Exceptional points of planar random walks

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joint work with

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Goal: Describe extremal/exceptional values of log-correlated processes

Completed for:

- Branching Brownian motion
- Two-dimensional Gaussian Free Field
- A couple of other related processes

Talk today: Local time of two-dimensional simple random walk

Erdős and Taylor (1960): How much time does simple random walk on $\mathbb{Z}^2$ of time-length n spent at its most frequented point?

For $\{X_n\colon n \geqslant 1\}$ SRW (with $X_0 = 0$), let $\ell_n(x) := \sum_{j=0}^n 1_{\{X_j = x\}}$

Then [ET60]: $\exists c_1, c_2 \in (0,\infty)$ s.t. w.h.p.

$$c_1(\log n)^2 \leqslant \max_{x\in\mathbb{Z}^d} \ell_n(x) \leqslant c_2(\log n)^2$$

Dembo, Peres, Rosen and Zeitouni (2001):

$$\frac{1}{(\log n)^2} \max_{x\in\mathbb{Z}^2} \ell_n(x) \;\underset{n\to\infty}{\overset{P}{\longrightarrow}}\; \frac{4}{\pi}$$

Q: Beyond leading order? Limit law?

In $d \geqslant 3$: $\max_x \ell_n(x) = O(\log n)$ with limit law Poisson

In $d = 1$: $\max_x \ell_n(x) = O(\sqrt{n})$, at most 3 maximizers! (B.Toth 2001)

Definition

Say $x \in \mathbb{Z}^2$ is a λ-*thick point* if

$$\ell_n(x) \geqslant \lambda \frac{4}{\pi} (\log n)^2$$

Dembo, Peres, Rosen and Zeitouni (2001):

$$\sum_{x \in \mathbb{Z}^2} 1_{\{\ell_n(x) \geqslant \lambda \frac{4}{\pi} (\log n)^2\}} = n^{2(1-\lambda)+o(1)}$$

Jego 2019: Full description of limit law for all $\lambda \in (0,1)$; cf previous talk

Dembo, Peres, Rosen and Zeitouni (2006): What does the set of avoided (a.k.a. late) points look like for random walk on torus?

SRW on torus $\mathbb{T}_n := (\mathbb{Z}/n\mathbb{Z})^2$. Define θ, n-*late points* as

$$\mathcal{L}_n(\alpha) := \Big\{x \in \mathbb{T}_n \colon \ell_{k(n)}(x) = 0 \text{ for } k(n) := \theta\frac{4}{\pi}(n\log n)^2\Big\}$$

Then w.h.p. $\mathcal{L}_n(\theta) = \emptyset$ for $\theta > 1$ and, for $\theta \in (0,1)$,

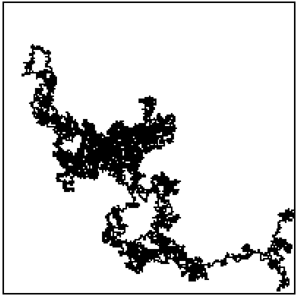
$$\big|\mathcal{L}_n(\theta)\big| = n^{2(1-\theta)+o(1)}$$

where $o(1) \to 0$ in probability

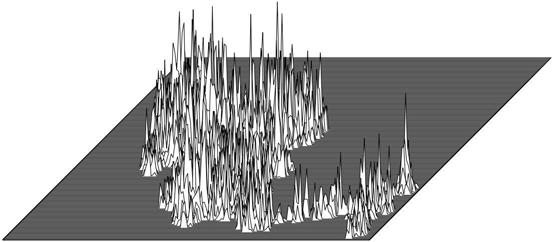
Q: Beyond leading order? Limit law?

