On Three Techniques for Rigorous Proofs of First-Order Phase Transitions

een wetenschappelijke proeve op het gebied van de Natuurwetenschappen, Wiskunde en Informatica

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To Iva

... for not letting me forget that life is
more than a mere mathematical formula.

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INTRODUCTION

1. PHASE TRANSITIONS IN EQUILIBRIUM SYSTEMS

Phase transitions are phenomena manifested by jumps or infinitely sharp peaks occurring in otherwise smooth physical quantities. There are plenty of phase transitions around us: a flu epidemic suddenly paralyzes the population in a large city, the financial market experiences a collapse of prices, water turns into ice. Whereas the first two examples are *dynamic*, involving a flow of time, the third example is *static*: water under cooling freezes to ice no matter how fast or slow the cooling process actually is. This is so because both substances are specific *phases* of the chemical H_2O in *thermal equilibrium*, i.e., in the idealized state where all the memory about the initial state is lost due to natural relaxation processes.

The concept of equilibrium and its probabilistic description was pioneered in the last century by Boltzmann, to whom we owe the crucial discovery that the probability to find a configuration (e.g., an arrangement of particles with given positions and velocities) is proportional to $\exp(-\beta \times \text{energy})$, where β is the inverse temperature. The connection with existing empirical thermodynamics was later established by Gibbs, who also discovered the importance of convexity properties of thermodynamic functions and used them to formulate the second law of thermodynamics in terms of a *variational principle*. The latter entails that, amongst all *a priori* states, equilibrium corresponds to the minimum or maximum of some thermodynamic potential (e.g., the Helmholtz free energy or pressure), whose selection depends on the parametrization.

Despite an early criticism on conceptual grounds, the Boltzmann-Gibbs theory was very successful in providing a theoretical background to many experimental facts. Nevertheless, attempts demonstrate mathematically within this description the occurrence of a phase transition failed, the reason being that the exponential weight factor did not seem to have any non-analyticities in β (at least not in the examples of realistic gases that had actually been tested). Only in the 1930's was it realized that an additional idealization is needed: the so-called *thermodynamic limit*. This is because the peaks signaling phase transitions are smooth in any finite system, and become sharper and narrower as the system size tends to infinity. Strictly speaking, it is then only in an infinite system that they become actual singularities; however, the typical size of a physicist's system is 10^{23} particles, a number fairly 'close' to infinity.

Mathematically, the procedure of the thermodynamic limit was put on a firm basis by Dobrushin, Lanford and Ruelle (DLR), who proposed to study directly the *infinite-volume measures*, with the conditional distributions inside any finite volume given the outside (exemplified by prescribing a *boundary condition*) being of the Boltzmann type. The measures arising thereby (and named after Gibbs) carry along a significant deal of information about the 'true' (i.e., finite) physical system. They embody the bulk, i.e., macroscopic, properties like the particle density or the degree of spontaneous magnetization, as well as the microscopic inter-dependence structure (i.e., the correlations). However, the most interesting aspect is their multiplicity: The occurrence of *more than one* distinct Gibbs measures in association with a *single* Hamiltonian and temperature indicates coexistence of different phases of the system. In such a case, the system is said to undergo a *first-order* phase transition. Higher-order phase transitions also exist, the distinction being whether it is the first or a higher derivative of the thermodynamic potential that exhibits a jump.

A famous, but mathematically yet rather unfounded, example of a first-order phase transition is the above water/ice transition. There are two Gibbs measures: one describing ice with particles arranged in a crystalline order, the other featuring essentially chaotic particle arrangements. Notably, the two phases pertain to one and the same particle system: it is the boundary condition that chooses between water and ice[1]. Due to the passage to the thermodynamic limit, no finite part of the boundary condition really matters—it is its *tail* behavior that determines the resulting phase. For this reason, the phase coexistence goes also by another name, *long-range order*, to express that the tendency to assume a certain order is global, notwithstanding the locality of interactions.

Over the last thirty years, long-range order has been established in a plethora of classical lattice systems, wherein the DLR-characterization of the equilibrium phases has proved to be extremely fertile. In fact, the theoretical foundations seem to be by and large complete, at least for systems with not too ill-behaved interactions. Some progress has been made also for continuum systems (e.g., non-ideal classical gases), but there a lot of 'fundamental' questions are still wide open (like the existence of the liquid-vapor transition[2]). In the area of lattice systems, finer questions concerning the broad variety of phases, critical behavior and the quantitative aspects of the universality are currently under intensive study.

DLR-formalism.

The DLR-formalism is a leitmotif for this thesis. Let us therefore quickly recount its most important definitions and claims. For a detailed exposition and proofs of the claims made below we refer to Georgii's monograph [Ge].

We begin by restraining the set of systems of interest. We shall concentrate on models with an underlying lattice structure and with a finite state space at each of the lattice sites.

[1] This distinctive role of the boundary condition applies only to the system with no particle conservation, as described by the grandcanonical ensemble. If the system does not allow for the particle exchange, the situation is more complicated. We refer the reader to [vEFS, Section 2.6.5] for a discussion of these issues.

[2] Progress in the case of a realistic interaction between particles has recently been made in [LMP] (see also [P]), nevertheless, the two-particle forces of Lenard-Jones type are still beyond reach.

Furthermore, we shall suppose that the state space is the same at all lattice sites. This is just a mild limitation, since the extension to the more general case is rather straightforward.

Thus let $\mathcal{S}$ be a single-site state space and let the underlying spatial structure be the d-dimensional hypercubic lattice $\mathbb{Z}^d$. The configuration space is then $\Omega = \mathcal{S}^{\mathbb{Z}^d}$. We denote configurations in $\mathbb{Z}^d$ by $\boldsymbol{\sigma} \in \Omega$, configurations in $\Lambda \subset \mathbb{Z}^d$ by $\boldsymbol{\sigma}_\Lambda \in \mathcal{S}^\Lambda$. There is a natural topology on Ω, the product topology, which in turn generates a natural σ-algebra $\mathcal{F}$, namely, the Borel σ-algebra of quasi-local observables. Let $\mathcal{F}_\Lambda$ be the projection of $\mathcal{F}$ onto a finite set $\Lambda \subset \mathbb{Z}^d$ and let $\mathcal{T}_\Lambda$ be the projection of $\mathcal{F}$ onto the complement $\Lambda^{\mathrm{c}} = \mathbb{Z}^d \setminus \Lambda$. Then the measure μ satisfies the DLR-condition with interaction Hamiltonians $(H_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ if

$$\mu\big(\boldsymbol{\sigma}_\Lambda|\mathcal{T}_\Lambda\big)(\tilde{\boldsymbol{\sigma}}) = \frac{e^{-\beta H_\Lambda(\boldsymbol{\sigma}_\Lambda|\tilde{\boldsymbol{\sigma}}_{\Lambda^{\mathrm{c}}})}}{Z_\Lambda^{\tilde{\boldsymbol{\sigma}}}(\beta)} \qquad \text{for } \boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}} \in \Omega \text{ and all } \Lambda \text{ finite,} \tag{1.1}$$

where $\tilde{\boldsymbol{\sigma}}$ is the boundary condition. The volume dependence of the Hamiltonians is to be chosen such that $H_{\Lambda'} - H_\Lambda$, for any $\Lambda \subset \Lambda'$, does not depend on the configuration in Λ. This is to ensure so-called *consistency*, asserting that $E_\mu\big(E_\mu(A|\mathcal{T}_\Lambda)|\mathcal{T}_{\Lambda'}\big) = E_\mu(A|\mathcal{T}_{\Lambda'})$ for any event $A \in \mathcal{F}$, where E_μ stands for the expectation w.r.t. μ. In other words, consistency means that also the conditional measures satisfy the DLR-condition in any smaller volume.

It follows from the Backward Martingale Theorem that the 'thermodynamic limit' of (1.1) exists, i.e.,

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} E_\mu(A|\mathcal{T}_\Lambda) = E_\mu(A|\mathcal{T}) \qquad \mu\text{-a.s. } \forall A \in \mathcal{F}, \tag{1.2}$$

where

$$\mathcal{T} = \bigcap_{\substack{\Delta\subset\mathbb{Z}^d\\ \text{finite}}} \mathcal{T}_\Delta \tag{1.3}$$

is the *tail* σ-algebra. The significance of the measure $E_\mu(\,\cdot\,|\mathcal{T})$ lies in that: (1) it is an infinite-volume Gibbs measure, (2) it is *extremal* in the sense that it is $\{0,1\}$-valued on the events in $\mathcal{T}$. In fact, there is an important decomposition theorem claiming that any Gibbs measure can be *uniquely* decomposed into extremal Gibbs measures. Naturally, physical phases are then associated with the extremal Gibbs measures[3], rather than with non-trivial convex combinations (the coefficients in the latter represent our 'degree of uncertainty' about the phase of the system).

The denominator in (1.1) is the *partition function*, with the superscript indicating the boundary condition. Physically, this is one of the most important quantities, and it grows or decays exponentially with $|\Lambda|$. The rate of this decay

$$f_\beta = -\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log Z_\Lambda^{\tilde{\boldsymbol{\sigma}}}(\beta) \tag{1.4}$$

[3] In fact, the extremal *translation-invariant* measures, in order to account for possible breaking of translation symmetry that is not reflected in the appearance of physically 'non-equivalent' phases. See [vEFS, Section 2.4.9] for a commentary on these subtleties.

is the *free energy* (or *pressure*, according to the context). It turns out that for not too wild interactions the limit exists and is independent of the boundary condition. In macroscopic thermodynamics, free energy (or pressure) are important thermodynamic potentials connecting up the various internal parameters of the system (e.g., in the thermodynamic *equation of state*). In fact, one of the main achievements of Boltzmann's probabilistic formalism is that it provides a way to compute these hitherto *ad hoc* given quantities on the basis of microscopic considerations. Unfortunately, such a computation is often a highly cumbersome task, insofar completed only for a couple of simple, predominantly two-dimensional, models.

The contents of this thesis.

This thesis presents four papers, cited hereafter by the first letter in the authors' names, i.e., [BdH], [BCK], [B], and [BBCK], demonstrating on examples of specific systems three basic techniques to deal with phase-transition phenomena:

(1) Variational principles and large deviation theory.
(2) Correlation inequalities, in particular, FKG and reflection positivity.
(3) Perturbative expansion techniques exemplified by the Pirogov-Sinai theory.

In the remainder of this introductory chapter, we endeavor to explain the pivotal notions of these techniques and to outline how they are used in the four papers. For better orientation of the reader, the papers are made clearly visible by the appearance of ♣ in the heading.

2. VARIATIONAL PRINCIPLES AND SYSTEMS WITH FROZEN RANDOMNESS

Drawing our attention back to equilibrium thermodynamics, let us recall that in the original Gibbs description of a gas characterized by volume V and temperature T, the pressure was obtained by taking the infimum of the quantity $\frac{E-TS}{V}$, where E denotes the energy and S the entropy, over all admissible[4] volumes V. This suggests the presence of an underlying variational principle. And, indeed, underneath lies a variation principle for the specific energy[5]: in the class of translation-invariant measures (and Hamiltonians), an alternative characterization of (1.1) is that the Gibbs measures are *minimizers* of the functional $\mu \mapsto f(\mu) = e(\mu) - Ts(\mu)$, where

$$e(\mu) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} E_\mu(H_\Lambda) \quad \text{and} \quad s(\mu) = -\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{k_\mathrm{B}}{|\Lambda|} E_\mu\bigl(\log \mu_\Lambda\bigr), \tag{2.1}$$

with μ_Λ denoting the projection of μ onto $\mathcal{F}_\Lambda$ and k_B denoting the Boltzmann constant. Here $e(\mu)$ is the specific energy of μ, whereas $s(\mu)$ is the *entropy* associated with the mea-

[4] In physics jargon, the word 'volume' refers tacitly to some regular container, e.g., a cylinder with a piston. Having the 'shape' fixed, the infimum then runs over the numerical volume of the enclosed gas.

[5] This applies to the case when only one *external* parameter of the system is to be controled, namely, the specific energy. We refer to Wightman's excellent introduction to Israel's book [I] for a comprehensive treatment of these variational principles and their interpretation in classical thermodynamics.

sure μ, which roughly equals the logarithm of the effective number of configurations per-site on which μ is supported.

The significance of the variational principle for $\mu \mapsto f(\mu)$ lies in its large deviation content: under the counting measure on $\mathcal{S}^\Lambda$, conditioned on having a fixed specific energy (this gives rise to the *microcanonical ensemble*), the probability that the periodized empirical measure λ_Λ is 'close' to some measure ν decays exponentially in $|\Lambda|$, with the rate $\beta\big(f(\mu) - f_\beta\big)$. Here β is adjusted such that a Gibbs measure corresponding to β has the same specific energy as the microcanonical measure above. The minimum of $f(\mu)$ is indeed the very number f_β in (1.1) and it is attained by the measures satisfying the DLR condition, which implies that the microcanonical measure above converges weakly as $\Lambda \nearrow \mathbb{Z}^d$ to a Gibbs measure defined in the DLR-sense. This procedure requires the indentification $\beta = (k_\mathrm{B}T)^{-1}$, whereby originates the interpretation of β as the inverse temperature.

In a sense, the function $\beta \mapsto f_\beta$ comprises all the information about possible phase transitions in β variable: At the points where $\beta \mapsto f_\beta$ is not continuously differentiable (by convexity, f_β has always the right and left derivatives), there are at least two distinct extremal Gibbs measures, distinguished by the value of the specific energy (which is essentially given by the right or left derivative $\frac{\mathrm{d}f_\beta}{\mathrm{d}\beta}$). This becomes understandable if one views $\beta \mapsto f_\beta$ as the Legendre transform of the rate function for large deviations in the energy. Namely, a cusp in $\beta \mapsto f_\beta$ is reflected by a flat piece in the graph of this rate function, yielding that two distinct values of energy are equally probable, at least on an exponential scale. With a little bit of additional effort, the existence of two distinct minimizers of $\mu \mapsto f(\mu)$ (characterized by these two energies) can be shown.

This observation offers a consistent strategy for establishing first-order phase transitions: prove that $\beta \mapsto f_\beta$ exhibits a cusp and use some general arguments to prove the existence and study the properties of the distinct phases. This was first adopted in the well-known Peierls argument, and it lies is the cornerstone of the Pirogov-Sinai theory outlined below. On the other hand, the large deviation principle and some specific knowledge on the free energy can also be used to *disprove* first-order phase transitions. In fact, this is what happens in [BdH].

♣ [BdH]—Uniqueness of the Gibbs state.

In the paper 'A Heteropolymer near a Linear Interface' [BdH], we study a polymer with hydrophobic and hydrophilic monomers in the vicinity of an oil-water interface. The interaction Hamiltonian is of the form

$$H_\Lambda = -\sum_{i\in\Lambda} (\omega_i + h)\mathrm{sign}(S_i), \tag{2.2}$$

where ω_i indicates whether the i-th monomer is hydrophobic ($\omega_i = +1$) or hydrophilic ($\omega_i = -1$), $S_i \in \mathbb{Z}$ is the vertical coordinate of the i-th monomer w.r.t. the interface, and h expresses the asymmetry between the effects of the two media (i.e., oil in $\Lambda \times \mathbb{Z}^+$ and water in $\Lambda \times \mathbb{Z}^-$).

The spatial configuration of the polymers is modeled by directed paths on $\mathbb{Z}^2$ (see Fig. 1). The analysis of the model is delicate because of the distribution of the monomer species along the path—their types are fixed and sampled from i.i.d. sequences of ±1's with equal densities.

The interaction favors hydrophobic monomers placed in oil ($S_i > 0, \omega_i = +1$) and hydrophilic monomers placed in water ($S_i < 0$, $\omega_i = -1$), i.e., a fairly restricted arrangement, whereas entropy favors as much freedom for fluctuations as possible. The free-energy minimization can, in fact, exhibit either of these options, depending on the parameter values: it has been shown in [BodH] that there is a non-trivial localization/delocalization transition (see Fig. 2).

In the *delocalized phase*, which occurs for h large enough, the fact that there are two types of monomers favoring different sides of the interface is irrelevant: the free energy corresponds to the entire path being on one side of the interface, namely $f_\beta = |h|$. The *localized phase*, which occurs for h close to zero, is then characterized by $f_\beta > |h|$, i.e., an *excess* over the 'delocalized' value. This indicates that the path stays close to the interface. However, as our analysis shows, the mathematical justification of this fact is a non-trivial issue.

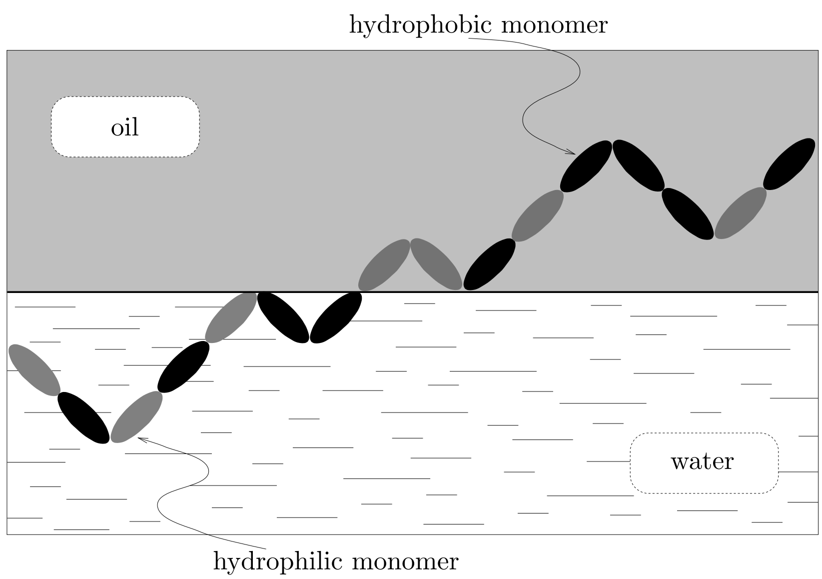

FIG. 1. A configuration of the heteropolymer near an oil-water interface.

