Extrema of log-correlated processes

with introduction to Extreme Value Theory

Marek Biskup (UCLA)

Baby Steps Beyond the Horizon
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Goals

Introduce and spawn interest in:

- Extreme value theory
- A class of processes "moving" current probability theory

We will also touch upon fancy buzzwords such as

- scaling limit
- extremal process
- branching random walk
- Gaussian free field
- etc

Let $X_1, \ldots, X_n$ be random variables (no assumptions).

Absolute maximum

$$M_n := \max_{i=1,\ldots,n} X_i$$

Questions:

- Order of magnitude
- Typical fluctuation
- Centered distribution

Quantitative version: Are there a_n and b_n such that

$$t \mapsto \mathbb{P}\big(M_n - a_n \leqslant b_n t\big)$$

tends, as $n \to \infty$, to a non-trivial function?

Some real life examples

- Ages of individuals (people) in a population
- Wildfire, earthquake, flood or tornado sizes, insurance losses
- Infrastructure failures (power grid, pipelines, cell network etc)
- Athletic achievements (100 m sprint runs)
- Annual temperature maxima

Assume: $X_1, X_2, \ldots, X_n$ independent and identically distributed (i.i.d)

Basic calculation:

$$\mathbb{P}\big(M_n - a_n \leqslant b_n t\big) = \mathbb{P}\Big(\bigcap_{i=1}^{n}\{X_i \leqslant a_n + b_n t\}\Big)$$

$$\overset{\text{i.i.d.}}{=} \prod_{i=1}^{n} \mathbb{P}\big(X_i \leqslant a_n + b_n t\big) \overset{\text{i.i.d.}}{=} \mathbb{P}\big(X_1 \leqslant a_n + b_n t\big)^n$$

As non-degeneracy requires $\mathbb{P}(X_1 \leqslant a_n + b_n t) \to 1$, we are justified to write

$$\mathbb{P}\big(X_1 \leqslant a_n + b_n t\big)^n = \big[1 - \mathbb{P}(X_1 > a_n + b_n t)\big]^n$$

$$\approx \mathrm{e}^{-n\mathbb{P}(X_1 > a_n + b_n t)}$$

and so we need ...

… to find a_n and b_n so that

$$F_n(t) := n\mathbb{P}(X_1 > a_n + b_n t)$$

tends to a non-degenerate limit. This requires a certain amount of regularity. The limit law is then quite constrained:

Theorem (Fisher-Trippet-Gnedenko)

Suppose $F(t) := \lim_{n\to\infty} F_n(t)$ *exists for all* $t \in \mathbb{R}$. *Then* $G(t) := \mathrm{e}^{-F(t)}$ *is, up to shift and scaling, one of the functions:*

- (Weibull class) $G(t) = \mathrm{e}^{-|t|^\alpha}$ *for* $t \leqslant 0$ *and* $G(t) = 1$ *for* $t \geqslant 0$
- (Fréchet class) $G(t) = \mathrm{e}^{-t^{-\alpha}}$ *for* $t \geqslant 0$ *and* $G(t) = 0$ *for* $t < 0$
- (Gumbel class) $G(t) = \mathrm{e}^{-\mathrm{e}^{-t}}$

Fréchet (1924), Fisher & Trippet (1928), von Mises (1936), Gnedenko (1943), …

Suppose that $X_1, X_2, \dots$ are i.i.d. normal $\mathcal{N}(0,1)$. Then for $t \gg 1$,

$$\mathbb{P}(X_1 > t) \approx \frac{1}{t} e^{-t^2/2}$$

and so

$$n\mathbb{P}\left(X_1 > \underbrace{\sqrt{2\log n} - \frac{\log\log n}{2\sqrt{2\log n}}}_{a_n} + b_n t\right) = \frac{1}{\sqrt{2}} e^{-\sqrt{2\log n}\, b_n t (1+o(1))}$$

Now set $b_n := \frac{1}{\sqrt{2\log n}}$ to get

$$n\mathbb{P}(X_1 > a_n + b_n t) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}} e^{-t}$$

The centered maximum is asymptotically Gumbel, $G(t) = e^{-\frac{1}{\sqrt{2}} e^{-t}}$.

Question: How about the second, third, etc (local) maxima?

Extremal process: random point measure

$$\eta_n := \sum_{i=1}^{n} \delta_{(X_i - a_n)/b_n}$$

captures the whole set of near-maximal values.

Theorem

Let G be the limit CDF in previous Theorem. Then for $F(t) := -\log G(t)$,

$$\eta_n \xrightarrow[n\to\infty]{\text{law}} \text{PPP}\,(\mathrm{d}F)$$

where $\text{PPP}\,(\mu) :=$ *Poisson point process with intensity* μ.

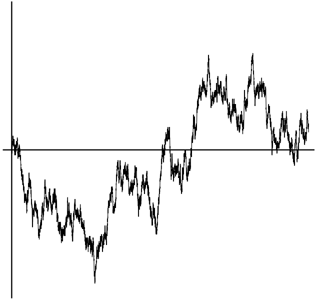
L. De Haan, A. Ferreira. *Extreme value theory: an introduction*. Springer Verlag.