In $d \geqslant 3$: strong coupling to i.i.d. (Miller and Souzi 2017)

Cover time problem: Ding, Lee and Peres (2012), Ding (2014), ..., Cortines, Louidor and Saglietti (2019)



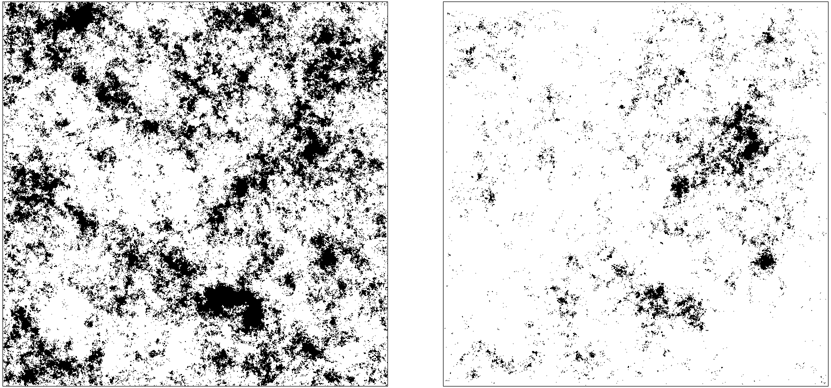
Set of visited sites



Local time profile

Note: Fractal on range of SRW

SRW run until θ-multiple of cover time



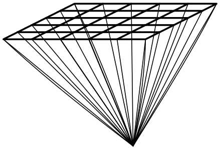
$N := 2000$, $\theta := 0.1$ (left) and $\theta := 0.3$ (right)

Planar domains: $D \subseteq \mathbb{R}^2$ bounded, open, "nice"

Discretization:

$$D_N := \left\{x \in \mathbb{Z}^2 \colon d_\infty\left({}^{x}/_{N}, D^{\mathrm{c}}\right) > \tfrac{1}{N}\right\}$$

Return mechanism: "boundary vertex" ϱ



$P^x :=$ law of SRW on $D_N \cup \{\varrho\}$ started from x

Local time:

$$L_t(x) := \frac{1}{\deg(x)} \ell_{\lfloor t \deg(\overline{D}_N)\rfloor}(x)$$

where $\deg(\overline{D}_N) := \sum_{x \in D_N \cup \{\varrho\}} \deg(x)$

Theorem (Max/min of local time)

Let $g := \frac{1}{2\pi}$ *and assume* $\{t_N\}_{N\geqslant 1}$ *is s.t., for some* $\theta > 0$,

$$\lim_{N\to\infty} \frac{t_N}{(\log N)^2} = 2g\theta,$$

Then for any $x_N \in D_N$, *in* P^{x_N}*-probability:*

$$\frac{1}{(\log N)^2} \max_{x\in D_N} L_{t_N}(x) \underset{N\to\infty}{\longrightarrow} 2g\big(\sqrt{\theta}+1\big)^2$$

and

$$\frac{1}{(\log N)^2} \min_{x\in D_N} L_{t_N}(x) \underset{N\to\infty}{\longrightarrow} 2g\big[(\sqrt{\theta}-1)\vee 0\big]^2$$

- Random walk is run for actual time $t_N \deg(\overline{D}_N) \asymp N^2(\log N)^2$
- $\theta = 1$ marks the leading-order scale of the cover time

Theorem (Points avoided by SRW)

There exist random a.s.-diffuse Borel measures

$$\{\mathcal{Z}^D_\lambda \colon \lambda \in (0,1)\}$$

on D satisfying $P(0 < \mathcal{Z}^D_\lambda(O) < \infty) = 1$ for all $O \subseteq D$ non-empty open s.t. for all $\theta \in (0,1)$ and any $\{t_N\}_{N\geqslant 1}$ with

$$t_N \sim 2g\theta(\log N)^2$$

we have

$$\frac{1}{\widehat{W}_N}\sum_{x\in D_N} 1_{\{L_{t_N}(x)=0\}}\,\delta_{x/N} \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \mathcal{Z}^D_{\sqrt{\theta}}$$

where $\widehat{W}_N := N^2 \mathrm{e}^{-\frac{t_N}{g\log N}}$

Recall: $\max_{x \in D_N} L_{t_N}(x) \sim 2g(\sqrt{\theta}+1)^2(\log N)^2$ and $\min_{x \in D_N} L_{t_N}(x) \sim 2g[(\sqrt{\theta}-1) \vee 0]^2(\log N)^2$