The next question we address in [BdH] concerns the *ergodic path behavior*, i.e., whether Cesaro averages along the polymer converge a.s. to deterministic quantities. Both the question of localization and ergodicity are answered in the affirmative, relying extensively on the large deviation nature of the free energy. Namely, if the path shoots away for an atypically long excursion, then it pays for doing so by an amount per monomer given by the excess free energy $\beta(f_\beta - |h|)$ on an exponential scale. With the help of this idea, estimates on the typical excursion length and height are derived, leading via a coupling argument to an ergodic theorem for all local observables. As a consequence of using non-perturbative analysis based on large deviation theory, the result applies to the *entire* localized region, all the way up to the transition line. This would hardly have been the case if, for instance, perturbative methods of the Pirogov-Sinai type had been invoked.

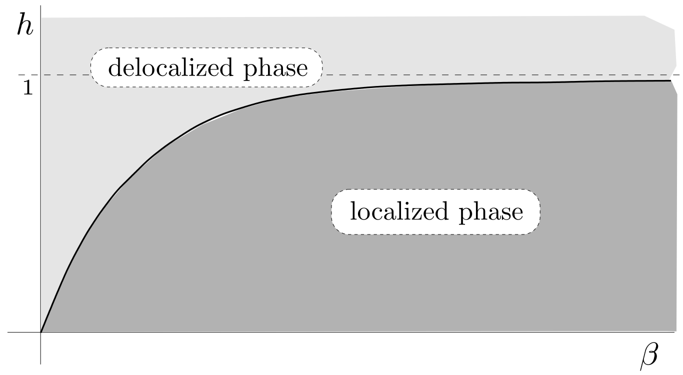

FIG. 2. Phase diagram for the heteropolymer near a linear interface as was found in [BodH]. The localized phase corresponds to $f_\beta > h$, the delocalized phase to $f_\beta = h$. The $h \le 0$ part of the phase diagram is completely symmetric.

In order to formulate the ergodic theorem, we have to operate with infinite-volume measures that are Gibbsian for the Hamiltonian (2.2) in the sense of DLR. Here arises a non-trivial obstacle due to the underlying randomness: the Gibbs measures are (measure-valued) random variables depending on the entire monomer configuration $\boldsymbol{\omega} = (\omega_i)$. On the one hand, it is not *a priori* clear how one should 'sample' such random measures (i.e., how to obtain measurable functions $\boldsymbol{\omega} \mapsto \mu_{\boldsymbol{\omega}}$ such that $\mu_{\boldsymbol{\omega}}$ is Gibbsian for every $\boldsymbol{\omega}$), because even the number of possible Gibbs measures is in principle $\boldsymbol{\omega}$-dependent. On the other hand, since we are interested only in *typical* $\boldsymbol{\omega}$, and this is a notion that cannot be quantified without having measurability, we can from the start restrict ourselves to 'samples' that are measurable w.r.t. the σ-algebra of $\boldsymbol{\omega}$. Thus, we put forward the concept of *measurable Gibbsian sections*, denoting measurable functions from $\{-1,+1\}^{\mathbb{Z}}$ to the set of Gibbs measures. Within this class we then prove our ergodicity and uniqueness assertions.

3. CORRELATION INEQUALITIES

In the realm of lattice models, the orchestrated use of a system's special symmetries at times offers a 'miraculous' solution to an otherwise intractable problem like evaluating the free energy of the model in a 'closed form'. Apart from the scarcity of such results, the disadvantage of this 'exact approach' lies in its low adaptability—the symmetries often sit too deeply in the arguments to allow for more than cosmetic changes. However, other properties like various correlation inequalities, discovered originally also as particular symmetries, have over the years proved to be a little more robust. Here we shall discuss some of the latter, namely, the FKG-inequality and reflection positivity.

FKG-inequality.

The FKG-inequality, named after Fortuin, Kasteleyn and Ginibre, is based upon the concept of *monotone events*. Let us suppose that $\mathcal{S}$ possesses some *a priori* ordering, and let us extend this to a *partial order* on $\mathcal{S}^{\mathbb{Z}^d}$ by putting

$$\boldsymbol{\sigma} \preceq \boldsymbol{\omega} \quad \text{iff} \quad \sigma_x \leq \omega_x \quad \forall x \in \mathbb{Z}^d. \tag{3.1}$$

Note that having a partial order on configurations enables us to work with monotone functions: we say that f is *monotone increasing* if $f(\boldsymbol{\sigma}) \leq f(\boldsymbol{\omega})$ for any $\boldsymbol{\sigma} \preceq \boldsymbol{\omega}$. Monotone events are then those events whose indicator function is monotone. Moreover, the partial order can be raised also to measures: we say that $\mu_1 \geq \mu_2$ if $\mu_1(A) \geq \mu_2(A)$ for all monotone *increasing* events A.

The measure μ is said to be *FKG* if it exhibits positive correlations of increasing events, i.e., if

$$\mu(A \cap B) \geq \mu(A)\mu(B) \qquad \text{for all } A, B \text{ increasing.} \tag{3.2}$$

This concept is strengthened to the *strong FKG*-property by requiring that $\mu(\,\cdot\,|A)$ be FKG for any A that enforces a *single* configuration on a finite set of sites. The advantage of this stronger notion is the following necessary and sufficient condition:

Proposition III.1. *For any two configurations $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, let $\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2$ denote their minimum and $\boldsymbol{\sigma}_1 \vee \boldsymbol{\sigma}_2$ their maximum. Then μ is strong FKG if and only if*

$$\mu(\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2)\mu(\boldsymbol{\sigma}_1 \vee \boldsymbol{\sigma}_2) \geq \mu(\boldsymbol{\sigma}_1)\mu(\boldsymbol{\sigma}_2) \qquad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2.$$

Moreover, the same is true if $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are required to differ on at most two positions.

Proof. See [FKG]. □

Despite its rather simple formulation, the FKG-inequality has some rather deep consequences. For instance, it implies that the thermodynamic limit of finite-volume Gibbs states with extreme boundary conditions exists and gives rise to extremal and ergodic measures. All other Gibbs measures (defined by taking thermodynamic limits of finite-volume measures) are sandwiched between these two extremes in the partial order on measures. In particular, if the two extremes coincide then there is uniqueness. Important is also the easy accessibility of a domination argument:

Proposition III.2. *On a finite lattice Λ, let μ_1 and μ_2 be two probability measures such that $\mu_1(\,\cdot\,) = \frac{1}{Z_1} W_1(\,\cdot\,)$ and $\mu_2(\,\cdot\,) = \frac{1}{Z_2} W_2(\,\cdot\,)$, where W_1 and W_2 are non-negative functions and Z_1 and Z_2 are normalizing constants. If at least one of the measures is FKG and if $\frac{W_1}{W_2}$ is monotone increasing, then $\mu_1 \geq \mu_2$.*

Proof. Suppose μ_2 is FKG. Then for any monotone increasing event A we have

$$\mu_1(A) = E_{\mu_1}(\mathbb{1}_A) = \frac{Z_2}{Z_1} E_{\mu_2}\left(\mathbb{1}_A \frac{W_1}{W_2}\right) \geq \frac{Z_2}{Z_1} E_{\mu_2}(\mathbb{1}_A)\, E_{\mu_2}\left(\frac{W_1}{W_2}\right) = \mu_2(A), \tag{3.3}$$

where the last equality follows from the observation that $E_{\mu_2}\bigl(\frac{W_1}{W_2}\bigr) = \frac{Z_1}{Z_2}$. The case when only μ_1 is known to be FKG is analogous. $\square$

Reflection positivity.

Reflection positivity (RP) is a technique, dating back to the early days of constructive quantum field theory, that uses 'correlations of half-spaces'. Indeed, it stood as one of the cornerstones in the Osterwalder-Schrader axiomatic scheme [OS], ensuring positive definiteness of certain scalar products. Later it was revived in lattice statistical mechanics in the context of classical-spin (either discrete or continuum) and quantum-spin systems. For instance, it provides the only currently known way to establish rigorously the symmetry breaking in the quantum Heisenberg antiferromagnet [DLS], and its lack for the quantum Heisenberg ferromagnet is the main cause why this model is still unresolved. Moreover, it finds nice applications to discrete classical systems, where the perturbative methods either do not work (e.g., in the Heilmann-Lieb models of liquid crystals [HL]) or are more complicated (e.g., in the Potts ferromagnet [KS]).

Let us explain the use of RP for proofs of phase transitions in lattice systems. For that it is convenient to study equlibrium measures on finite tori. Let $\mathbb{T}_N$ be a lattice torus of size N, with N even, and let $P \subset \mathbb{T}_N$ be a hyperplane perpendicular to one of the coordinate directions. Here P is thought to be composed of two antipodal components (see Fig. 3). In this way, $\mathbb{T}_N$ splits into two equal components $\mathbb{T}_N^{\mathcal{L}}$ and $\mathbb{T}_N^{\mathcal{R}}$ (where P belongs to both).

Analogously one also considers observables that reside only in these half-spaces. Given P, we can define σ-algebras $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\mathcal{R}}$ of events depending only on configurations in the left and right half-spaces[6]. The latter are mirror symmetric w.r.t. each other, so we can define the reflection operator ϑ_P acting on functions as $[\vartheta_P f](\boldsymbol{\sigma}) = f(\vartheta_P \boldsymbol{\sigma})$, where $(\vartheta_P \boldsymbol{\sigma})_x = \sigma_{\vartheta_P x}$ with $\vartheta_P x$ denoting the mirror image of x w.r.t. P.

Definition III.3. We say that the measure μ on $\mathbb{T}_N$ is *reflection positive* if

$$E_\mu\bigl(f\,[\vartheta_P g]\bigr) = E_\mu\bigl([\vartheta_P f]\, g\bigr) \tag{3.4}$$

$$E_\mu\bigl(f\,[\vartheta_P f]\bigr) \geq 0, \tag{3.5}$$

[6]It is convenient to use the same symbol to denote the sets of functions measurable w.r.t. the σ-algebra. Hence, in particular, $f \in \mathcal{F}_{\mathcal{L}}$ means that f depends only on the configuration in the left half-space.

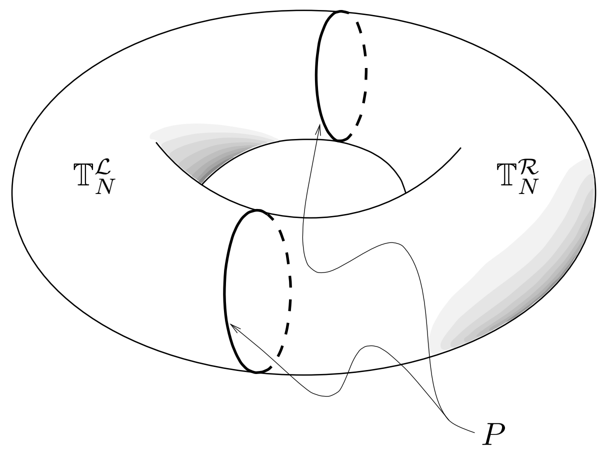

FIG. 3. Lattice torus $\mathbb{T}_N$ with two components of P.

for all $f, g \in \mathcal{F}_{\mathcal{L}}$ and all planes $P \subset \mathbb{T}_N$.

The properties (3.4) and (3.5) are reminiscent of a scalar product. Indeed, the map $f, g \mapsto E_\mu\bigl(f\,[\vartheta_P g]\bigr)$ can be viewed as a non-negative definite bilinear form and, in particular, the Cauchy-Schwarz relation holds:

$$\bigl|E_\mu\bigl(f\,[\vartheta_P g]\bigr)\bigr| \le E_\mu\bigl(f\,[\vartheta_P f]\bigr)^{1/2} E_\mu\bigl(g\,[\vartheta_P g]\bigr)^{1/2}. \tag{3.6}$$

This relation is the starting point of the so-called *chessboard estimates*, through which reflection positivity enters the phase-transition proofs in short-range discrete-spin models[7]. Roughly, the idea is that events of the type $\prod_x f_x$, where f_x depends only on a neighborhood of x, can be 'split from each other' by a repeated use of (3.6) w.r.t. all possible planes P, and at the same time can be 'spread out' all over the torus. In this way, local correlations can be efficiently estimated in terms of global events, which often constitutes a rather significant simplification for the analysis of the former.

The precise statement for f_x being the indicator of a 'behavioral pattern' in a box centered at x, i.e., the indicator of a set of configurations on the sites of this box, is given in the following proposition, which is originally due to [FL].

Proposition III.4 (Chessboard estimates). *Let τ be the shift operator on $\mathbb{T}_N$, with N even, and let μ be a reflection-positive measure on $\mathbb{T}_N$. Let $\{c_x\}$ be a collection of distinct hypercubes of size 1, with c_x centered at x, and associate with c_x a particular behavioral pattern $\mathcal{B}_x$. Then*

$$E_\mu\Bigl(\prod_x \mathbb{1}_{\mathcal{B}_x}\Bigr) \le \prod_x E_\mu\bigl(\chi_{\mathcal{B}_x}^{\mathbb{T}_N}\bigr)^{\frac{1}{|\mathbb{T}_N|}}.$$

[7] Long-range models cannot be treated by the chessboard estimates because the interaction between half-spaces cannot be separated. An alternative way to prove symmetry breaking in these models is by using spin-wave arguments, see [Ge, Chapter 20].

Here $|\mathbb{T}_N|$ denotes the number of sites in $\mathbb{T}_N$, and

$$\chi_{\mathcal{B}_x}^{\mathbb{T}_N} = \prod_{y\text{-even}} \mathbb{1}_{\mathcal{B}_x \circ \tau^y} \prod_{y\text{-odd}} \mathbb{1}_{\mathcal{B}_x^* \circ \tau^y}$$

induces $\mathcal{B}_x$ on all even translates of c_x and its mirror image $\mathcal{B}_x^*$ to all odd translates of c_x.

Proof. See [FL]. □

Chessboard estimates are useful in establishing items (i) and (ii) of Proposition III.5 below, taken over from [KS]. The idea behind the claim is as follows: If we know that two distinct behavioral patterns occur with a probability very close to one under some translation-invariant measure μ, then the typical configurations are actually 'composed' mostly of these two patterns (as follows from the Ergodic Theorem). Moreover, if we can show that the two behavioral patterns 'avoid' each other in the typical configurations under μ, then a long-range order is inevitable.

Proposition III.5. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ denote two behavioral patterns on a cube $c \in \mathbb{T}_N$, and let μ_N^α be a translation-invariant Gibbs measure on $\mathbb{T}_N$, with the Hamiltonian depending continuously on a parameter $\alpha \in [\alpha_\mathcal{X}, \alpha_\mathcal{Y}]$. Let $A \in (\frac{1}{2}, 1]$ and $B \in [0, \frac{1}{4}]$ be such that $B \le \left[\frac{1}{2} - \sqrt{\frac{1}{2} - \frac{A}{2}}\right]^2$, and let $\epsilon_a, \epsilon_b \in (0, \frac{1}{2})$. If for all $\alpha \in [\alpha_\mathcal{X}, \alpha_\mathcal{Y}]$ and all N large enough*

(0) $\mathcal{X} \cap \mathcal{Y} = \emptyset$,

(1) $\mu_N^\alpha(\mathcal{X} \cup \mathcal{Y}) \ge A$,

(2) $\mu_N^\alpha\big(\mathcal{X} \cap (\mathcal{Y} \circ \tau^y)\big) \le B, \qquad \forall y \in \mathcal{T}_N$

and

(3a) $\mu_N^{\alpha_\mathcal{X}}(\mathcal{X}) \ge 1 - \epsilon_a$,

(3b) $\mu_N^{\alpha_\mathcal{Y}}(\mathcal{Y}) \ge 1 - \epsilon_b$,

then there exists $\alpha_c \in [\alpha_\mathcal{X}, \alpha_\mathcal{Y}]$ and two distinct infinite-volume translation-invariant extremal Gibbs states $\mu_{\alpha_c}^\mathcal{X}$ and $\mu_{\alpha_c}^\mathcal{Y}$, such that

$$\mu_{\alpha_c}^\mathcal{X}(\mathcal{X}) \ge 1 - \bar\epsilon \qquad \text{and} \qquad \mu_{\alpha_c}^\mathcal{Y}(\mathcal{Y}) \ge 1 - \bar\epsilon,$$

where $\bar\epsilon = \bar\epsilon(A, B)$ is such that $\bar\epsilon \to 0$ as $A \to 1$ and $B \to 0$.

Proof. See [KS]. □

♣ **[BCK]—Partially ordered/disordered phases.**

In the paper 'Coexistence of Partially Disordered/Ordered Phases in an Extended Potts Model' [BCK], we study the Gibbs states for the following Hamiltonian:

$$H(\boldsymbol{\sigma}) = -J \sum_{\langle x,y\rangle} \delta_{\sigma_x\sigma_y} - \kappa \sum_{\langle x,y\rangle} (\delta^R_{\sigma_x}\delta^R_{\sigma_y} + \delta^S_{\sigma_x}\delta^S_{\sigma_y}) - h \sum_x (\delta^S_{\sigma_x} - \delta^R_{\sigma_x}). \tag{3.7}$$

Here $\sigma_x \in R \cup S$ with $R = \{1, \dots, r\}$ and $S = \{r+1, \dots, r+s\}$, while $\delta_{\sigma_x \sigma_y}$ indicates that $\sigma_x = \sigma_y$, $\delta^R_{\sigma_x}$ indicates that $\sigma_x \in R$, and $\delta^S_{\sigma_x}$ indicates that $\sigma_x \in S$. The parameters $J, \kappa > 0$ are fixed and $h \in \mathbb{R}$ is arbitrary. Due to the first term in (3.7), the model is an extension of the $(r+s)$-state Potts model. Note that the rest of the interaction introduces an attraction (since $\kappa > 0$) of the spins within the two spin families R and S.