Assume $Y_1, Y_2, \ldots$ are i.i.d. and that $X_1, X_2, \ldots$ are given by

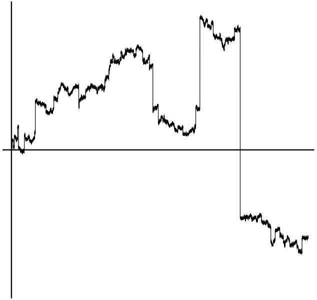
$$X_k := \sum_{i=1}^{k} Y_i$$

This makes $k \mapsto X_k$ a **random walk.**

Typical plots (assuming $\mathbb{E}Y_1 = 0$ for simplicity):



$\mathbb{E}(Y_1^2) < \infty$



$\mathbb{E}(Y_1^2) = \infty$

Theorem (Donsker's Invariance Principle)

Assume the above setting with $\mathbb{E}(Y_1) = 0$ and $\mathbb{E}(Y_1^2) < \infty$. Then, as $n \to \infty$, the distribution of $t \mapsto W_t^{(n)}$ on $C[0,\infty)$ where

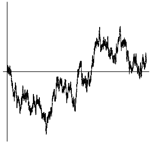
$$W_t^{(n)} := \frac{1}{\sqrt{n}}\Big(X_{\lfloor nt \rfloor} + (tn - \lfloor nt \rfloor)X_{\lfloor nt \rfloor + 1}\Big)$$

tends to the law of Brownian motion $t \mapsto B_t$ with $E(B_t^2) = \mathbb{E}(Y_1^2)t$.

Brownian motion := random continuous function with independent centered Gaussian increments such that $E(B_t B_s) = \min\{t, s\}$

Law of Brownian motion on $C[0,\infty)$ is called **Wiener measure**

An example of a **scaling limit:** at global scale, a non-trivial limit process is obtained



Convergence in law := expectations of bounded continuous functions converge. So, in particular, we have:

Corollary

Assume the above setting with $\mathbb{E}(Y_1) = 0$ *and* $\mathbb{E}(Y_1^2) < \infty$. *Then*

$$\frac{1}{\sqrt{n}} \max_{k=1,\dots,n} X_k \xrightarrow[n\to\infty]{\text{law}} \max_{0\leqslant t\leqslant 1} B_t \overset{\text{law}}{=} \left|\mathcal{N}(0, \mathbb{E}(Y_1^2))\right|$$

Note: Limit law **universal** modulo scaling, where (vaguely)

universal := independent of particulars of the model

Still, very different structure than for Model I!

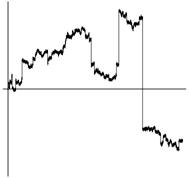
Changes for $\mathbb{E}(Y_1) < 0$:

$$\max_{k=1,\dots,n} X_k \text{ converges in law without centering or scaling}$$

The limit distribution is NOT universal, individual values matter

Changes for $\mathbb{E}(Y_1^2) = \infty$, the so called **heavy tailed** regime:

- Limit process has **stable law**
- Need to scale by $n^{1/\alpha}$ for $\alpha \in (0, 2)$ instead of $\sqrt{n}$



Intermediate regime?

Model I: determined by local properties (individual entries matter)
Model II: determined by global properties (only averages matter)

Universal behavior obtained, stable under perturbations.

Question: Is there a regime where both local and global properties matter?

Let $b \in \mathbb{N}$ obey $b \geqslant 2$ and set $\mathbb{L}_0 := \{\varrho\}$ and $\mathbb{L}_n := \{1, \dots, b\}^n$ for $n \geqslant 1$.

A **b-ary tree of depth** n := graph with vertex set $\mathbb{T}_n := \bigcup_{k=0}^{n} \mathbb{L}_k$ and an edge between any vertices of the form $(\sigma_1, \dots, \sigma_k)$ and $(\sigma_1, \dots, \sigma_k, \sigma_{k+1})$.

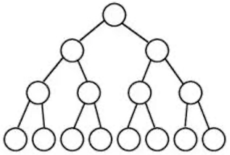
Definition

Given i.i.d. r.v.'s $\{Y_\sigma : \sigma \in \mathbb{T}_n\}$, a **branching random walk** of depth n and step distribution Y is the family $\{X_\sigma : \sigma \in \mathbb{L}_n\}$ where, for $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{L}_n$,

$$X_\sigma := Y_\varrho + \sum_{k=1}^{n} Y_{(\sigma_1, \dots, \sigma_k)}$$

Key facts:

- Along each root-to-leaf path, X_σ is a random walk.
- Walks along different path are correlated (by common part).
- There are exponentially many root-to-leaf paths.