At typical points: $L_{t_N}(x) \sim t_N \sim 2g(\log N)^2$

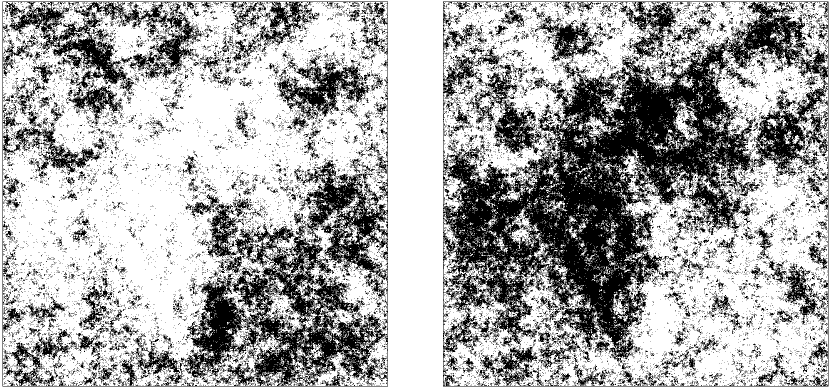
λ-thick point: for $\lambda \in [0,1]$,

$$L_{t_N}(x) \geqslant 2g\big(\sqrt{\theta}+\lambda\big)^2(\log N)^2$$

λ-thin point: for $\lambda \in [0, \sqrt{\theta} \wedge 1]$,

$$L_{t_N}(x) \leqslant 2g\big(\sqrt{\theta}-\lambda\big)^2(\log N)^2$$

More simulations: thick/thin points



$N := 1000$, $\theta := 10$ and $\lambda := \pm 0.1$ (same run of the walk)

Theorem (Random walk thick/thin points)

Suppose $t_N \sim 2g\theta(\log N)^2$ *for* $\theta > 0$ *and let* $\{a_N\}_{N\geqslant 1}$ *be s.t.*

$$\lim_{N\to\infty} \frac{a_N}{(\log N)^2} = 2g(\sqrt{\theta}+\lambda)^2$$

for some $\lambda \neq 0$ *with* $-(1 \wedge \sqrt{\theta}) < \lambda < 1$. *Then*

$$\zeta_N^D := \frac{1}{W_N} \sum_{x\in D_N} \delta_{x/N} \otimes \delta_{(L_{t_N}(x)-a_N)/\sqrt{2a_N}}$$

where

$$W_N := \frac{N^2}{\sqrt{\log N}} \mathrm{e}^{-\frac{(\sqrt{2t_N}-\sqrt{2a_N})^2}{2g\log N}}$$

obeys, for $\alpha := 2/\sqrt{g}$,

$$\zeta_N^D \ \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \ \sqrt{\frac{\sqrt{\theta}}{\sqrt{\theta}+\lambda}}\, \mathrm{e}^{-\alpha^2\lambda^2/16}\, \mathcal{Z}_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{\alpha\lambda h}\mathrm{d}h$$

Convenient notation: $\mathcal{Z}_{-\lambda}^D := \mathcal{Z}_\lambda^D$

Q: What are the $\mathcal{Z}_\lambda^D$?