Due to the second and the third term in (3.7), our model can be viewed as an interpolation between the Potts and the Ising model with an external field (the latter is obtained by setting $J = 0$ and shifting the field by the amount $\frac{1}{\beta} \log \frac{r}{s}$). The Potts model itself is one of the systems where a good control is possible over the entire regime of temperatures, featuring so-called *ordered* states at low temperatures and *disordered* states at high temperatures, with a sharp first-order transition in between. This transition was first rigorously established in [KS], employing reflection positivity. On the other hand, the Ising model exhibits a phase transition when the corresponding external field changes from negative to positive values. Therefore, a rather rich phase structure can be expected for the Hamiltonian (3.7). Indeed, the phase diagrams are as depicted in Fig. 4.

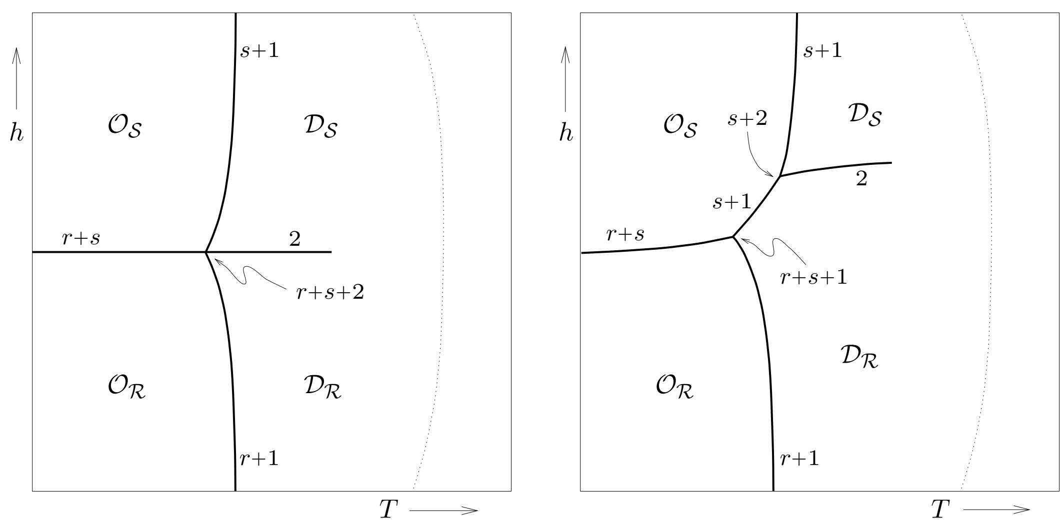

FIG. 4. Phase diagram for the extended Potts model. The left picture corresponds to the symmetric case when $r = s \gg 1$, the right picture to the strongly asymmetric case $r \gg s \gg 1$. The figures attached to the lines and the points denote the number of proved coexisting translation-invariant Gibbs states. The dotted line shows the region where the Gibbs state is known to be unique.

Let first $r = s$. Then both families R and S are completely symmetric with respect to the flip symmetry $h \leftrightarrow -h$ and, consequently, the coexistence between R-states and S-states (i.e., the transition governed by the Ising part of the interaction, i.e., by the second and the third term in (3.7)) can occur only on the symmetry line $h = 0$. This line is crossed by the order/disorder transition line and the parameter space is thus divided into four (not disconnected) regions: the $\mathcal{O}_\mathcal{S}$-region with s ordered states, the $\mathcal{O}_\mathcal{R}$-region with r ordered states, the $\mathcal{D}_\mathcal{S}$-region where a single disordered state concentrated on spins of the S-family exists, and, finally, the $\mathcal{D}_\mathcal{R}$-region with a single disordered state concentrated on the spins in the R-family. All these regions meet at a quadruple point where there are altogether $r+s+2$ states.

It is interesting to ponder about what happens with the phase diagram when we let r grow above s. There are two possibilities: either the whole picture undergoes a 'smooth deformation' but does not change qualitatively, or some new coexistence line starts forming. Though we cannot really answer this question for r, s, $r-s$ small, the second option seems to be more plausible. Namely, we prove for $r \gg s \gg 1$ that the quadruple coexistence point splits into two triple points, which are connected by a non-trivial piece of a transition line directly between the $\mathcal{O}_\mathcal{S}$ and $\mathcal{D}_\mathcal{R}$-states. Of interest in both cases is also the coexistence between $\mathcal{D}_\mathcal{R}$ and $\mathcal{D}_\mathcal{S}$. These two phases are not strictly separated in the phase diagram: there is a path in the parameter space connecting the two phases (presumably) without encountering a phase transition on the way. This is a situation analogous to the liquid-vapor transition. As is argued below, some characterization is still possible in terms of a graphical representation, but no thermodynamic 'order' parameter is available in this case.

Graphical representation.

It is crucial for our analysis that the model offers a beautiful reformulation in terms of a *graphical representation*. The latter is best explained by a coupling with a Bernoulli bond process [ES]: Let $p = 1 - e^{-\beta J}$ and assume only the first term in (3.7). Let each bond of $\mathbb{Z}^d$ be *occupied* with probability p and *vacant* with probability $1-p$ if the spins at the end-points agree, but let it be vacant with probability 1 if they disagree. The choice is made independently of all the other bonds. The joint measure on spin and bond configuration is the *Edwards-Sokal measure*, and the bond marginal is the *Fortuin-Kasteleyn representation* (FK) of the Potts model [FK]. This graphical representation inherits various properties of the spin system, but has interesting features on its own too (for instance, it is strong FKG). This is of importance since phase transition can often be characterized in terms of *percolation* of occupied bonds (i.e., the existence of an infinite cluster).

In the full model (3.7), the graphical representation is constructed along very much the same lines. However, two terms are now playing the role of the Potts interaction. Consequently, the number of possible 'bond states' has to be larger. There are five types of bonds: ordered bonds $\mathcal{O}_\mathcal{S}$ and $\mathcal{O}_\mathcal{R}$, disordered bonds $\mathcal{D}_\mathcal{S}$ and $\mathcal{D}_\mathcal{R}$, and the 'vacant' bond state $\mathcal{V}$. Whereas the $\mathcal{D}_\mathcal{S}$-state can occur whenever the end-points belong both to the S-family, the $\mathcal{O}_\mathcal{S}$ state can occur only when the end-points are equal and belong to S. The vacant bond state is independent of the spins at the end-points.

The important feature of this representation is that the corresponding Gibbs states are strong FKG with respect to the order $\mathcal{O}_{\mathcal{S}} \succ \mathcal{D}_{\mathcal{S}} \succ \mathcal{V} \succ \mathcal{D}_{\mathcal{R}} \succ \mathcal{O}_{\mathcal{R}}$. Moreover, as h increases, the measures are increasing in the sense of FKG. However, this being applicable without regard of the particular values of J and κ, monotonicity alone does not suffice to make the above conclusion about phase coexistence—some additional arguments are needed. For instance, we need to know that while ascending along a vertical line, an R–S transition is to be encountered. As a glance at Proposition III.4 shows, this can be converted to a question about correlations, which is in turn well suited for chessboard estimates. Thus, if reflection positivity is also available, a consistent scheme for proving all the above phenomena is established.

♣ **[B]—Reflection positivity of the random-cluster measures.**

As already mentioned, in the process of proving various properties of the phase diagram in [BCK], a variant of the FK-representation has been used. It is an interesting and frequently used feature of the latter that it allows a natural continuation to non-integer values of the total number of possible spins $q = r + s$. In this way the so-called *random-cluster* model is recovered: Take a Bernoulli bond process, where bonds are occupied with probability p and vacant with probability $1-p$, independently of each other, and assign each configuration an additional weight q per connected component. The appropriately normalized measure then corresponds, for q positive integer, to the FK-representation of the q-state Potts model. Moreover, this continuation respects the natural FKG-property of the FK-representation.

Unfortunately, the other important tool, namely, reflection positivity, could not be proved for other than integer values of q since the existing argument used a mapping back onto the q-state Potts model. This situation was settled negatively in the paper 'Reflection Positivity of the Random-Cluster Measure Invalidated for Noninteger q' [B], where a counterexample to reflection positivity is constructed whenever q is not integer. An interesting feature of the proof is the use of analytic continuation: there is a representation (derived in Proposition 2 of [B]) whose terms are either positive or *vanish* when q is integer, but whose analytic continuation to non-integer q is no longer positive[8]. In this way, reflection positivity of the Potts model can be viewed as the result of extensive cancelations due to the special choice of q. Or, if seen from the perspective of field theory where reflection positivity supplements certain convexity properties, the random-cluster measure is not a 'good field theory' unless q is integer.

4. PIROGOV-SINAI THEORY

For finite systems, as the temperature goes to zero ($\beta \to \infty$) the Gibbs measures tend to be more and more concentrated around the minima of the Hamiltonian. For infinite systems, however, there is for any $\beta < \infty$ a positive density of *fluctuations* (i.e., 'spots' with energy larger than the absolute minimum). It turns out, and this is the subject of Pirogov-Sinai

[8] This provides another example to the series worked out in [GG], pointing out several instances in models of statistical mechanics where an 'obvious' analytic continuation fails.

theory whose rudiments we intend to explain below, that these fluctuations can be controlled by analytic methods.

Pirogov-Sinai theory [PSa,b] in its original form, developed in the late 1970's and reformulated by means of inductive procedures in the mid 1980's [KP1,Z,BI], requires that the model has only a *finite* number of *ground states*, i.e., configurations whose energy cannot be lowered by changing a finite number of spins. Given a finite set $\mathcal{A}$ of spin-states at each site, we shall assume for simplicity that these ground states are actually just the *constant* configurations, i.e., those having all spins equal to the same value. Let the Hamiltonian in Λ be given by

$$H_\Lambda = \sum_{A \cap \Lambda \neq \emptyset} \varPhi_A, \tag{4.1}$$

where $(\varPhi_A)$ are *interaction potentials* such that $\varPhi_A \equiv 0$ if $\operatorname{diam}(A) > r$ (i.e., r is the interaction *range*). Given an arbitrary configuration $\boldsymbol{\sigma}$, we define its *contours* as the connected components of the union of all r-cubes, where $\boldsymbol{\sigma}$ is not equal to any of the ground states. Each contour is actually a connected set of sites with a configuration on it. At the boundary of the contour this configuration necessary 'blends' into a ground state. To the ground states at the exterior boundary we refer as the *external color*, to the ground states at the internal boundary components we refer as the *internal colors*.

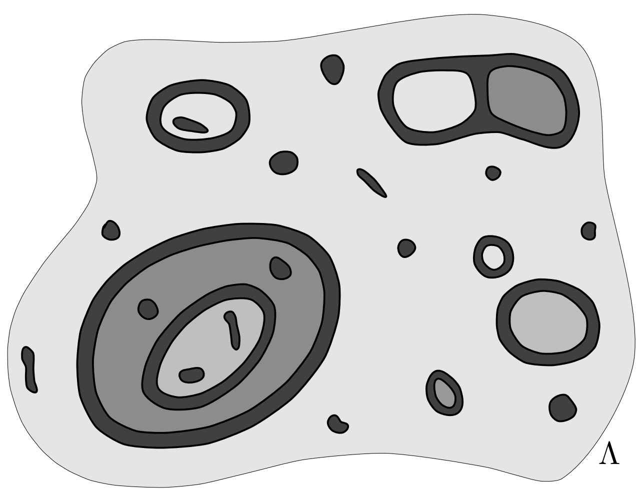

FIG. 5. A contour configuration in Λ with three distinct ground states, distinguished by different shades of gray. Observe the possibility of contours inside contours, as well as contours having more than one internal color. The external color is always unique.

The introduction of contours is typically accompanied by the following rewrite of energies. Let Λ_q denote the set of non-contour sites in Λ where the configuration equals the ground state q. Then

$$H_\Lambda = \sum_q e_q|\Lambda_q| + \sum_{\Gamma_\alpha} E(\Gamma_\alpha), \tag{4.2}$$

where e_q is the energy density in the q-th ground state[9] and $E(\Gamma_\alpha)$ is the energy associated with contour Γ_α.

With (4.2) in the hand, the partition function in volume Λ with a constant boundary condition q can be written as

$$Z_\Lambda^q = \sum_{(\Gamma_\alpha)} \exp\Bigl(-\beta\sum_{q'} e_{q'}|\Lambda_{q'}|\Bigr)\prod_\alpha \Psi(\Gamma_\alpha), \tag{4.3}$$

where $\Psi(\Gamma_\alpha) = e^{-\beta E(\Gamma_\alpha)}$ and where the sum runs over all sets of mutually *compatible* contours in Λ_r, the r-neighborhood of Λ, where compatibility is defined by requiring that the contours do not overlap and that their internal/external colors match.

In order to get started, one is in need of the so-called *Peierls condition*

$$e^{\beta e_{\mathfrak{e}(\Gamma)}|\Gamma|}\,\Psi(\Gamma) \le e^{-\beta\tau|\Gamma|}, \quad \text{for all } \Gamma \text{ and } \tau > 0, \tag{4.4}$$

where $|\Gamma|$ denotes the number of sites in Γ and $\mathfrak{e}(\Gamma)$ denotes its exterior color. This condition ensures that contours are themselves sufficiently improbable. However, in order to show that the 'whole fluctuation', comprising the contour itself and the ground states (with other contours) in its interior, is unlikely to appear, we need such an exponential bound rather for the functionals

$$\mathfrak{Z}_{\mathfrak{e}(\Gamma)}(\Gamma) = \frac{\prod_q Z^q_{\mathrm{Int}_q(\Gamma)}}{Z^{\mathfrak{e}(\Gamma)}_{\mathrm{Int}(\Gamma)} e^{-\beta e_{\mathfrak{e}(\Gamma)}|\Gamma|}}\Psi(\Gamma), \tag{4.5}$$

where $\mathrm{Int}_q(\Gamma)$ denotes the components of the interior for which the interior color equals q. Note that (4.5) is the ratio between the weight of the contour+interior and the weight of the configuration where Γ is removed and the interior has the external color $\mathfrak{e}(\Gamma)$.

With (4.5) one can recast (4.3) as

$$Z_\Lambda^q = e^{-\beta e_q|\Lambda|}\sum_{(\Gamma_\alpha)}\prod_\alpha \mathfrak{Z}_q(\Gamma_\alpha), \tag{4.6}$$

where the sum now runs over the collections of *spatially non-overlapping* contours whose external color is q (in particular, the matching condition for the internal/external colors of the contours is not needed).

---

[9]Here we make a formal distinction between the e_q's even though the fact that they correspond to ground state configurations implies that they are all equal. Actually, the rewrite (4.2) enables us to treat the ground state energies as independent parameters, which is particularly convenient while parametrizing the phase diagram. When e_q's are no longer equal, some of the q's are not ground states any more—one then often talks about *reference states* to avoid confusion.

The significance of the rewrite in (4.6) lies in the fact that the multitude of colors (and the related matching rules) as well as the explicit dependence on the energies (e_q) have vanished from the description. Thus, (4.6) gives rise to an explicit 'decomposition' of Z_Λ^q into the contribution due to energy (the exponential factor) and entropy (the sum thereafter). As Proposition IV.2 below asserts, if q gives rise to a low temperature phase or, equivalently, if the functionals $\mathfrak{z}_q$ satisfy a bound of the type (4.4), then this decomposition provides a consistent way to compute the free energy. In order to show how we need some more technical definitions.

We first introduce the *truncated functionals* $\tilde{\mathfrak{z}}$ by requiring

$$\tilde{\mathfrak{z}}_q(\Gamma) = \begin{cases} \mathfrak{z}_q(\Gamma) & \mathfrak{z}_q(\Gamma) \le e^{-\frac{1}{3}\beta\tau|\Gamma|} \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

(in the first case we call Γ *stable*, in the second case *unstable*) and define, for each q, the *metastable* partition function $\mathcal{Z}_\Lambda^q$ and free energy f_q by the formulas

$$\mathcal{Z}_\Lambda^q = \sum_{(\Gamma_\alpha)} \prod_\alpha \tilde{\mathfrak{z}}_q(\Gamma_\alpha) \tag{4.8}$$

$$f_q = e_q - \frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \mathcal{Z}_\Lambda^q. \tag{4.9}$$

The reason for using the truncated functionals is that both $\mathcal{Z}_\Lambda^q$ and f_q can be controlled by the techniques of *cluster expansion*. In Proposition IV.1 below we formulate a theorem due to [KP2] on the convergence criterion for the latter. In this claim, the 'sans-serif' capital letters B, C, etc., denote sets of contours and $\mathcal{Z}(\mathsf{L}, \tilde{\mathfrak{z}})$ denotes the partition function where only the contours from L are allowed to appear. In particular, if L is the set of contours in Λ_r, then $\mathcal{Z}_\Lambda^q = \mathcal{Z}(\mathsf{L}, \tilde{\mathfrak{z}}_q)$. The symbol K is reserved for the set of all contours in $\mathbb{Z}^d$, and $\mathcal{B}$ is the set of all finite subsets of K. The set C is said to be a *cluster* if it cannot be decomposed into two subsets such that each contour in one is compatible with any contour in the other.

