\lambda)\theta_{2^{n+1}} \sim [\lambda + (1-\lambda)\gamma^{-1}]\frac{1}{2d-s}\gamma^{-n}$$

Hence

$$L_\beta\left(\beta^{[\lambda+(1-\lambda)\gamma^{-1}]\frac{1}{2d-s}}\right) \sim \lambda + (1-\lambda)2$$

and so

$$\phi_\beta\left(\beta^{[\lambda+(1-\lambda)\gamma^{-1}]\frac{1}{2d-s}}\right) \sim \frac{\lambda + (1-\lambda)2}{[\lambda + (1-\lambda)\gamma^{-1}]^\Delta}(2d-s)^\Delta \frac{1}{(\log\beta)^\Delta}$$

Now pick $t \in [0,1]$ so that $\lambda + (1-\lambda)\gamma^{-1} = \gamma^{-t}$. Then $\lambda = \frac{2d}{2d-s} - \frac{s}{2d-s}\gamma^{-t}$ and, after a calculation,

$$\phi_\beta\left(e^{\gamma^{-t}\frac{u(\beta)}{2d-s}}\right) \sim 2^{-t}\left[\frac{s}{2d-s}\gamma^{-t} - 2\frac{s-d}{2d-s}\right](2d-s)^\Delta \frac{1}{(\log\beta)^\Delta}$$

Shifting the argument and rewriting in terms of ψ_β, we get the claim.

Intuitive explanation

- Minimizing paths retain dyadic structure all the way to lattice scale.
- It thus matters when the dyadic refinements reach the scale $\beta^{-1/s}$ at which direct connections become prevalent.
- This mechamism dominates the limit $\beta \to \infty$; for β fixed ϕ_β contains all the local optimization effects.

After all of this, we only have the leading order scale.

No info obtained on shapes of large balls: $L_\beta(2r) = L_\beta(r) + o(1)$

Conjecture

For each $x \in \mathbb{Z}^d$, define Y_x by

$$D(0,x) = \phi_\beta(|x|)\Big[\log|x| - \frac{1}{2(2d-s)}\log\log|x| + Y_x\Big]^\Delta$$

Then

$$Y_x \xrightarrow[|x|\to\infty]{\text{law}} \text{Gumbel law}$$

So ball of radius r should be close to $\{x \in \mathbb{Z}^d\colon |x| \leqslant e^{r^{1/\Delta}\varphi(r)+O(1)}\}$ for some slowly varying φ. Long edges make this set very fuzzy, though.

THANK YOU!