Gaussian Free Field: centered Gaussian $\{h_x \colon x \in D_N\}$ with

$$\mathrm{Cov}(h_x, h_y) := G^{D_N}(x,y)$$

where $G^{D_N}(x,y) := \frac{1}{\deg(y)} E^x\big[\sum_{k=0}^{\tau_\rho - 1} 1_{\{X_k = y\}}\big]$ is the Green function

Large-N asymptotic: For $r^D(z) :=$ conformal radius of D from $z \in D$,

$$G^{D_N}(x,x) = g \log N + c_0 + g \log r^D(x/N) + o(1)$$

while, for $x \neq y$,

$$G^{D_N}(x,y) = \widehat{G}^D(x/N, y/N) + o(1)$$

where $\widehat{G}^D :=$ the (continuum) Green function on D

Setting scales:

Bolthausen, Deuschel and Giacomin (2001):

$$\max_{x\in D_N} h_x \sim 2\sqrt{g}\log N$$

Daviaud (2004): GFF thick points ($\lambda \in (0,1)$)

$$\#\{x \in D_N : h_x \geqslant 2\sqrt{g}\lambda \log N\} = N^{2(1-\lambda^2)+o(1)}$$

where $o(1) \to 0$ in probability

Biskup and Louidor (2019): Set

$$\eta_N^D := \frac{1}{K_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x - \widehat{a}_N}$$

where

$$K_N := \frac{N^2}{\sqrt{\log N}} \mathrm{e}^{-\frac{(\widehat{a}_N)^2}{2g \log N}}$$

and $\{\widehat{a}_N\}_{n \geqslant 1}$ obeys $\widehat{a}_N \sim 2\sqrt{g}\lambda \log N$ with $\lambda \in (0,1)$. Then

$$\eta_N^D \xrightarrow[N \to \infty]{\text{law}} \mathfrak{c}(\lambda)\, Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h} \mathrm{d}h$$

where Z_λ^D is the (normalized) LQG measure at parameter λ and $\mathfrak{c}(\lambda) := \lambda^{-1} \mathrm{e}^{2c_0 \lambda^2/g} / \sqrt{8\pi}$

Jego (2019): Similar result for λ-thick points of SRW

Limit law of $\max_{x \in D_N} h_x$ known too: Bramson, Ding and Zeitouni (2015), Biskup and Louidor (2015,2018,2020)

Theorem (Thick points of zero-average DGFF)

The LQG measure admits a decomposition

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\lambda\alpha\mathfrak{d}(x)Y}\, Z_\lambda^{D,0}(\mathrm{d}x)$$

where $Y \perp\!\!\!\perp Z_\lambda^{D,0}$ *with* $Y = \mathcal{N}(0,\sigma_D^2)$ *for*

$$\sigma_D^2 := \int_{D\times D} \mathrm{d}x\mathrm{d}y\, \widehat{G}^D(x,y)$$

and

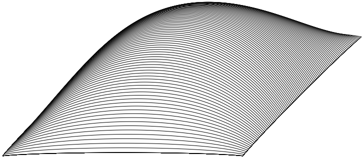
$$\mathfrak{d}(x) := \mathrm{Leb}(D)\frac{\int_D \mathrm{d}y\, \widehat{G}^D(x,y)}{\int_{D\times D} \mathrm{d}z\,\mathrm{d}y\, \widehat{G}^D(z,y)}$$

In addition,

$$\Bigl(\eta_N^D \Bigm| \sum_{x\in D_N} h_x = 0\Bigr) \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathfrak{c}(\lambda)\, Z_\lambda^{D,0}(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h$$

Intuitive form: $Z_\lambda^{D,0} \overset{\text{law}}{=} \Bigl(Z_\lambda^D \Bigm| \int_D h(x)\mathrm{d}x = 0\Bigr)$

Plot of $\mathfrak{d}$-function:



Theorem (Limit measure formula)

Limit laws for avoided/thick/thin points hold with

$$\mathcal{Z}_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathfrak{c}(\lambda)\mathrm{e}^{\lambda\alpha(\mathfrak{d}(x)-1)Y} Z_\lambda^{D,0}(\mathrm{d}x)$$

where $Y \perp\!\!\!\perp Z_\lambda^{D,0}$ *with* $Y = \mathcal{N}(0, \sigma_D^2)$