Proposition IV.1 (Cluster expansion). *Let* $\mathfrak{a}\colon \mathsf{K} \to [0,\infty)$, $\mathfrak{b}\colon \mathsf{K} \to [0,\infty)$, *and* $\tilde{\mathfrak{z}}_q\colon \mathsf{K} \to \mathbb{C}$ *be such that*

$$\sum_{\Gamma'\colon \Gamma' \cup \Gamma \neq \emptyset} e^{\mathfrak{a}(\Gamma') + \mathfrak{b}(\Gamma')} |\tilde{\mathfrak{z}}_q(\Gamma')| \le \mathfrak{a}(\Gamma) \qquad \textit{for each} \quad \Gamma \in \mathsf{K}. \tag{4.10}$$

Then $\mathcal{Z}(\mathsf{L}, \tilde{\mathfrak{z}}_q) \neq 0$ *for each finite* $\mathsf{L} \subset \mathsf{K}$, *and there exists a unique function* $\mathfrak{z}_q^{\mathrm{T}}\colon \mathcal{B} \to \mathbb{C}$ *such that*

$$\log \mathcal{Z}(\mathsf{L}, \tilde{\mathfrak{z}}_q) = \sum_{\mathsf{C}\colon \mathsf{C} \subset \mathsf{L}} \mathfrak{z}_q^{\mathrm{T}}(\mathsf{C}) \qquad \textit{for each} \quad \mathsf{L} \in \mathcal{B}. \tag{4.11}$$

Moreover, the function $\mathfrak{z}_q^{\mathrm{T}}$ *is given by the formula*

$$\mathfrak{z}_q^{\mathrm{T}}(\mathsf{C}) = \sum_{\mathsf{B}:\mathsf{B}\subset\mathsf{C}} (-1)^{|\mathsf{B}|-|\mathsf{C}|} \log \mathcal{Z}(\mathsf{B}, \tilde{\mathfrak{z}}_q), \tag{4.12}$$

with

$$\sum_{\substack{\mathsf{C}\in\mathcal{B} \\ \Gamma'\cap\Gamma\neq\emptyset\ \forall\Gamma'\in\mathsf{C}}} |\mathfrak{z}_q^{\mathrm{T}}(\mathsf{C})| e^{\mathfrak{b}(\mathsf{C})} \le \mathfrak{a}(\Gamma) \qquad \text{for each} \quad \Gamma \in \mathsf{K}, \tag{4.13}$$

and

$$\mathfrak{z}_q^{\mathrm{T}}(\mathsf{C}) = 0 \qquad \text{whenever} \;\; \mathsf{C} \;\; \text{is not a cluster.} \tag{4.14}$$

Proof. See [KP2]. □

Note that (4.10) is implied by the definition in (4.7), provided τ is large enough. Also note that the condition $\mathcal{Z}(\mathsf{L}, \tilde{\mathfrak{z}}_q) \neq 0$ ensures that (4.9) is 'uniformly' well defined (including taking the right branch of the logarithm, when the weights are complex). For practical purposes formula (4.11) is the most important, stating (when rephrased back into the language of contours) that $\mathcal{Z}_\Lambda^q$ is given by the exponential of the sum of all clusters that can be produced in Λ. Formula (4.14) then implies that the exponent grows proportionally to $|\Lambda|$, because the sum of the contributions coming from the clusters touching a finite contour (i.e., a finite connected set of sites) is finite. If the cluster weights are translation invariant, then this also yields the succinct expression for the free energy

$$f_q = e_q - \frac{1}{\beta} \sum_{\mathsf{C}:\mathsf{C}\ni 0} \frac{1}{\|\mathsf{C}\|} \mathfrak{z}_q^{\mathrm{T}}(\mathsf{C}), \tag{4.15}$$

where the sum goes over the clusters 'containing the origin', i.e., those C with $0 \in \cup_{\Gamma\in\mathsf{C}}\Gamma$, and where $\|\mathsf{C}\|$ denotes the size of the latter set.

Having assorted the main concepts and explained why they are well defined, we can now state the main proposition of the (current version of) Pirogov-Sinai theory. This proposition is originally due to [Z].

Proposition IV.2. *Let* $f = \min_q f_q$ *and let* $a_q = f_q - f$. *Then there exists a* $C = C(\beta)$, *satisfying* $C(\beta) \to 0$ *as* $\beta \to \infty$, *such that the following holds:*

(1) If Γ *is unstable, then*

$$a_{\mathfrak{e}(\Gamma)} |\mathrm{Int}(\Gamma)| \ge \tfrac{1}{3}\tau|\Gamma|. \tag{4.16}$$

In particular, if $a_q = 0$, *then all contours with external color* q *are stable.*

(2) For all Λ *and* q

$$\exp\Bigl(-\beta f_q|\Lambda| - C|\partial\Lambda|\Bigr) \le Z_\Lambda^q \le \exp\Bigl(-\beta f|\Lambda| + C|\partial\Lambda|\Bigr). \tag{4.17}$$

Moreover, the value f equals the free energy of the model.

Proof. See [Z]. □

Based on this assertion, the entire low-temperature phase diagram can be constructed. Namely, the phases with $f_q = f$ (i.e., $a_q = 0$) have all contours stable and so one can use cluster expansions (in particular, formula (4.11)) to estimate the probability that there is a contour encircling the origin in the Gibbs measure in Λ with boundary condition q. With the help of (4.13) and a little bit of additional combinatorics, this probability vanishes exponentially fast as $\beta \to \infty$, uniformly in Λ. Therefore, the limit $\Lambda \nearrow \mathbb{Z}^d$ gives rise to a measure concentrated on configurations that look like a 'sea' of q-spins with small 'islands' of other spin-states. The ergodic theorem then implies that the measures for $\{q\colon a_q = 0\}$ are distinct. Hence, if there is more than one such q, then we have a first-order phase transition.

We have indicated how *coexistence* of phases is deduced from the evaluation of metastable free energies f_q. However, we would also like to know that the ground states with $a_q > 0$ *cannot* yield a separate Gibbs state (note that $a_q \ge 0$ because (4.13) effectively leads to the partition sum over a smaller set of configurations). This can be answered in the affirmative based on the analysis leading to (4.16): by introducing a very large contour, any 'unstable' boundary condition gets flipped to a stable one, provided the volume is large enough. The proof of this *completeness* of the phase diagram is due to [Z].

♣ **[BBCK]—Two-phase coexistence line in the Potts model.**

In the paper 'Gibbs Structure and Phase Coexistence in the Potts model with External Fields' [BBCK], the above explained theoretical arsenal has been used to establish the basic features of the phase diagram. The model we study is similar to that of [BCK], however, the spins take values from the set $\{1,\dots,q+2\}$ and the Hamiltonian is

$$H(\sigma) = -J\sum_{\langle x,y\rangle}\delta_{\sigma_x,\sigma_y} - \sum_x\sum_{j=1}^{q+2} h_j\delta_{\sigma_x,j}. \tag{4.18}$$

The reasons for an interest in this model are very much like the ones we have already given when describing [BCK]. However, there are two novel points here. First, the presence of external fields causes all sorts of technical difficulties (for instance, it undermines the FKG-order on which the analysis in [BCK] was based). Second, there are certain intriguing subtleties in the phase diagram, especially when $h_j = 0$ for all $j \ne 1$ (the *one-field* case). Let us elaborate slightly more on the latter case, while referring to the respective chapter for a comprehensive discussion of more general cases. Throughout, it is assumed that q is large.

When only $h_1 \ne 0$, both the aforementioned methods (i.e., reflection positivity and Pirogov-Sinai theory) are doomed to fail when h_1 is large positive and β is not close to 0 and ∞.

Since the large-positive values of h_1 can be handled by a *high-fugacity* expansion (a sort of Pirogov-Sinai expansion around the point $h_1 = \infty$), there is only a *bounded region* in the (h_1, β)-plain left unexplored (see Fig. 1 of [BBCK]). Interestingly, the phase transition line, which presumably terminates in a *critical point* in this region, marks coexistence between only *two* phases: a low-temperature ordered phase and a high-temperature disordered phase.

From the physics point of view, such a behavior is reminiscent of the *liquid-vapor* transition, both because of the driving mechanism (order v. disorder) and because of the termination in a critical point. In fact, to the best of our knowledge, such transition has not been rigorously described before. Apart from applying Pirogov-Sinai theory to the appropriate graphical representation (which has been done, for the case of the Potts model with a special kind of external field, in [vEFK]), the main objective of the paper is to develop an alternative means for analyzing the phase-coexistence line inside the region where the perturbative methods fail.

Gibbs structure of Edwards-Sokal measures.

As in [BCK], our major tool of study is an appropriate graphical representation (see [H] for a review of graphical representations in the Potts and related models). However, in this case we actually take deliberately the advantage of working in the coupled system, i.e., in the Edwards-Sokal coupling of the spin system and the graphical representation.

We begin by defining the notion of Gibbs measures for such combined systems. In particular, we allow the boundary spin/bond configurations in the Gibbsian specification to be prescribed in *independent* sets of sites/bonds. Even though this might sound formal, there are some advantages of this approach—certainly in the use of conditioning and the whole 'extremal-decomposition' theory. For such measures, we establish lots of general properties. In particular, first we prove that there is actually a one-to-one correspondence between the spin Gibbs measures and the Edwards-Sokal Gibbs measures. Moreover, since we are able to prove that certain random-cluster marginals have nice FKG properties, the standard results upon the convergence of finite-volume measures with wired and free boundary conditions are shown to carry over for the Edwards-Sokal measures.

Having built a certain framework, we can relate various percolation properties in the graphical representation to the existence and/or the non-existence of the phase transition in the spin marginal. Finally, we also make a link to the DLR states of the graphical representations defined in the sense of [Gr]. In particular, we prove that the DLR states of the graphical representation are in close correspondence with the Edwards-Sokal measures having at most one infinite cluster in the graphical marginal. This extends the recent partial solution given by [Gr,PVdV] to the question raised in [vEFS] whether (and for what boundary conditions) the measures obtained via a thermodynamic limit admit a DLR characterization.

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A HETEROPOLYMER NEAR A LINEAR INTERFACE

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Abstract. We consider a quenched-disordered heteropolymer, consisting of hydrophobic and hydrophilic monomers, in the vicinity of an oil-water interface. The heteropolymer is modeled by a directed simple random walk $(i, S_i)_{i\in\mathbb{N}}$ on $\mathbb{N}\times\mathbb{Z}$ with an interaction given by the Hamiltonians $H_n^\omega(S) = \lambda\sum_{i=1}^n(\omega_i + h)\,\mathrm{sign}(S_i)$ $(n \in \mathbb{N})$. Here, λ and h are parameters and $(\omega_i)_{i\in\mathbb{N}}$ are i.i.d. ±1-valued random variables. The $\mathrm{sign}(S_i) = \pm1$ indicates whether the i–th monomer is above or below the interface, the $\omega_i = \pm1$ indicates whether the i–th monomer is hydrophobic or hydrophilic. It was shown by Bolthausen and den Hollander [BodH] that the free energy exhibits a localization/delocalization phase transition at a curve in the (λ, h)–plane.

In the present paper we show that the free-energy localization concept is equivalent to pathwise localization. In particular, we prove that free-energy localization implies exponential tightness of the polymer excursions away from the interface, strictly positive density of intersections with the interface, and convergence of ergodic averages along the polymer. We include an argument due to G. Giacomin, showing that free-energy delocalization implies that there is pathwise delocalization in a certain weak sense.

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1. INTRODUCTION

Heteropolymers near an interface between two solvents are intriguing because of the possibility of a localization/delocalization phase transition. A typical example is a polymer consisting of hydrophobic and hydrophilic monomers in the presence of an oil-water interface.

In the bulk of a single solvent, the polymer is subject to thermal fluctuations and therefore is rough on all space scales. However, near the interface the polymer can benefit from the fact that part of its monomers prefer to be in one solvent and part in the other. The energy it may gain by placing as many monomers as possible in their preferred solvent can, at least for low temperatures, tame the entropy-driven fluctuations. Consequently, the polymer becomes captured by the interface and therefore is smooth on large space scales. The two regimes of characteristic behavior are separated by a phase transition.

1.1. The model. The polymer is modeled by a random walk path $(i, S_i)_{i\in\mathbb{L}}$, where $\mathbb{L} \subseteq \mathbb{Z}$ indexes the monomers, $S_i \in \mathbb{Z}$ and $S_i - S_{i-1} = \pm 1$. The interface is the horizontal in $\mathbb{L} \times \mathbb{Z}$. We distinguish two cases:

(1) the *singly-infinite* polymer, where $\mathbb{L} = \mathbb{N}$ and $S_0 = 0$
(2) the *doubly-infinite* polymer, where $\mathbb{L} = \mathbb{Z}$ and $S_0 \in 2\mathbb{Z}$.

The heterogeneity within the polymer is represented by assigning a random variable $\omega_i = \pm 1$ to monomer i for each $i \in \mathbb{L}$, where $\omega_i = +1$ means that monomer i is hydrophobic and $\omega_i = -1$ that it is hydrophilic.

Let $F(\mathbb{L})$ be the set of all finite connected subsets of $\mathbb{L}$. In the simplest model, the thermodynamics of the heteropolymer is governed by the family $(H^{\omega}_{\Lambda})_{\Lambda\in F(\mathbb{L})}$ of Hamiltonians

$$H^{\omega,\lambda,h}_{\Lambda}(S) = \lambda \sum_{i\in\Lambda} (\omega_i + h)\Delta_i(S) \tag{1.1}$$

w.r.t. the reference measure giving all paths $S = (S_i)_{i\in\mathbb{L}}$ equal probability, i.e., the measure P for simple random walk (SRW). Here, λ and h are parameters, $\omega = (\omega_i)_{i\in\mathbb{L}}$ is the disorder configuration, and

$$\Delta_i(S) = \begin{cases} \operatorname{sign}(S_i) & \text{if} \quad S_i \neq 0 \\ \operatorname{sign}(S_{i-1}) & \text{if} \quad S_i = 0. \end{cases} \tag{1.2}$$

The role of the Hamiltonian is that (for $\lambda > 0$) it favors the combinations $S_i > 0, \omega_i = +1$ and $S_i < 0, \omega_i = -1$, so hydrophobic monomers in the oil above the interface ($\mathbb{L} \times \mathbb{Z}_+$) and hydrophilic monomers in the water below the interface ($\mathbb{L} \times \mathbb{Z}_-$). (Note that the definition of $\Delta_i(S)$ actually corresponds to a bond model.) The parameter λ plays the role of the inverse temperature, whereas h expresses the asymmetry between the affinities of the monomer species with the solvents.

The Hamiltonian is $(S,\omega,h) \to (-S,-\omega,-h)$ symmetric. In view of this, we shall henceforth take

$$\mathcal{I} = \{(\lambda,h)\colon \lambda > 0,\, h \geq 0\} \tag{1.3}$$

as our parameter space.

1.2. The free energy and a phase transition. The singly-infinite quenched i.i.d. random model with Hamiltonian (1.1) and with a symmetric disorder distribution was recently analyzed in detail by Bolthausen and den Hollander [BodH]. For the reader's convenience we describe some of the results obtained in that paper.

The localization/delocalization phase transition is established by estimating the *free energy*

$$\phi(\lambda,h) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log E\big(e^{H_{\Lambda_n}^{\omega,\lambda,h}}\big), \tag{1.4}$$

where $\Lambda_n = \{1,\dots,n\}$ and where E stands for the expectation w.r.t. SRW starting at 0. The limit is shown to exists and to be ω-independent a.s. by the subadditive ergodic theorem.

It was observed that $\phi(\lambda,h) \geq \lambda h$, with the lower bound attained for delocalized paths. Indeed, $P(S_i \geq 0 \;\forall\; 0 \leq i \leq n) \sim C/\sqrt{n}$ $(n\to\infty)$, and conditioned on this event

$$\frac{1}{|\Lambda_n|} H_{\Lambda_n}^{\omega,\lambda,h} = \frac{1}{|\Lambda_n|}\lambda \sum_{i\in\Lambda_n} (\omega_i + h) = \lambda h(1+o(1)) \quad \omega\text{-a.s.} \tag{1.5}$$

For this reason, it is natural to work with the *excess free energy*

$$\psi(\lambda,h) = \phi(\lambda,h) - \lambda h \tag{1.6}$$

and to put forward the following concept of a phase transition:

Definition 1. [BodH] *The polymer is said to be*

(a) *localized if* $\psi > 0$,
(b) *delocalized if* $\psi = 0$.

As indicated by (1.5), (b) is justified by noting that delocalized paths yield no contribution to ψ. Conversely, (a) is justified by noting that only those excursions that move below the interface can give a positive contribution to ψ. Nonetheless, Definition 1 makes no claims as to the actual path behavior. The present paper shows that, in fact, a bit of work is needed to obtain a path statement from (a) and (b).

Let us define

$$\mathcal{L} = \{\psi > 0\} \cap \mathcal{I} \tag{1.7}$$

$$\mathcal{D} = \{\psi = 0\} \cap \mathcal{I} \tag{1.8}$$

as the sets of parameters for which the model is localized, respectively, delocalized in the sense of Definition 1. Neither of these sets is trivial, as shown by the following theorem.

Theorem 1. [BodH] *There is a continuous non-decreasing function* $h_c\colon (0,\infty) \mapsto (0,1)$ *such that*

$$\mathcal{L} = \{(\lambda, h) \in \mathcal{I}\colon 0 \le h < h_c(\lambda)\}. \tag{1.9}$$

Moreover,

$$\lim_{\lambda\to\infty} h_c(\lambda) = 1 \qquad \textit{and} \qquad \lim_{\lambda\downarrow 0} \frac{h_c(\lambda)}{\lambda} = K_c, \tag{1.10}$$

where $0 < K_c < \infty$ *is a number related to a Brownian version of the model.*

Theorem 1 asserts that $\mathcal{L}$ and $\mathcal{D}$ are separated by a phase transition line $\lambda \mapsto h_c(\lambda)$ (which extends over all temperatures). Although it is relatively easy to establish the existence and uniqueness of $h_c(\lambda)$ (essentially via the convexity of ϕ) and to evaluate the limit $\lambda \to \infty$ (through an appropriate lower bound on the expectation in (1.4)), the scaling law for $\lambda \downarrow 0$ is a rather involved problem. The intuitive reason why a Brownian constant should appear for $\lambda \downarrow 0$ is that for high temperatures the polymer excursions are large. Therefore, from a coarse-grained point of view, both the excursions and the disorder inside the excursions may be approximated by their Brownian counterparts. However, the details of this approximation are quite delicate.