Theorem (Aïdekon 2013)

Suppose that Y_1 is continuously distributed with

$$\mathbb{E}e^{Y_1} = \frac{1}{b} \quad \text{and} \quad \mathbb{E}\left(Y_1 e^{Y_1}\right) = 0$$

Then there exists a non-degenerate, positive random variable Z such that

$$\mathbb{P}\left(\max_{\sigma \in \mathbb{L}_n} X_\sigma \leqslant -\frac{3}{2}\log n + t\right) \underset{n\to\infty}{\longrightarrow} E\left(e^{-Ze^{-t}}\right)$$

Main differences: For i.i.d.'s we'd get

- Z constant a.s.
- $1/2$ instead of $3/2$

As before, denote the (empirical) extremal process by

$$\eta_n := \sum_{\sigma \in \mathbb{L}_n} \delta_{(X_\sigma - a_n)/b_n}$$

Theorem (Madaule 2017)

Under assumptions of the previous theorem,

$$\eta_n \xrightarrow[n\to\infty]{\text{law}} \mathrm{PPP}\left(Z\mathrm{e}^{-t}\mathrm{d}t\right)$$

where the Poisson point process is defined conditionally on Z.

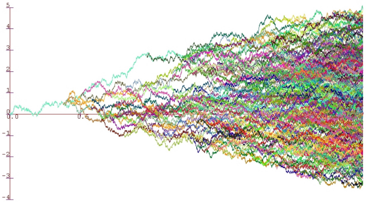
Punchline:

maximum/extremal process: Gumbel with a **random** shift by $\log Z$

The random shift arises from early "generations" of the process.

Branching Brownian motion: Early work by Fisher (1937), Kolmogorov, Petrovsky and Piskunov (1937) on **Fisher-KPP equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$$



(stolen from N. Berestycki's notes)

for reaction-diffusion processes in sciences.

McKean (1975): probabilistic interpretation

$$u(t,x) = P\big(\text{all BBM particles at time } t \text{ left of } x\big)$$

for "step" initial condition $u(0,x) = 1_{[0,\infty)}(x)$

Maximum: Bramson (1978,1983), Lalley and Selke (1987)

Extremal process is randomly shifted Gumbel: Arguin, Bovier and Kistler (2011,2012), Aïdekon, Berestycki, Brunet and Shi (2013)

Gaussian free field in $\Lambda \subseteq \mathbb{Z}^d$: Gaussian process $\{h_x : x \in \Lambda\}$ with

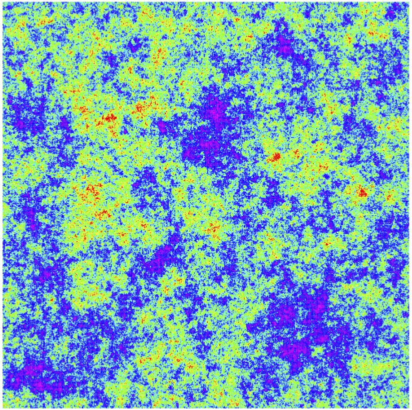
$$\mathbb{E}(h_x) = 0 \quad \text{and} \quad \mathbb{E}(h_x h_y) = G^{\Lambda}(x,y)$$

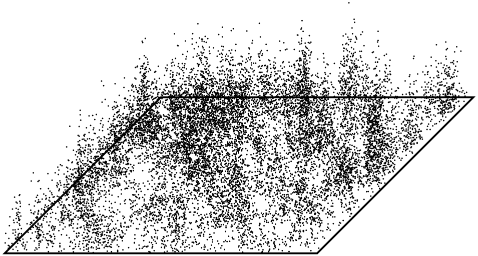
where $G^{\Lambda}(x,y) :=$ expected number of visits to y by simple random walk started at x before exiting Λ.

$d = 2$ special due to **logarithmic correlations**: for $\Lambda_N := (0,N)^d \cap \mathbb{Z}^d$,

$$G^{\Lambda_N}(x,y) = \frac{2}{\pi}\log\left(\frac{N}{1+|x-y|}\right) + O(1)$$

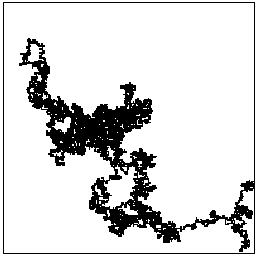
Caused by marginal-recurrence of simple random walk in $d = 2$

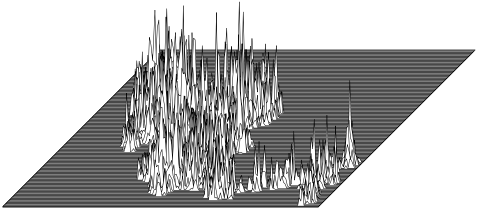




Level lines (SLE_4): Schramm and Sheffield (2009)
Maximum: Bramson, Ding and Zeitouni (2015)
Extremal process: B.-Louidor (2015, 2018, 2020)

Most frequent point of simple random walk: How much time does a simple random walk of given time length spend at its most visited point?





leading order: Erdős & Taylor (1960), Dembo, Peres, Rosen & Zeitouni (2001)
towards actual limit: Jego (2020), B.-Louidor (2021), ...

Strong connection to scaling limit of the **cover time**, etc

Take away message

- Log-correlated processes form a **universality class** where both local and global correlation structures matter.
- They are **ubiquitous** in (particularly, two-dimensional) probability
- Their maximum/extremal process has a **randomly shifted Gumbel law**

THANK YOU!