Note: RHS kind of “log-normal interpolation” of Z_λ^D and $Z_\lambda^{D,0}$

General strategy:

1. Prove tightness
2. Characterize the law of (subsequential) weak limits

Ad (1): Away from "endpoint" values of parameters, tightness is verified by a first-moment calculation

Ad (2): Earlier work (not giving sharp results) used 2nd moment calculations. This requires truncation which is doable for GFF but difficult for local time. An alternative approach goes via *2nd Ray-Knight Theorem/Dynkin Isomorphism*

Let $\{X_t \colon t \geqslant 0\}$:= constant-speed cont. time RW on $V \cup \{\varrho\}$. Set

$$\widehat{\ell}_t(x) := \frac{1}{\deg(x)} \int_0^t \mathrm{d}s\, 1_{\{X_s = x\}}$$

Denote $\widehat{\tau}_\varrho(t) := \inf\{s \geqslant 0 \colon \widehat{\ell}_s(\varrho) \geqslant t\}$ and set

$$\widehat{L}_t(x) := \ell_{\widehat{\tau}_\varrho(t)}(x)$$

Note: $\widehat{L}_t(\varrho) = t \dots$ parametrization by time at ϱ!

Theorem (Eisenbaum, Kaspi, Marcus, Rosen and Shi (2000))

For each $t \geqslant 0$, there is a coupling of $\widehat{L}_t^V$ and two GFFs h and $\tilde{h}$ on V (with zero boundary conditions at ϱ) s.t.

$$h^V \text{ and } \widehat{L}_t^V \text{ are independent}$$

and, for all $x \in V$,

$$\widehat{L}_t^V(x) + \frac{1}{2}(h_x^V)^2 = \frac{1}{2}\big(\tilde{h}_x + \sqrt{2t}\big)^2$$

Assume we wish to describe points where $\widehat{L}_{t_N}(x) \approx a_N$. If GFF on LHS can be neglected, meaning

$$\widehat{L}_{t_N}(x) \approx \frac{1}{2}(\tilde{h}_x + \sqrt{2t_N})^2$$

this corresponds to

$$\text{either} \quad \tilde{h} \approx \sqrt{2a_N} - \sqrt{2t_N} \quad \text{or} \quad \tilde{h} \approx -\sqrt{2a_N} - \sqrt{2t_N}$$

Only one eventuality relevant for thick/thin (not avoided) points. This reduces the problem to GFF thick point asymptotic at scale

$$\widehat{a}_N := \sqrt{2a_N} - \sqrt{2t_N}$$

For $t_N \sim 2g\theta(\log N)^2$ and $a_N \sim 2g(\sqrt{\theta} + \lambda)^2(\log N)^2$ we get

$$\widehat{a}_N \sim 2\sqrt{g}\lambda \log N$$

so results of Biskup and Louidor (2019) are applicable.

Key problem: Control the effect of GFF on LHS! This leads to a *convolution identity* that, fortunately, can be solved.

Light points: $\widehat{L}_{t_N}(x) = O(1)$

Key difference: For light/avoided points, GFF on LHS of 2nd Ray-Knight formula acts as a *thinning process* with intensity $O(1/\sqrt{\log N})$!

Light point measure:

$$\vartheta_N^D := \frac{1}{\widehat{W}_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{\widehat{L}_{t_N}(x)}$$

Lemma (Light point thinning)

Suppose ϑ_N^D tends in law to ϑ^D along subsequence $\{N_k\}$. Then

$$\frac{\sqrt{\log N}}{\widehat{W}_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{\widehat{L}_{t_N}(x)} \otimes \delta_{h_x} \ \underset{\substack{N=N_k \\ k\to\infty}}{\overset{\text{law}}{\longrightarrow}} \ \vartheta^D \otimes \frac{1}{\sqrt{2\pi g}} \text{Leb}$$