1.3. Earlier path results. As already alluded to, Theorem 1 characterizes the phase transition in terms of the free energy rather than the path. One would like to prove that, indeed, $\mathcal{L}$ corresponds to a localized path and (the interior of) $\mathcal{D}$ to a delocalized path. Moreover, one would like to learn more about the path characteristics, e.g., the length and the height of a typical excursion. Progress in this direction has been made by Sinai (1993) [Si], who proved pathwise localization in the symmetric case $h = 0$ for all $\lambda > 0$.

Sinai introduces a (Gibbsian) probability distribution $Q_n^{\omega,\lambda,0}$ in the volume $\Lambda_n = \{1,\dots,n\}$, defined by

$$\frac{dQ_n^{\omega,\lambda,0}}{dP_n}(S) = \frac{e^{H_{\Lambda_n}^{\omega,\lambda,0}(S)}}{Z_{\Lambda_n}^{\omega,\lambda,0}}, \tag{1.11}$$

where the reference measure P_n is the projection onto Λ_n of the SRW-measure P. His result reads:

Theorem 2. [Si] *Let $h = 0$ and $\lambda > 0$. Then there exist a deterministic number $\zeta = \zeta(\lambda) > 0$ and two random variables $n(\omega) \in \mathbb{N}$, $m(\omega) \in \mathbb{N}$ such that for almost all ω*

$$\sup_{0 \le i \le n} Q_n^{\omega,\lambda,0}(|S_i| > s) \le e^{-\zeta s} \quad (n \ge n(\omega),\, s \ge m(\omega)). \tag{1.12}$$

Theorem 2 states that the path measure is *exponentially tight*. This result has been extended by Albeverio and Zhou (1996) [AZ], who show that the length of the longest excursion in Λ_n is of order $\log n$ and so is the height of the highest excursion.

1.4. Path results in the present paper—outline. The goal of the present paper is to give a complete description of the path for all $(\lambda, h) \in \mathcal{L}$. We in fact adopt a more comprehensive attitude by discussing the entire Gibbsian structure associated with the Hamiltonian (1.1). (Theorems 1 and 2 are in this respect statements about the Gibbs measures generated by the free boundary condition.)

We begin by singling out a class of 'regular' Gibbs measures (Section 2). For this, measurability and moderate growth of the boundary condition are the key concepts. Within this class we establish, for all $(\lambda, h) \in \mathcal{L}$, uniqueness of the Gibbs measure, exponential tightness of the path in the vertical direction, and ergodicity in the horizontal direction (Sections 3 and 5). The proof requires three preparatory lemmas, leading up to positivity of the lower density of intersections with the interface, which is the key ingredient in the proof (Section 4). The paper is concluded by showing that for $(\lambda, h) \in \mathcal{D}$ the path is delocalized in a weak sense, namely, it spends a zero fraction of its time in any finite layer around the interface (Section 6).

The main results of the present paper are Theorem 3 (Section 3) and Theorem 4 (Section 6).

1.5. Literature remarks. The *annealed* model (i.e., the partition sum is averaged over ω) treated by Sinai and Spohn (1996) [SS] is exactly solvable when the ω_i's are i.i.d. or interact via an Ising Hamiltonian. It turns out that the annealed heteropolymer is delocalized even in the presence of an interface. To get localization, an additional binding potential at the interface has to be superimposed.

The *quenched* model (i.e., ω is kept frozen) is mathematically much harder. The *periodic* case (e.g., ω represents some periodic constraint within the polymer) has been successfully dealt with by using a transfer-matrix approach [GIN]. In that paper, the underlying random walk is three-dimensional, undirected and with Gaussian steps, while the interface is a two-dimensional plane. It turns out that the phase transition curve diverges at some finite value of λ. This may be attributed to the flexibility of the Gaussian random walk to keep its monomers in their preferred solvent (by making large steps when necessary).

Prior to [Si] and [BodH], the *random* quenched problem (e.g., ω i.i.d.) had been analyzed by Garel et al. [GHLO] using the replica method. The latter study draws a conclusion qualitatively similar to that of Theorem 1.

The free-energy localization concept has proved to be useful also in the study of higher-dimensional generalizations of the present model [BG]. The latter authors consider a d-dimensional Gaussian surface, pinned at the interface outside a finite box and weighted by the same type of Hamiltonian as in (1.1). A localization/delocalization phase transition in the sense of Definition 1 is found, but the properties of the phase transition line are not yet fully understood and are possibly different from the ones in Theorem 1.

Whittington [W1,W2] and Orlandini at al. [OTW] consider the model where the heteropolymer is confined to a half-space above the interface and has an attractive interaction at the interface. Both for periodic and random quenched disorder they establish the existence of a localization/delocalization phase transition.

2. PRELIMINARIES

2.1. Gibbsian structure. Let $(\omega_i)_{i\in\mathbb{L}}$ be an i.i.d. sequence of ±1-valued random variables defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$. Here $\Omega=\{-1,1\}^{\mathbb{L}}$, $\mathcal{B}$ is the σ-algebra generated by the cylinder sets, and $\mathbb{P}$ is the i.i.d. measure with $\mathbb{P}(\omega_i=+1)=\mathbb{P}(\omega_i=-1)=1/2$. Expectation w.r.t. $\mathbb{P}$ will be denoted by $\mathbb{E}$.

Let

$$\Sigma=\begin{cases}\big\{S=(S_i)_{i\in\mathbb{N}}\colon S_0=0,|S_i-S_{i-1}|=\pm1\ \forall i\in\mathbb{N}\big\}\\ \big\{S=(S_i)_{i\in\mathbb{Z}}\colon S_0\in2\mathbb{Z},\,|S_i-S_{i-1}|=\pm1\ \forall i\in\mathbb{Z}\big\}\cup\{S\equiv\pm\infty\}\end{cases}\tag{2.1}$$

be the space of SRW-paths for the singly-infinite ($\mathbb{L}=\mathbb{N}$) and the doubly-infinite ($\mathbb{L}=\mathbb{Z}$) case, respectively.

Let $\mathcal{F}$ be the σ-algebra generated by the cylinder sets. For $\Lambda\subset\mathbb{L}$, let $\mathcal{F}_\Lambda$ be the projection of $\mathcal{F}$ onto Λ, and let $\mathcal{T}=\cap_{\Lambda\in F(\mathbb{L})}\mathcal{F}_{\Lambda^c}$ be the tail σ-field (remember that $F(\mathbb{L})$ denotes the set of all finite connected subsets of $\mathbb{L}$). We use $\mathcal{P}(\Sigma,\mathcal{F})$ to denote the space of all probability measures on $(\Sigma,\mathcal{F})$. Note that $\mathcal{P}(\Sigma,\mathcal{F})$ is compact in the weak topology for both the singly-infinite and the doubly-infinite case. (This is why we added $\pm\infty$ in (2.1) for the doubly-infinite case. In Section 5 we shall see that when $(\lambda,h)\in\mathcal{L}$ the Gibbs measures assign zero probability to $\{S\equiv\pm\infty\}$.) Let P,E denote probability and expectation under SRW.

We define Gibbs measures by means of the *Gibbsian specification* (for details see Chapter 1 of Georgii [Ge])

$$\gamma_\Lambda^{\omega,\lambda,h}(S|\tilde S)=\frac{e^{H_\Lambda^{\omega,\lambda,h}(S)}}{Z_\Lambda^{\omega,\lambda,h}(\tilde S)}P(S_\Lambda|\tilde S_{\Lambda^c})1_{\{S_{\Lambda^c}=\tilde S_{\Lambda^c}\}}\quad(\Lambda\in F(\mathbb{L})).\tag{2.2}$$

This specification is a probability measure on infinite paths $S = S_\Lambda \vee \tilde{S}_{\Lambda^c} \in \Sigma$ (with $S_\Lambda = (S_i)_{i\in\Lambda}$ and $\tilde{S}_{\Lambda^c} = (\tilde{S}_i)_{i\in\Lambda^c}$), absolutely continuous w.r.t. the conditional measure $P(S_\Lambda|\tilde{S}_{\Lambda^c})$ corresponding to the SRW-bridge, and a measurable function of the boundary condition $\tilde{S} \in \Sigma$. The partition function $Z_\Lambda^{\omega,\lambda,h}(\tilde{S})$ is the normalizing constant (which actually only depends on $\tilde{S}_{\partial\Lambda} = (\tilde{S}_i)_{i\in\partial\Lambda}$, with $\partial\Lambda$ being the outer boundary of Λ). It is easy to verify that the specifications $(\gamma_\Lambda^{\omega,\lambda,h})_{\Lambda\in F(\mathbb{L})}$ form a consistent family.

Given $\omega \in \Omega$ and $(\lambda, h) \in \mathcal{I}$, the Gibbs measures are defined as follows:

$$\mathcal{G}_\omega^{\lambda,h} = \big\{\mu \in \mathcal{P}(\Sigma, \mathcal{F})\colon\, \mu = \mu\gamma_\Lambda^{\omega,\lambda,h}\ \forall\Lambda \in F(\mathbb{L})\big\}, \tag{2.3}$$

i.e., $\gamma_\Lambda^{\omega,\lambda,h}$ is the conditional expectation of μ in Λ given the boundary condition in Λ^c. By compactness, any weak (subsequential) limit of $\gamma_\Lambda^{\omega,\lambda,h}(\tilde{S})$ as $\Lambda \to \mathbb{L}$, with a fixed boundary condition $\tilde{S}$, leads to a Gibbs measure (because the specifications are consistent). Hence $\mathcal{G}_\omega^{\lambda,h} \neq \emptyset$.

2.2. Regular measures. As is typical for Gibbs measures with unbounded single-component state spaces, an extreme boundary condition may overrule the effect of the interaction itself. In our setting, for the singly-infinite case and for any $(\lambda, h) \in \mathcal{I}$, there is a whole class of Gibbs measures (of at least countably-infinite cardinality) for which delocalized behavior is enforced when $\tilde{S}_i$ grows linearly with i. Similarly for the doubly-infinite case.

One can analyze this situation by looking at the *lower free energy* $\phi_{\tilde{S}}$ corresponding to $\tilde{S}$, defined by

$$\phi_{\tilde{S}}(\lambda, h) = \liminf_{n\to\infty} \frac{1}{|\Lambda_n|}\mathbb{E}\Big(\log Z_{\Lambda_n}^{\omega,\lambda,h}(\tilde{S})\Big) \tag{2.4}$$

$$\psi_{\tilde{S}}(\lambda, h) = \phi_{\tilde{S}}(\lambda, h) - \lambda h. \tag{2.5}$$

Lemma 1. *Consider the singly-infinite case. Let* $\lim_{i\to\infty} \tilde{S}_i/i = 0$*. Then* $\psi_{\tilde{S}}(\lambda, h) \geq \psi(\lambda, h)$*. Similarly for the doubly-infinite case.*

Proof. To find a lower bound on $\phi_{\tilde{S}}(\lambda, h)$, we pick Λ_{2n} and restrict the summation in $Z_{\Lambda_{2n}}^{\omega,\lambda,h}(\tilde{S})$ to paths that end by hitting the interface and subsequently moving at maximal speed. More precisely, if $c_n = \tilde{S}_{2n}/2n > 0$, then the path moves from height 0 at position $2n(1 - c_n)$ to height $2nc_n$ at position $2n$. This gives

$$Z_{\Lambda_{2n}}^{\omega,\lambda,h}(\tilde{S}) \geq Z_{\Lambda_{2n(1-c_n)}}^{\omega,\lambda,h}(0)\exp\Big[\lambda\sum_{i=2n(1-c_n)+1}^{2n}(\omega_i + h)\Big]\frac{\binom{2n(1-c_n)}{n(1-c_n)}}{\binom{2n}{n(1-c_n)}}, \tag{2.6}$$

where $Z^{\omega,\lambda,h}_{\Lambda_{2m}}(0)$ denotes the partition sum with boundary condition $\tilde{S}_{2m} = 0$ ($m \in \mathbb{N}$). The binomial factors come from the fact that the path must match the boundary condition (recall that the partition sum is defined w.r.t. the SRW-bridge). Now, it was shown in [BodH] that the ratio of $Z^{\omega,\lambda,h}_{\Lambda_{2n}}(0)$ and the partition function with free boundary condition, which was used to define $\phi(\lambda, h)$, is of linear order in n. Therefore the claim follows after taking logarithms, dividing by $2n$, letting $n \to \infty$, using that $c_n \to 0$, and using the relation between ϕ and ψ in (1.6). The case $c_n < 0$ and the doubly-infinite case are completely analogous. □

Lemma 1 shows that any sublinear boundary condition cannot destroy localization in the sense of Definition 1. Thus, a natural distinction between sublinear and linear boundary conditions arises. This leads us to the following definition.

Definition 2. *Given $(\lambda, h) \in \mathcal{I}$, the regular Gibbs measures are those $\mu \in \mathcal{G}^{\lambda,h}_{\omega}$ for which $\lim_{i\to\pm\infty} S_i/i = 0$ μ-a.s. The set of regular Gibbs measures is denoted by $\mathcal{G}^{R,\lambda,h}_{\omega}$.*

The theory of Gibbs measures guarantees that all regular measures lie in the closed convex hull of the weak limits generated by sublinear boundary conditions.

2.3. Measurable Gibbsian sections. As we noted earlier, $\mathcal{G}^{\lambda,h}_{\omega} \neq \emptyset$ for all ω by compactness. However, although $\mu_\omega \in \mathcal{G}^{\lambda,h}_{\omega}$ can be arranged into a measure-valued function of ω, it is not *a priori* clear that this can be done in a measurable way, because of possible non-uniqueness of the Gibbs measure. Formally, if we put $\mathcal{G}^{R,\lambda,h} = \bigcup_{\omega\in\Omega}\{(\omega,\mu)\colon \mu \in \mathcal{G}^{R,\lambda,h}_{\omega}\}$, then the question is whether or not there are *measurable* sections $(\omega, \mu_\omega)_{\omega\in\Omega} \in \mathcal{G}^{R,\lambda,h}$. We shall answer this question affirmatively when $(\lambda, h) \in \mathcal{L}$. Measurability will be important later on because we shall want to integrate over ω.

Define

$$\hat{\mathcal{G}}^{R,\lambda,h} = \left\{ \mu_{(\cdot)}\colon \Omega \mapsto \mathcal{P}(\Sigma,\mathcal{F}) \;\Big\langle\; \begin{matrix} \mu_\omega \in \mathcal{G}^{R,\lambda,h}_{\omega} \ \forall \omega \in \Omega \\ \mu_{(\cdot)}(A) \ \mathcal{B}\text{-measurable } \forall A \in \mathcal{F} \end{matrix} \right\} \tag{2.8}$$

to be the set of regular measurable Gibbsian sections. Observe that $\mu \in \hat{\mathcal{G}}^{R,\lambda,h}$ implies that μ, when regarded as a measure-valued function on Ω, is measurable w.r.t. the Borel σ-algebra associated with the weak topology on $\mathcal{P}(\Sigma,\mathcal{F})$.

Lemma 2. *Let $(\lambda, h) \in \mathcal{L}$. Then*
(a) $\hat{\mathcal{G}}^{R,\lambda,h}$ is non-empty both for the singly-infinite and the doubly-infinite case.
(b) For the doubly-infinite case there is a $\mu_{(\cdot)}\colon \Omega \mapsto \mathcal{P}(\Sigma,\mathcal{F})$ such that $\mu_{(\cdot)}(A)$ is $\mathcal{B}$-measurable for all $A \in \mathcal{F}$ and

(1) $\mu_\omega(|S_0| < \infty) = 1$
(2) μ_ω *is regular Gibbsian, i.e.,* $\mu_\omega \in \mathcal{G}^{R,\lambda,h}_{\omega}$
(3) $\mu_{\sigma\omega}(\sigma A) = \mu_\omega(A)$ *for all* $A \in \mathcal{F}$

hold for $\mathbb{P}$-almost all ω. Here σ denotes the left-shift by two (!) lattice sites, acting on path and disorder.

The proof of Lemma 2 is given in Section 5 and requires a large deviation estimate on the partition function appearing in (2.2), which is derived in Lemma 3 (Section 4). The point here is to rule out that mass escapes to infinity under the doubly-infinite measure μ_ω (i.e., $\{S \equiv \pm\infty\}$ has non-zero probability). This is in fact likely to happen when $(\lambda, h) \in \mathcal{D}$, but here we are only considering $(\lambda, h) \in \mathcal{L}$.

3. UNIQUENESS AND POSITIVE DENSITY IN THE LOCALIZATION REGIME

It is intuitively clear that $\psi > 0$ implies recurrence, i.e., the path hits the interface infinitely often. Namely, for any regular boundary condition $\tilde{S}$ we have $Z_\Lambda^{\omega,\lambda,h}(\tilde{S})e^{-\lambda h|\Lambda|} = e^{|\Lambda|\psi_{\tilde{S}}(\lambda,h)+o(|\Lambda|)} \to \infty$ as $\Lambda \to \mathbb{L}$, which implies that the set $\{S \in \Sigma\colon S_i > 0\ \forall i \geq n\}$ has zero probability for all n (recall (1.1) and (2.1)). Below we shall in fact prove more, namely, that all regular Gibbs measures are *positively recurrent*, i.e., the path visits every height with a certain positive frequency.

For $a \in \mathbb{Z}$, let

$$\varrho_a^-(S) = \liminf_{\Lambda\to\mathbb{L}} \frac{2}{|\Lambda|} \sum_{i\in\Lambda} 1_{\{S_i=a\}}, \tag{3.1}$$

where the factor 2 takes care of the parity of SRW. We shall say that $(\omega, \mu_\omega)_{\omega\in\Omega} \in \hat{\mathcal{G}}^{R,\lambda,h}$ is *localized* if $\mathbb{E}\mu_\omega(\varrho_0^- > 0) = 1$, i.e., if $\mu_\omega(\varrho_0^- > 0) = 1$ for $\mathbb{P}$-almost all ω, by Fubini's theorem. Now we are ready to state the main theorem of our paper:

Theorem 3. *Let $(\lambda, h) \in \mathcal{L}$. Then*
(a) $\hat{\mathcal{G}}^{R,\lambda,h}$ is a singleton both for the singly-infinite and the doubly-infinite case.
(b) The unique doubly-infinite Gibbsian section $(\omega, \mu_\omega)_{\omega\in\Omega}$ is localized and is jointly translation invariant (i.e., $\mu_{\sigma\omega}(\sigma A) = \mu_\omega(A)$ for $\mathbb{P}$-almost all ω and all $A \in \mathcal{F}$).
(c) The unique singly-infinite Gibbsian section $(\omega, \nu_\omega)_{\omega\in\Omega}$ is localized and is asymptotically equal to $(\omega, \mu_\omega)_{\omega\in\Omega}$:

$$\lim_{n\to\infty} \sup_{A\in\mathcal{F}} \left|\nu_\omega(\sigma^n A) - \mu_\omega(\sigma^n A)\right| = 0 \quad \textit{for } \mathbb{P}\textit{-almost all } \omega. \tag{3.2}$$

(d) Both Gibbsian sections have a.s. constant densities, i.e., for $\mathbb{P}$-almost all ω and all $A \in \mathcal{F}$

$$\lim_{\Lambda\to\mathbb{L}} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} 1_{\sigma^i A}(S) = \mathbb{E}\mu_\omega(A) \qquad \textit{for } \mu_\omega\textit{-almost all } S \in \Sigma, \tag{3.3}$$

and similarly for ν_ω.

(e) Both Gibbsian sections are exponentially tight: for any $s \in \mathbb{Z}$ and $\varepsilon > 0$ there exists a random number $n_0(s, \epsilon, \omega)$ such that

$$\nu_\omega(S_n = s) \leq \mathcal{O}(1) e^{-(\zeta_s - \varepsilon)|2s|} \quad (n \geq n_0(s, \varepsilon, \omega)), \tag{3.4}$$

with $\zeta_s = \psi(\lambda, h)$ when $s > 0$ and $\zeta_s = \psi(\lambda, h) + \lambda h$ when $s < 0$.

Assertions (a) and (b) establish uniqueness within the class of regular Gibbsian sections (in $\mathcal{L}$). The measures ν_ω resp. μ_ω should be understood as describing the behavior of the polymer near the endpoint, respectively, away from the endpoints. Assertion (c) claims that these two blend into each other at infinity. Assertion (d) corresponds to ergodicity along the polymer. (Note that the probabilities $\mu_\omega(\sigma^i A)$ typically vary a great deal with i according to the local disorder.) Assertion (e) provides an extension of Sinai's result cited in Theorem 2, with explicit bounds on the decay rate.

4. THREE PREPARATORY LEMMAS

In order to prove Lemma 2 and Theorem 3, we first have to state a couple of technical lemmas that establish exponential growth of the partition function (Lemma 3), exponential tightness of the interarrival times to the interface (Lemma 4), and—most importantly—a.s. strict positivity of the lower density of intersections with the interface under both μ_ω and ν_ω (Lemma 5). To avoid confusion, we emphasize that the proof of Lemma 2 requires only the result of Lemma 3, hence there is no problem with measurability in Lemmas 4 and 5. Throughout the sequel we assume $(\lambda, h) \in \mathcal{L}$ and suppress these parameters from the notation.

4.1. Large deviations for the partition sum. The assertion of Lemma 3 is a large deviation estimate for the partition sum that will be needed later on, in particular, in conjunction with a Borel-Cantelli argument.

Lemma 3. *Let $Z_{2n}^\omega = Z_{\Lambda_{2n}}^\omega(0)$ be the partition function for the boundary condition $\tilde{S}_0 = \tilde{S}_{2n} = 0$. Then for each $\varepsilon \in (0, \psi)$ there is a $\delta_\varepsilon > 0$ such that*

$$\mathbb{P}\Big(\frac{1}{2n} \log Z_{2n}^\omega < \psi + \lambda h - \varepsilon\Big) \leq \mathcal{O}(1) e^{-\delta_\varepsilon 2n} \quad (n \to \infty). \tag{4.1}$$

Proof. Given $\varepsilon > 0$, there is an m large enough such that

$$\frac{1}{2m} \mathbb{E}(\log Z_{2m}^\omega) \geq \psi + \lambda h - \varepsilon/2. \tag{4.2}$$

This follows from the fact that a sublinear boundary condition does not lower the free energy (see Lemma 1). Pick such an m and put $k = \lfloor n/m \rfloor$. Then, by restricting the path to return to 0 at positions $2m, 4m, \ldots, 2km$ ($\leq 2n$), we obtain

$$Z_{2n}^{\omega} \geq \frac{\binom{2m}{m}^k \binom{2(n-km)}{n-km}}{\binom{2n}{n}} \Big[\prod_{j=0}^{k-1} Z_{2m}^{\sigma^{jm}\omega}\Big] Z_{2n-2km}^{\sigma^{km}\omega}. \tag{4.3}$$

Here the binomial factor reflects the fact that the partition sum is defined w.r.t. the SRW-bridge. After taking logarithms and dividing by $2n$ we get

$$\mathbb{P}\Big(\frac{1}{2n}\log Z_{2n}^{\omega} < \psi + \lambda h - \varepsilon\Big) \leq \mathbb{P}\Big(\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{2m}\log Z_{2m}^{\sigma^{jm}\omega} < \psi + \lambda h - 3\varepsilon/4\Big), \tag{4.4}$$

where we have assumed n so large that the factors outside the square brackets in (4.3) give rise to a correction less than $\varepsilon/4$. Now, $(1/2m)\log Z_{2m}^{\sigma^{jm}\omega}$ ($j = 0, \ldots, k-1$) are i.i.d. bounded random variables. Therefore a standard large deviation estimate gives that the r.h.s. of (4.4) is bounded by $\mathcal{O}(1)e^{-\delta'_\varepsilon 2k}$ for some $\delta'_\varepsilon > 0$. From this the claim easily follows by choosing $\delta_\varepsilon = \delta'_\varepsilon/m$ (we neglect the additional correction coming from rounding off $\lfloor n/m \rfloor$, which is absorbed into the $\mathcal{O}(1)$-term). $\square$

4.2. Tightness of interarrival times. Let us introduce the notion of *arrival times*, defined as the positions where the path hits the interface, i.e.,

$$\ldots < N_{-1} < N_0 \leq 0 < N_1 < N_2 < \ldots, \tag{4.5}$$

are specified by $S_{2N_k} = 0$ ($k \in \mathbb{Z}$) and $S_{2r} \neq 0$ if $r \notin (N_k)$. Let $\xi_k = N_{k+1} - N_k$ ($k \in \mathbb{Z}$) be the *interarrival times*. (Both sequences end when no further arrivals occur.) Note that only the even sites are counted in the excursions.

Lemma 4. *If μ_ω is a measurable Gibbsian section corresponding to the disorder ω, then there is a $\kappa > 0$ such that for any $i \in \mathbb{Z}$, $L \in \mathbb{Z}$, $K \in \mathbb{N}$ and any $m_{i+j} \in \mathbb{N}$ ($j = 0, \ldots, K-1$)*

$$\mathbb{E}\Big[\mu_\omega\big(\xi_{i+j} = m_{i+j}\ \forall j = 0, \ldots, K-1 \big| N_i = L\big)\Big] \leq \mathcal{O}(1)\prod_{j=0}^{K-1} e^{-\kappa m_{i+j}}. \tag{4.6}$$

Proof. Fix i, L, K. The event

$$A = \{\xi_{i+j} = m_{i+j}\ \forall j = 0, \ldots, K-1\}, \tag{4.7}$$

if conditioned on $\{N_i = L\}$, means that $S_{2k_j} = 0$ for $k_j = L + \sum_{l=0}^{j-1} m_{i+l}$ and $S_{2r} \neq 0$ for $k_j < r < k_{j+1}$ $(j = 0, \ldots, K-1)$. Since μ_ω is Gibbsian, we can apply conditioning to write (recall (1.1))

$$\mu_\omega(A|N_i = L) = \Big[\prod_{j=0}^{K-1} \frac{1 + e^{-2\lambda(\Omega_{I_j} + h|I_j|)}}{2Z_{I_j}^\omega e^{-\lambda(\Omega_{I_j} + h|I_j|)}} P_{I_j} \Big] \times \mu_\omega(S_{2k_{j+1}} = 0 \;\forall j = 0, \ldots, K-1 | N_i = L), \tag{4.8}$$

where $I_j = (2k_j, 2k_{j+1}] \cap \mathbb{Z}$, $\Omega_{I_j} = \sum_{l \in I_j} \omega_l$, and P_{I_j} is the probability that SRW conditioned on $S_{2k_j} = 0 = S_{2k_{j+1}}$ never touches the interface in between. By neglecting the last factor we obtain

$$\mu_\omega(A|N_i = L) \leq \prod_{j=0}^{K-1} \frac{1 + e^{-2\lambda(\Omega_{I_j} + h|I_j|)}}{2Z_{I_j}^\omega e^{-\lambda(\Omega_{I_j} + h|I_j|)}} P_{I_j}. \tag{4.9}$$

Pick $\varepsilon > 0$. By Lemma 3, there exists a $\delta_\varepsilon > 0$ such that

$$\mathbb{P}\Big(Z_{I_j}^\omega e^{-\lambda h|I_j|} < e^{(\psi - \varepsilon)|I_j|} \Big) \leq \mathcal{O}(1) e^{-\delta_\varepsilon |I_j|} \qquad \text{for all } j. \tag{4.10}$$

Moreover, a standard large deviation estimate gives that there exists a $\delta'_\varepsilon > 0$ such that

$$\mathbb{P}\big(|\Omega_{I_j}| > \frac{\varepsilon}{\lambda} |I_j| \big) \leq \mathcal{O}(1) e^{-\delta'_\varepsilon |I_j|} \qquad \text{for all } j. \tag{4.11}$$

Hence, by using (4.11) to estimate the numerator of the fractions in (4.9), and (4.10) to estimate the denominator on the complement of the event in (4.11), we get

$$\mathbb{E}\big[\mu_\omega(A|N_i = L)\big] \leq \mathcal{O}(1) \prod_{j=0}^{K-1} \Big[e^{-\delta_\varepsilon |I_j|} + e^{-\delta'_\varepsilon |I_j|} + e^{-(\psi - 2\varepsilon)|I_j|} \Big], \tag{4.12}$$

where we also used that each factor in the r.h.s. of (4.9) is ≤ 1. The desired estimate (4.6) is now obtained by setting $\kappa = 2 \sup_{\varepsilon > 0} \min\{\delta_\varepsilon, \delta'_\varepsilon, \psi - 2\varepsilon\}$. Since $\psi > 0$, we obviously have $\kappa > 0$. $\square$

4.3. Positive lower density of intersections. Now comes the most important lemma, which establishes a.s. positivity of ϱ_0^- (recall (3.1)). As explained in Section 5, this result will make accessible certain coupling techniques that will be used to prove Theorem 3.

Lemma 5. *There is a $\hat{\varrho} > 0$ such that for any measurable Gibbsian section $(\omega, \mu_\omega)_{\omega\in\Omega} \in \hat{\mathcal{G}}^{R,\lambda,h}$: $\mu_\omega(\varrho_0^- \geq \hat{\varrho}) = 1$ for $\mathbb{P}$-almost all ω.*

Proof. Let us concentrate on the doubly-infinite case. (The singly-infinite case can be handled analogously.) Let

$$A_{n;k} = \Big\{ \sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} \leq k \Big\}. \tag{4.13}$$

Let $-2(n+n_-)$ label the last arrival before $-2n$ and $2(n+n_+)$ the first arrival after $2n$. Since Lemma 4 provides an estimate for interarrival times in a row, we have for $0 \leq k \leq n$

$$\begin{aligned}\mathbb{E}\mu_\omega(A_{n;k}) &\leq \mathcal{O}(1)\Big[\sum_{l=0}^{k}\binom{2n+1}{l}\Big]\sum_{n_+,n_-=1}^{\infty} e^{-\kappa(2n+n_-+n_++1)} \\ &\leq \mathcal{O}(n)\binom{2n+1}{k}e^{-\kappa n},\end{aligned} \tag{4.14}$$

where the binomial factor accounts for all possible positions of the k arrivals within $[-2n, 2n]$.

Pick $0 < \hat{\varrho} < 1/2$ and pick $k = k(n) = \lfloor (2n+1)\hat{\varrho} \rfloor$. Then, using Stirling's formula, we obtain

$$\mathbb{E}\mu_\omega(A_{n;k(n)}) \leq \mathcal{O}(n)\big[e^{-\kappa/2}\hat{\varrho}^{-\hat{\varrho}}(1-\hat{\varrho})^{-(1-\hat{\varrho})}\big]^{2n}. \tag{4.15}$$

So if $\hat{\varrho}$ satisfies $\hat{\varrho}\log\hat{\varrho} + (1-\hat{\varrho})\log(1-\hat{\varrho}) + \kappa/2 > 0$, then the r.h.s. is summable on n, and hence

$$\mathbb{E}\mu_\omega(A_{n;k(n)} \text{ i.o.}) = 0 \tag{4.16}$$

by the Borel-Cantelli lemma. Consequently,

$$\sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} > \lfloor (2n+1)\hat{\varrho} \rfloor \tag{4.17}$$

eventually under $\mathbb{E}\mu_\omega$, and hence under μ_ω for $\mathbb{P}$-almost all ω. Therefore the claim follows (recall (3.1)). $\square$

5. PROOFS

5.1. Proof of Lemma 2. Fix $(\lambda, h) \in \mathcal{L}$ and suppress these parameters from the notation. We shall consider the doubly-infinite case and construct a measure-valued function $\omega \mapsto \mu_\omega$ with the desired properties. The existence proof in the singly-infinite case is analogous.

We start the construction by defining a finite-volume jointly translation-invariant Gibbs measure on SRW-paths and disorder configurations, and then identify a thermodynamic limit thereof. The construction guarantees that the limit fulfils the requirements stated in Lemma 2(b). Lemma 4 will be used to show that no mass escapes to infinity.

Consider a finite string $\Lambda \subset \mathbb{Z}$, with $|\Lambda|$ even and $\Lambda \ni 0$. Let $\tilde{\gamma}^\omega_\Lambda(S_\Lambda)$ be the specification in Λ defined by

$$\tilde{\gamma}^\omega_\Lambda(S_\Lambda) = \frac{1}{2^{|\Lambda|}} \frac{e^{H^{\omega,\lambda,h}_\Lambda(S_\Lambda)}}{\tilde{Z}^{\omega,\lambda,h}_\Lambda} 1_{\{\exists(2i)\in\Lambda:\, S_{2i}=0\}} 1_{\{S_{\max\Lambda}-S_{\min\Lambda}=\pm 1\}}. \tag{5.1}$$

Note that here we force the path to intersect the interface somewhere and that we impose periodic boundary conditions. Clearly, $\tilde{\gamma}^{\sigma\omega}_\Lambda(\sigma A) = \tilde{\gamma}^\omega_\Lambda(A)$ for any $A \in \mathcal{F}_\Lambda$ (where σ acts cyclically).

Pick a sequence (Λ_{2n}) of such intervals with $|\Lambda_{2n}| = 2n$. Now define $\mu^{(n)}_B(A)$ by

$$\mu^{(n)}_B(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(\omega) \tilde{\gamma}^\omega_{\Lambda_{2n}}(A) \qquad (A \in \mathcal{F}_{\Lambda_{2n}}, B \in \mathcal{B}_{\Lambda_{2n}}). \tag{5.2}$$

By compactness, we have $\mu^{(n_k)}_B(A) \to \mu_B(A)$ along a subsequence (n_k) for all $A \in \cup_n \mathcal{F}_{\Lambda_{2n}}$, all $B \in \cup_n \mathcal{B}_{\Lambda_{2n}}$, and for some $\mu_B(A)$. Since $\mu_B(A)$ is σ-additive on $\cup_n \mathcal{B}_{\Lambda_{2n}} \times \cup_n \mathcal{F}_{\Lambda_{2n}}$, it has a unique extension $\bar{\mu}_{(.)}(\cdot)$ to $\mathcal{B} \times \mathcal{F}$ by the Caratheodory theorem. Before we extract μ_ω from $\bar{\mu}_\Omega$, we first verify that $\bar{\mu}_\Omega$ assigns zero probability to $\{S \equiv \pm\infty\}$, which is Lemma 2(b)(1). This will follow if $\bar{\mu}_\Omega\big(|S_0| \geq a\big) \to 0$ for $a \to \infty$ (i.e., the path cannot escape to infinity). Indeed, since $\tilde{\gamma}^\omega_\Lambda$ is Gibbsian we may estimate

$$\begin{aligned} \mu^{(n_k)}_\Omega\big(|S_0| \geq 2a\big) &\leq \mathbb{E}\, \tilde{\gamma}^\omega_{\Lambda_{2n_k}}(N_0 \leq -a,\, N_1 \geq a) \\ &\leq \mathcal{O}(1) \sum_{i_1,i_2=a}^{\infty} e^{-\kappa(i_1+i_2+1)} \to 0 \qquad (a \to \infty) \end{aligned} \tag{5.3}$$

uniformly in n_k, with the aid of (4.6). Now $\mu_B(A) \leq \mathbb{P}(B)$ implies $\bar{\mu}_B(A) \leq \mathbb{P}(B)$, so by the Radon-Nikodym theorem there exists a unique μ_ω such that

$$\bar{\mu}_B(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(\omega) \mu_\omega(A) \qquad (A \in \mathcal{F}). \tag{5.4}$$

Clearly, by (5.3), $\mu_\omega(|S_0| < \infty) = 1$ for $\mathbb{P}$-almost all ω, so Lemma 2(b)(1) is established.