Proof: Conditional 2nd moment calculation

Recall: $\widehat{L}_{t_N} + \frac{1}{2}h^2 \overset{\text{law}}{=} \frac{1}{2}(\tilde{h} + \sqrt{2t_N})^2$. Evaluating the measure on LHS against functions of $\ell + \frac{1}{2}h^2$ yields:

Lemma (Convolution identity)

Given $f \in C_{\text{c}}(D \times [0,\infty))$ *with* $f \geqslant 0$ *denote*

$$f^{*\text{Leb}}(x,\ell) := \frac{1}{\sqrt{2\pi g}} \int_{\mathbb{R}} \mathrm{d}h\, f\bigl(x, \ell + \tfrac{h^2}{2}\bigr)$$

Then for every weak subsequential limit ϑ^D *of* ϑ^D_N*,*

$$\bigl\langle \vartheta^D, f^{*\text{Leb}} \bigr\rangle \overset{\text{law}}{=} \mathfrak{c}(\sqrt{\theta}) \int Z^D_{\sqrt{\theta}}(\mathrm{d}x) \otimes \mathrm{e}^{\alpha\sqrt{\theta}\,h} \mathrm{d}h\, f\bigl(x, \tfrac{h^2}{2}\bigr)$$

simultaneously for all f *as above*

Use analytic arguments to prove unique solution. This yields:

$$\vartheta_N^D \xrightarrow[N\to\infty]{\text{law}} \sqrt{2\pi g}\,\mathfrak{c}(\sqrt{\theta})\; Z^D_{\sqrt{\theta}}(\mathrm{d}x)\otimes\mu(\mathrm{d}h)$$

where

$$\mu(\mathrm{d}h) := \delta_0(\mathrm{d}h) + \Biggl(\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\Bigl(\frac{\alpha^2\theta}{2}\Bigr)^{n+1}h^n\Biggr)1_{(0,\infty)}(h)\,\mathrm{d}h$$

Restricting to atom at 0 gives

$$\frac{\sqrt{\log N}}{\widehat{W}_N}\sum_{x\in D_N}1_{\{\widehat{L}_{t_N}(x)=0\}}\delta_{x/N} \xrightarrow[N\to\infty]{\text{law}} \sqrt{2\pi g}\,\mathfrak{c}(\sqrt{\theta})\; Z^D_{\sqrt{\theta}}(\mathrm{d}x)$$

This gives law of avoided points in time parametrization by local time at ϱ

Need to reduce time parametrization to:

1. the actual time of cont.-time walk
2. discrete-time random walk

Ad (1): Need to control the fluctuations of total time at a given time at ϱ. Key observation: Summing 2nd-RN identity we get

$$\sum_{x\in D_N}\big(L_t(x)-t\big)+\frac{1}{2}\sum_{x\in V}\big(h_x^2-Eh_x^2\big)=\frac{1}{2}\sum_{x\in V}\big(\tilde h_x^2-E\tilde h_x^2\big)+t\sum_{x\in D_N}\tilde h_x$$

A covariance calculation shows that the GFF-squared terms are both order $|D_N|^2$. Since $t=t_N\to\infty$, they can be neglected. Hence:

- Fluctuations of total time are asymptotically Gaussian with mean zero and variance $\sim\sigma_D^2 t|D_N|^2$
- Fixing the fluctuations is tantamount to fixing $\sum_{x\in D_N}\tilde h_x$

Put together, this step amounts to

$$\text{replacing } Z^D_{\sqrt{\theta}} \text{ by } \mathrm{e}^{\lambda\alpha(\mathfrak{d}(x)-1)Y}\,Z^{D,0}_\lambda(\mathrm{d}x)$$

Ad (2): No effect for limit for avoided points, but additional factor due to fluctuations of sums of exponentials appears for thick/thin points.

Goal: Random walk with periodic/reflecting b.c.

Key issue: GFF now pinned at starting point; picks up additional $\sqrt{N}$ near-constant fluctuation. This shifts levels by term that diverges as $N \to \infty$.

THANK YOU!