The uniqueness of the representation in (5.4) implies that μ_ω is a σ-additive probability measure and that $\mu_\omega \in \mathcal{G}^{R,\lambda,h}_\omega$ for $\mathbb{P}$-almost all ω. To prove the latter property, which is

Lemma 2(b)(2), pick $\Lambda \subset \Delta \in F(\mathbb{Z})$, $D \in \mathcal{F}$, $\tilde{\omega} \in \Omega$, take $B = \{\omega \in \Omega \colon \omega|_\Delta = \tilde{\omega}|_\Delta\}$, and subtract (5.2) with $A = D$ from (5.2) with A replaced by the function $\gamma_\Lambda^{\tilde{\omega}}(D|\cdot)$ (i.e., the specification defined in (2.2)), to get

$$\int_\Omega \mathbb{P}(d\omega) 1_{\{\omega\in\Omega \colon \omega|_\Delta = \tilde{\omega}|_\Delta\}} \left[\mu_\omega(D) - E_{\mu_\omega} \gamma_\Lambda^{\tilde{\omega}}(D|\cdot)\right] = 0, \tag{5.5}$$

where E_{μ_ω} stands for expectation w.r.t. μ_ω. Here we have been able to use Gibbsianness of $\tilde{\gamma}_{\Lambda_{2n}}^\omega$ (with $\Delta \subset \Lambda_{2n}$) even under integration over ω, because ω is equal to $\tilde{\omega}$ inside Λ. Now proceed by letting $\Delta \to \mathbb{Z}$ to show that $\mu_{\tilde{\omega}}(D) = E_{\mu_{\tilde{\omega}}} \gamma_\Lambda^{\tilde{\omega}}(D|\cdot)$ for $\mathbb{P}$-almost all $\tilde{\omega}$. Hence, μ_ω is a Gibbs measure for the Hamiltonian in (1.1). It follows from (5.3) and a straightforward Borel-Cantelli argument that $\bar{\mu}_\Omega$ is regular and, consequently, μ_ω is regular $\mathbb{P}$-almost surely, which completes the proof of Lemma 2(b)(2).

Finally, Lemma 2(b)(3) follows from the periodicity of the specification $\tilde{\gamma}_\Lambda^\omega(S_\Lambda)$, which is trivially jointly translation-invariant. □

5.2. Proof of Theorem 3. The proof will come in four steps. Step 1, which is the most technical, shows that for two arbitrary Gibbsian sections corresponding to the *same* medium the paths intersect infinitely often. Via a coupling argument in Step 2 this will prove items (a–c) of Theorem 3. Items (d) and (e) are established in Steps 3 and 4, respectively.

STEP 1. Let $(\omega, \mu_\omega)_{\omega\in\Omega} \in \hat{\mathcal{G}}^{R,\lambda,h}$ be the measurable Gibbsian section whose existence was established in Lemma 2. Let $(\omega, \nu_\omega)_{\omega\in\Omega} \in \hat{\mathcal{G}}^{R,\lambda,h}$, either singly-infinite or doubly-infinite. In order to make coupling possible, we have to show that paths intersect infinitely often under a joint measure. The result of Lemma 5 allows us to choose the product measure.

Label the paths under μ_ω by 1, the paths under ν_ω by 2. Let

$$C_\infty = \{(S^1, S^2) \colon S_n^1 = S_n^2 \text{ i.o.}\} \tag{5.6}$$

be the set of pairs of paths that intersect infinitely often. We shall show that $(\mu_\omega \times \nu_\omega)(C_\infty) = 1$ for $\mathbb{P}$-almost all ω. The proof goes as follows.

As was shown in Lemma 5, both measures have a strictly positive lower density of intersections with the interface. Hence the function $f_M = 1_{\{N_1 - N_0 \geq M\}}$, i.e., the indicator of the event that 0 belongs to an excursion larger than M (recall (4.5)), is well defined on a set of full measure. Since $f_M \in L^1(\Sigma, \mathcal{F}, \mathbb{E}\mu_\omega)$, we have by the ergodic theorem (recall that $\mathbb{E}\mu_\omega$ is σ-invariant) that there exists an $\bar{f}_M$ such that

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \sigma^j f_M = \bar{f}_M \quad \mathbb{E}\mu_\omega\text{-a.s.}, \tag{5.7}$$

where σ acts on Σ, and $\mathbb{E}E_{\mu_\omega}(\bar{f}_M) = \mathbb{E}E_{\mu_\omega}(f_M)$. Moreover, $\sigma\bar{f}_M = \bar{f}_M$. Hence, $\mu_\omega(\bar{f}_M > a)$ is constant $\mathbb{P}$-a.s. (by ergodicity w.r.t. the disorder) and

$$\mu_\omega(\bar{f}_M > a) = \mathbb{E}\mu_\omega(\bar{f}_M > a) \le \frac{\mathbb{E}E_{\mu_\omega}(\bar{f}_M)}{a} = \frac{\mathbb{E}E_{\mu_\omega}(f_M)}{a}. \tag{5.8}$$

The r.h.s. can be further estimated with the help of Lemma 4, namely,

$$\mathbb{E}E_{\mu_\omega}(f_M) \le \mathcal{O}(1)\sum_{n=M}^{\infty} n\, e^{-\kappa n} = \mathcal{O}(M)e^{-\kappa M} \qquad (M \to \infty), \tag{5.9}$$

where we use that $\mathbb{E}\mu_\omega(f_M = 1)$ is bounded by the sum over $n \ge M$ of the l.h.s. of (4.6) with $i = 0$, $K = 1$, and L running from $-n+1$ to 0. Therefore, combining (5.8) and (5.9), we have

$$\mu_\omega(\bar{f}_M > a) \le \frac{\mathcal{O}(M)}{a}e^{-\kappa M} \qquad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{5.10}$$

Now, on $\{\bar{f}_M < b\}$ the fraction of sites of $2\mathbb{Z}$ covered by excursions of length $\ge M$ is less than b. Hence, on $\{\bar{f}_M < \hat{\varrho}/2\} \times \Sigma$ at least half of the arrivals of S^2 occur within the S^1-excursions of length $< M$ (recall that $\hat{\varrho}$ is a lower bound for ϱ_0^- defined in (3.1)). If the two paths (S^1, S^2) are to avoid each other, then the first has to stay either above or below the other during *all* of these (infinitely many) excursions.

To show that the probability of the latter event is zero we introduce some definitions. Let

$$p(n,\omega) = \frac{\gamma_{\Lambda_{2n}}(S_i > 0\, \forall 1 \le i < 2n|0)}{\gamma_{\Lambda_{2n}}(S_i \ne 0\, \forall 1 \le i < 2n|0)} \tag{5.11}$$

and put $p_M = \max_{n<M}\max_{\omega\in\Omega}\max\big\{p(n,\omega), 1-p(n,\omega)\big\}$. An easy computation shows that $p_M = (1+e^{-2\lambda(1+h)(M-1)})^{-1} < 1$. This is the least price to pay (when conditioning upon the arrivals) to avoid that the path S^1 be swapped to $-S^1$ during an excursion of length $< M$.

Next, define the remotest intersection time as

$$\tau = \begin{cases} \max\{k\colon S^1_{2k} = S^2_{2k} \text{ or } S^1_{-2k} = S^2_{-2k}\} & (S^1,S^2) \notin C_\infty \\ \infty & (S^1,S^2) \in C_\infty. \end{cases} \tag{5.12}$$

Also define $\mathcal{N}_{k,M}(n) = \#\{i\colon k \le i \le n+k,\ S^2_{2i} = 0,\ f_M(\sigma^i S^1) = 0\}$ and $A_M = \{\bar{f}_M < \hat{\varrho}/2\} \times \Sigma$. On A_M we have

$$\liminf_{n\to\infty} \frac{\mathcal{N}_{k,M}(n)}{n} \ge \hat{\varrho}/2 > 0 \qquad \mathbb{E}\nu_\omega\text{-a.s.} \tag{5.13}$$

as follows from Lemma 5 and the reasoning below (5.10). Therefore we get

$$(\mu_\omega \times \nu_\omega)\big(A_M \cap [C_\infty]^c\big) \le \sum_{k=1}^{\infty}\Big[2(\mu_\omega \times \nu_\omega)\big(\{\lim_{n\to\infty} p_M^{\mathcal{N}_{k,M}(n)} 1_{A_M}\}1_{\{\tau=k\}}\big)\Big] = 0, \quad (5.14)$$

Here we decompose according to the values of τ, condition upon the arrivals of both S^1 and S^2 in $[\tau, \tau+n]$, then bound by p_M the interarrival probabilities of S^1 for excursions of length $< M$ containing at least one arrival of S^2, and bound by 1 otherwise, and finally use that $1_{A_M}\, p_M^{\mathcal{N}_{k,M}(n)} \le e^{n\frac{\hat{\varrho}}{4}\log p_M}$ for n large enough, as follows from (5.13). The factor 2 reflects whether S^1 stays above S^2 from τ onwards or vice versa. Note that there is no problem with 1_{A_M} in the conditioning, because A_M is a tail event.

The conclusion of (5.14) is that $(\mu_\omega \times \nu_\omega)\big(A_M \cap [C_\infty]^c\big) = 0$ for all M. On the other hand,

$$\mu_\omega\Big(\bigcup_{M=1}^{\infty}\{\bar{f}_M < \hat{\varrho}/2\}\Big) = 1 \quad (5.15)$$

by the Borel-Cantelli lemma and (5.10). Hence, combining (5.14) and (5.15), we find that $(\mu_\omega \times \nu_\omega)\big([C_\infty]^c\big) = 0$, i.e., the paths S^1 and S^2 intersect infinitely often $(\mu_\omega \times \nu_\omega)$-almost surely.

STEP 2. We show by a coupling inequality that any two measures μ_ω and ν_ω have to agree on the tail σ-field $\mathcal{T}$. Besides other things, this implies uniqueness. The proof is done for ν_ω singly-infinite, the doubly-infinite case requiring only formal alterations.

Let $k \in \mathbb{N}$ and $A \in \mathcal{F}_{\Lambda_k^c}$ (A should be thought of as approximating a tail event). Define

$$\tau = \inf\{n \ge 0\colon S_n^1 = S_n^2\}. \quad (5.16)$$

Note that $\tau < \infty$. Let E_ω denote the expectation w.r.t. the product measure $\mu_\omega \times \nu_\omega$. Then we can write

$$\begin{aligned}\big|\mu_\omega(A) - \nu_\omega(A)\big| &= \big|E_\omega(A \times \Sigma) - E_\omega(\Sigma \times A)\big| \\ &\le \big|E_\omega(1_{\{\tau>k\}}1_{A\times\Sigma}) - E_\omega(1_{\{\tau>k\}}1_{\Sigma\times A})\big| \\ &\le E_\omega(1_{\{\tau>k\}}),\end{aligned} \quad (5.17)$$

where we use that $E_\omega(1_{\{\tau\le k\}}1_{A\times\Sigma}) = E_\omega(1_{\{\tau\le k\}}1_{\Sigma\times A})$ because μ_ω and ν_ω have the same conditional probabilities. Hence

$$\sup_{A\in\mathcal{F}_{\Lambda_k^c}} \big|\mu_\omega(A) - \nu_\omega(A)\big| \le E_\omega(1_{\{\tau>k\}}). \quad (5.18)$$

By Step 1 the r.h.s. tends to 0 as $k \to \infty$. Consequently, μ_ω and ν_ω agree on the tail σ-field $\mathcal{T}$. In particular,

$$\lim_{k\to\infty} |\nu_\omega(\sigma^k A) - \mu_\omega(\sigma^k A)| = 0 \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{5.19}$$

STEP 3. The a.s. convergence of ergodic averages under ν_ω can be proved through a comparison with the a.s. convergence under $\mathbb{E}\mu_\omega$, which is translation invariant. Namely, given a set $A \in \mathcal{F}$, let

$$A_> = \Big\{\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{\sigma^k A} > \mathbb{E}\mu_\omega(A)\Big\}. \tag{5.20}$$

Clearly, $A_>$ is a tail event, and $\mathbb{E}\mu_\omega(A_>) = 0$ by the translation invariance of $\mathbb{E}\mu_\omega$. But this implies $\mathbb{E}\nu_\omega(A_>) = 0$, since $\mathbb{E}\nu_\omega$ coincides with $\mathbb{E}\mu_\omega$ on $\mathcal{T}$. So

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{\sigma^k A} \le \mathbb{E}\mu_\omega(A) \quad \nu_\omega\text{-a.s. for } \mathbb{P}\text{-almost all } \omega. \tag{5.21}$$

The same argument works for the limes inferior, so the limit in (3.3) is established.

STEP 4. The last property to prove is that μ_ω and ν_ω are exponentially tight. Since we know by (5.19) that $|\nu_\omega(\sigma^n A) - \mu_\omega(\sigma^n A)| \to 0$ as $n \to \infty$, it suffices to study the tail of μ_ω. To that end, pick $s \in \mathbb{Z}, s > 0$. We have from Gibbsianness

$$\mu_\omega(S_0 = 2s) = \sum_{n_+,n_-=s}^{\infty} \frac{P_{n_+,n_-}(S_0 = 2s)}{Z^\omega_{I_{n_+,n_-}} e^{-\lambda(\Omega_{I_{n_+,n_-}} + h|I_{n_+,n_-}|)}} \mu_\omega(S_{-2n_-} = S_{2n_+} = 0), \tag{5.22}$$

where $I_{n_+,n_-} = (-2n_-, 2n_+] \cap \mathbb{Z}$, and $P_{n_+,n_-}(S_0 = 2s)$ is the probability that SRW, conditioned on hitting the interface at $-2n_-$ and $2n_+$, climbes to height $2s$ at 0 without ever touching the interface in between. By using Lemma 3 (and using the Borel-Cantelli lemma to get rid of $\mathbb{E}$ as in (5.10)), we have for any $\varepsilon > 0$

$$\big(Z^\omega_{I_{n_+,n_-}} e^{-\lambda h|I_{n_+,n_-}|}\big)^{-1} \le \mathcal{O}(1) e^{-|I_{n_+,n_-}|(\psi-\varepsilon)}, \tag{5.23}$$

so the r.h.s. of (5.22) is $\mathbb{P}$-a.s. absolutely summable and of order $e^{-4s(\psi-\varepsilon)}$ as $s \to \infty$ (note that $e^{-\lambda\Omega_I} = o(e^{\varepsilon|I|})$ for each $\varepsilon > 0$ as $|I| \to \infty$). After letting $\varepsilon \downarrow 0$ we obtain that the tail property in Theorem 3 is proved for $s > 0$, with $\zeta_s = \psi$. For $s \in \mathbb{Z}, s < 0$ there is an additional factor

$$\exp\Big[-2\lambda \sum_{l\in I_{n_-,n_+}} (\omega_l + h)\Big] \tag{5.24}$$

in the numerator of each summand. This raises ζ_s by λh. $\square$

6. ZERO DENSITY IN THE DELOCALIZATION REGIME

In this section we consider the singly-infinite case and present an argument due to G. Giacomin (private communication) showing that in the interior of the delocalization regime the path is delocalized in the following sense:

Theorem 4. *Let $(\lambda, h) \in int(\mathcal{D})$ and let $\nu_\omega \in \mathcal{G}_\omega^{R,\lambda,h}$ be an arbitrary singly-infinite regular Gibbs measure. Then for $\mathbb{P}$-almost all ω*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{S_i=a\}} = 0 \quad \textit{for all } a \in \mathbb{Z}, \tag{6.1}$$

in probability w.r.t. ν_ω.

Remark. Note that Theorem 4 makes a claim about *all* regular Gibbs measures under a typical disorder. Since we do not have Lemma 2 for $(\lambda, h) \in \mathcal{D}$, the notion of a measurable Gibbsian section is not available.

Proof. Fix $a \in 2\mathbb{Z}$ (without loss of generality). For $k, l \in \mathbb{N}$, define

$$A_{k,l}^a = \Big\{ \sum_{i=0}^{l} 1_{\{S_{2i}=a\}} \geq k+1 \Big\}. \tag{6.2}$$

We shall show that for any boundary condition $\tilde{S}$ and any $\varepsilon > 0$ the event $A^a_{\lfloor \varepsilon n \rfloor, n}$ has a probability decaying to zero under the finite-volume specification $\gamma^\omega_{\Lambda_{2n}}(\,\cdot\,|\tilde{S})$ in the limit as $n \to \infty$. The key ingredient is the well-known entropy inequality

$$\gamma^\omega_{\Lambda_{2n}}(A^a_{\lfloor \varepsilon n \rfloor, n}|\tilde{S}) \leq \frac{\log 2 + \mathcal{H}^\omega_{2n}}{\log\big(1/P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n}|\tilde{S})\big)}, \tag{6.3}$$

where $P_{2n}(\,\cdot\,|\tilde{S})$ is the SRW-bridge probability measure between 0 and $\tilde{S}_{2n}$, and

$$\mathcal{H}^\omega_{2n} = \mathcal{H}\big(\gamma^\omega_{\Lambda_{2n}}(\,\cdot\,|\tilde{S})\big|P_{2n}(\,\cdot\,|\tilde{S})\big) \tag{6.4}$$

denotes the relative entropy of the probability measure $\gamma^\omega_{\Lambda_{2n}}(\,\cdot\,|\tilde{S})$ w.r.t. $P_{2n}(\,\cdot\,|\tilde{S})$.

We first note that the specific relative entropy $\mathcal{H}^\omega_{2n}/2n$ vanishes in the thermodynamic limit:

$$\lim_{n\to\infty} \frac{\mathcal{H}^\omega_{2n}}{2n} = -\phi(\lambda,h) + \lambda \frac{\partial \phi}{\partial \lambda}(\lambda,h) = 0 \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{6.5}$$

Indeed, by (1.1) and (2.2)

$$\begin{aligned}
\mathcal{H}^\omega_{2n} &= \sum_{S_{\Lambda_{2n}}} \gamma^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}}|\tilde{S}) \log \frac{\gamma^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}}|\tilde{S})}{P_{2n}(S_{\Lambda_{2n}}|\tilde{S})} \\
&= -\log Z^\omega_{\Lambda_{2n}}(\tilde{S}) + \frac{1}{Z^\omega_{\Lambda_{2n}}(\tilde{S})} \sum_{S_{\Lambda_{2n}}} H^{\omega,\lambda,h}_{\Lambda_{2n}}(S_{\Lambda_{2n}} \vee \tilde{S}_{\Lambda^c_{2n}}) e^{H^{\omega,\lambda,h}_{\Lambda_{2n}}(S_{\Lambda_{2n}} \vee \tilde{S}_{\Lambda^c_{2n}})} P_{2n}(S_{\Lambda_{2n}}|\tilde{S}) \\
&= -\log Z^\omega_{\Lambda_{2n}}(\tilde{S}) + \lambda \frac{\partial}{\partial \lambda} \log Z^\omega_{\Lambda_{2n}}(\tilde{S}).
\end{aligned} \tag{6.6}$$

Hence, the first equality in (6.5) follows after letting $n \to \infty$ and interchanging the limit with $\partial/\partial\lambda$ (which is allowed because of the convexity and regularity of ϕ in $int(\mathcal{D})$), while the second equality in (6.5) holds because $\phi(\lambda, h) = \lambda h$ on $\mathcal{D}$. Thus, after we show that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) < 0 \qquad \text{for all } \varepsilon > 0, \tag{6.7}$$

it will follow from (6.3) and (6.5) that $\lim_{n\to\infty} \gamma^{\omega}_{\Lambda_{2n}}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) = 0$. Conditioning then implies the same for any (regular) Gibbs measure ν_ω, because (6.5) and (6.6) carry over.

Pick ν_ω and define τ_1 (τ_2) to be the leftmost (rightmost) site i with $0 \le i \le 2n$ such that $S_i = a$. If τ_1 or $\tau_2 = \infty$, then (6.1) is trivially satisfied, so we can suppose without loss of generality that $\tau_1, \tau_2 < \infty$. Then

$$P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) = \sum_{0 \le l_1 \le l_2 \le n} P_{2n}(\tau_1 = 2l_1,\, \tau_2 = 2l_2 | \tilde{S}) P_{2(l_2 - l_1)}(A^0_{\lfloor \varepsilon n \rfloor, l_2 - l_1} | 0), \tag{6.8}$$

where the last factor can be further estimated by the corresponding number for the free SRW, namely,

$$P_{2(l_2-l_1)}(A^0_{\lfloor \varepsilon n \rfloor, l_2 - l_1} | 0) \le \frac{P\big(A^0_{\lfloor \varepsilon n \rfloor, l_2 - l_1} \cap \{S_{2(l_2-l_1)} = 0\}\big)}{P(S_{2(l_2-l_1)} = 0)} \le \mathcal{O}(\sqrt{n}) P(A^0_{\lfloor \varepsilon n \rfloor, n}), \tag{6.9}$$

where we used that $l_2 - l_1 \le n$ and $P(S_{2n} = 0) \sim C/\sqrt{n}$. Thus

$$P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) \le \mathcal{O}(\sqrt{n}) P(A^0_{\lfloor \varepsilon n \rfloor, n}), \tag{6.10}$$

so we need only consider the case $a = 0$.

Next, similarily as in the proof of Theorem 3, let us define the interarrival time ξ_i as the duration between the i-th and the $(i+1)$-st intersection with the interface. Then we may write

$$A^0_{\lfloor \varepsilon n \rfloor, n} = \Big\{ \sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n \Big\}. \tag{6.11}$$

Now, under P the ξ_i are i.i.d. with distribution function satisfying

$$\sum_{l=1} P(\xi_1 = l) z^l = 1 - \sqrt{1 - z^2} \qquad \text{for all } 0 \le z < 1. \tag{6.12}$$

By the exponential Chebyshev inequality we have

$$P\Big(\sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n\Big) \le z^{-n} (1 - \sqrt{1 - z^2})^{\lfloor \varepsilon n \rfloor} \qquad \text{for all } z \in (0,1). \tag{6.13}$$

The r.h.s. attains its minimum at z such that $z^2 = (1-2\varepsilon'_n)(1-\varepsilon'_n)^{-2}$ with $\varepsilon'_n = \lfloor \varepsilon n \rfloor / n \le \varepsilon$. Consequently, using (6.10) and (6.13) we get the bound

$$\limsup_{n\to\infty} \frac{1}{2n} \log P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) \le (1-\varepsilon)\log(1-\varepsilon) - \frac{1}{2}(1-2\varepsilon)\log(1-2\varepsilon) \tag{6.14}$$

when ε is small enough. The r.h.s. is $\sim -\varepsilon^2$ as $\varepsilon \downarrow 0$. Hence (6.7) holds for all $\varepsilon > 0$ and the proof of (6.1) is complete. $\square$

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COEXISTENCE OF PARTIALLY DISORDERED/ORDERED PHASES IN AN EXTENDED POTTS MODEL

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Abstract. We consider a generalization of the standard Potts model in which there are $q = r + s$ states with an interaction that distinguishes the two subspecies. We develop a graphical representation (of the FK-type) for the system and show that this representation may be incorporated directly into reflection positivity arguments. Using the combinations of these techniques, we establish detailed properties of the phase diagram including the existence of sharp triple points. Whenever relevant, the phases are characterized by the percolation properties of the underlying representation.

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1. INTRODUCTION

In this paper we consider a variant of the Potts model. As usual, the spins σ can take on one of q values in a set Q, but here the set Q splits into disjoint subsets R and S containing, respectively, r and s elements, i.e., $Q = R \cup S$, $q = r + s$. The Hamiltonian is given by the expression:

$$H(\boldsymbol{\sigma}) = -J \sum_{\langle x,y\rangle} \delta_{\sigma_x \sigma_y} - \kappa \sum_{\langle x,y\rangle} (\delta^R_{\sigma_x}\delta^R_{\sigma_y} + \delta^S_{\sigma_x}\delta^S_{\sigma_y}) - h \sum_x (\delta^S_{\sigma_x} - \delta^R_{\sigma_x}), \tag{1.1}$$

with $\kappa, J > 0$, $h \in \mathbb{R}$, and the symbol $\langle x, y\rangle$ denoting a nearest-neighbor pair on $\mathbb{Z}^d$. Here $\delta_{\sigma\sigma'} = 1$ if $\sigma = \sigma'$ and zero otherwise, δ^R_σ is the indicator of the event $\sigma \in R$ (and similarly for δ^S_σ), implying that $\delta^R_\sigma \delta^R_{\sigma'} + \delta^S_\sigma \delta^S_{\sigma'}$ vanishes unless σ and σ' belong to the same family. Notice that the second term causes a *repulsion* for the neighboring pairs $\langle x, y\rangle$ with $\sigma_x \in S$ and $\sigma_y \in R$. The magnetic field h acts on the entire set R or S; hence as $h \to \infty(-\infty)$ we recover the usual s (r) state Potts model. Throughout, it will be assumed that $r \gg s \gg 1$.

It is plausible to expect that the Potts part of the interaction will govern the symmetry breaking *within* the families R and S. Thence, for β large and h increasing, the system will undergo a first-order transition between a regime dominated by the R set with r ordered states and a regime dominated by the S set with s ordered states. On the other hand, for h large and positive, the system will undergo an order-disorder transition reminiscent of the s-state Potts model as β varies, and similarly an r-state Potts transition for h large negative. Thus, as h increases, a turnover from R-dominated disordered state to S-dominated disordered state can be expected at higher temperatures, with a complete symmetry maintained within each group (R or S) in both states. This latter change may or may not be signaled by a phase transition, however, certainly not for $\beta \ll 1$. For r and s large, we prove that such a transition indeed occurs at intermediate temperatures.

In the symmetric case, i.e., for $r = s$, both of these '$R \to S$' transitions occur at $h = 0$, which pretty much completes the picture. An interesting feature of the asymmetric case is that it makes conceivable, for $r > s$, a direct transition from R-disordered to S-ordered phase. Indeed, for $r \gg s$, we establish this fact along with the existence of two (sharp) triple points suggested by the presence of such a phase boundary; i.e., essentially the entire phase diagram depicted in Figure 1 is established. As the insert shows, in the symmetric case, these triple points degenerate to a single 'quadruple' point with 4 coexisting phases representing altogether $q + 2$ different equilibrium states. In fact, a generalization of the symmetric case has already been discussed in [LMR], for Q decomposing into q_1 different families with each containing q_2 elements (i.e., $q = q_1 q_2$), but that only for $h = 0$. Different aspects (e.g., quasilocality) of the latter—so called 'fuzzy Potts'—model have been addressed in [MVdV].

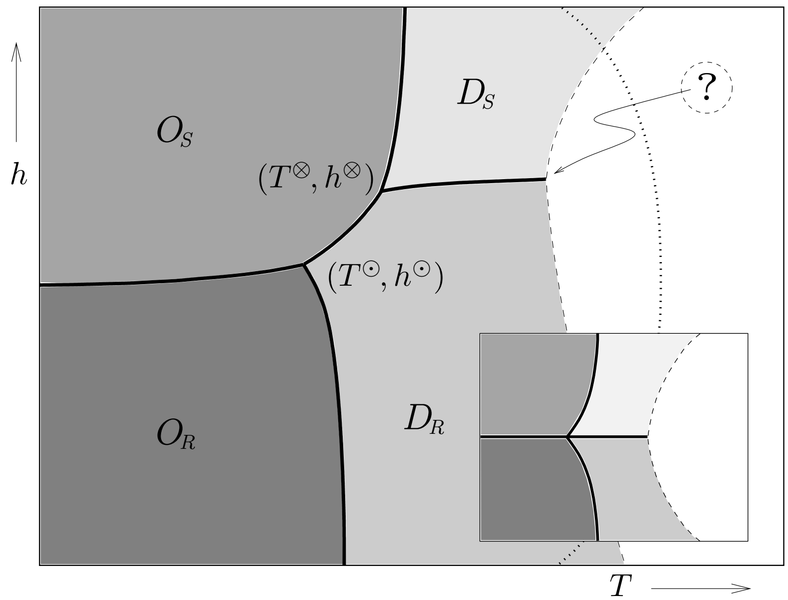

FIG. 1. Phase diagram for the model (1.1). Here $J, \kappa > 0$ are fixed, $r \gg s \gg 1$, and T is the temperature. The dashed line indicates the percolation thresholds of the 'weak' bonds and the dotted line marks the area where Gibbs uniqueness follows from the high-temperature expansion. Insert shows the symmetric case $r = s \gg 1$. The question mark prompts that the suggested behavior at the end-point of the transition line is hypothetical.

In order to facilitate our analysis, we have developed a graphical representation that is a natural generalization of the random-cluster model for the Potts system. Thus there are 'strong' and 'weak' bonds of both the R and S type. For example, a weak R-bond insists that both end-points are in the R-set while a strong R-bond induces a matching pair of spin states of R-type. This graphical representation enjoys an FKG monotonicity that is useful for various portions of our analysis. In particular, all of the phases can be described in terms of the percolation of the corresponding bonds. However, one can only go so far with graphical representations; estimates for the probabilities of contours are also needed. In this regard, reflection positivity (RP) methods combined with the chessboard estimate provide a rather effective service.

Somewhat to our surprise, we find that the graphical representations can be incorporated directly into the RP machinery. (*A priori*, one might imagine having to use RP on the spin system and processing this into a statement about the graphical representation.) This appears to be a promising technique—especially useful in higher dimensions—that is apparently generalizable. Thus, all in all, a rather seamless derivation is permitted. Unfortunately, there are certain limitations to this combination of RP and graphical representations. In particular, we can prove RP only for values of the parameters in the graphical representation that correspond to genuine spin-systems, i.e., integer values of r and s. In fact, this may represent a genuine 'limitation': It was recently shown by one of us [B] that the usual random-cluster model with non-integer q is *not* RP in the desired sense. Here is our main result:

Main Theorem. *Consider the system as described by the Hamiltonian (1.1) with $\kappa, J > 0$. Then for r, s and r/s large enough, there is a $\bar\beta < \infty$ and an ϵ small such that*

(I) there are four closed and connected regions O_R, O_S, D_R, and D_S covering the set $\mathcal{J} = \{(\beta, h);\, \beta \ge \bar\beta,\, h \in [-\infty, \infty]\}$, where the following translation-invariant states exist

- *(i) r 'ordered' states $\langle\,\cdot\,\rangle^j_{O_R}$, $j = 1, \dots, r$, in O_R,*
- *(ii) s 'ordered' states $\langle\,\cdot\,\rangle^k_{O_S}$, $k = r+1, \dots, r+s$, in O_S,*
- *(iii) a single 'disordered' state $\langle\,\cdot\,\rangle_{D_S}$ in D_S,*
- *(iv) a single 'disordered' state $\langle\,\cdot\,\rangle_{D_R}$ in D_R.*

These states are characterized by the relations

$$\langle \delta_{\sigma_x j}\delta_{\sigma_x\sigma_y}\rangle^j_{O_R} \ge 1-\epsilon, \qquad \langle \delta^S_{\sigma_x}\delta^S_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle_{D_S} \ge 1-\epsilon,$$
$$\langle \delta_{\sigma_x k}\delta_{\sigma_x\sigma_y}\rangle^k_{O_S} \ge 1-\epsilon, \qquad \langle \delta^R_{\sigma_x}\delta^R_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle_{D_R} \ge 1-\epsilon,$$

valid for any pair x, y of neighboring sites, any $1 \le j \le r$ and any $r+1 \le k \le r+s$.

(II) The intersections $(O_R \cup O_S) \cap (D_R \cup D_S) = \mathcal{C}_{OD}$ and $(O_R \cup D_R) \cap (O_S \cup D_R) = \mathcal{C}_{RS}$ constitute two continuous non-selfintersecting curves $\mathcal{C}_{OD}$ $\mathcal{C}_{RS}$, where the order/disorder and the R/S states coexist, respectively. The curve $\mathcal{C}_{OD}$ admits a parametrization by βh, whereas $\mathcal{C}_{RS}$ is parametrizable by β. On the complement of these curves, the inequalities

$$\bigl\langle \delta_{\sigma_x\sigma_y}\delta^R_{\sigma_y}\bigr\rangle \ge 1-\epsilon, \qquad \bigl\langle \delta^S_{\sigma_x}\delta^S_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\bigr\rangle \ge 1-\epsilon,$$
$$\bigl\langle \delta_{\sigma_x\sigma_y}\delta^S_{\sigma_y}\bigr\rangle \ge 1-\epsilon, \qquad \bigl\langle \delta^R_{\sigma_x}\delta^R_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\bigr\rangle \ge 1-\epsilon,$$

hold for every translation-invariant Gibbs state in the respective region and any pair x and y of nearest-neighbor sites.

(III) There are exactly two triple points, where $O_S+O_R+D_R$ and $O_S+D_S+D_R$ meet, respectively. These triple points are connected by a single line of coexistence between s-order and r-disorder.

(IV) In the high-temperature region (i.e., for $\beta\kappa, \beta J \ll 1$) the conditions of complete analyticity are met. In particular, the set of all Gibbs measures is a singleton in this region.

(V) The phases O_S and O_R can alternatively be characterized by spontaneous magnetization and/or percolation. In D_S and D_R, there is percolation of the corresponding disordered bonds.

The organization of the remainder of this paper is as follows. In Section 2 we introduce the graphical representation, establish some of its useful properties, and relate the percolation characterization of the respective phases to the non-vanishing of an order parameter. Section 3 is devoted to the proof of the Main Theorem. In this section, Lemma III.3 is of special interest for the asymetric case, because it directly rules out quadruple coexistence (whenever $r \gg s$). For reader's convenience, the computations based on chessboard estimates are performed here only in the two-dimensional case. We relegate the full (and somewhat clumsy) arguments to the Appendix.

2. THE GRAPHICAL REPRESENTATION

In this section we develop a graphical representation of the model and establish its various useful properties. To simplify our derivations, we will restrict ourselves to free (or periodic) boundary conditions in these preliminary discussions and defer the detailed analysis of boundary-condition issues to a later subsection.

Let us start with the identity

$$e^{\beta J\delta_{\sigma\sigma'}+\beta\kappa(\delta^S_\sigma\delta^S_{\sigma'}+\delta^R_\sigma\delta^R_{\sigma'})}=1+(e^{\beta\kappa}-1)(\delta^S_\sigma\delta^S_{\sigma'}+\delta^R_\sigma\delta^R_{\sigma'})+e^{\beta\kappa}(e^{\beta J}-1)(\delta^S_{\sigma\sigma'}+\delta^R_{\sigma\sigma'}), \quad (2.1)$$

where $\delta^R_{\sigma\sigma'}$ ($\delta^S_{\sigma\sigma'}$) indicates that both spins coincide and belong to R (S). The five terms on the r.h.s. give rise to five different species of bonds—vacant, s-disorded, r-disorded, s-ordered, and r-ordered, with the prefactors representing the *a priori* weights of the corresponding bonds. For notational simplicity, we will introduce a paralel notation in terms of colors: the five types of bonds above are called vacant, light-blue, light-red, dark-blue, and dark-red, in the order of their appearance.

We may label the five species by $\alpha \in I = \{v, s^{\mathrm{d}}, r^{\mathrm{d}}, s^{\mathrm{o}}, r^{\mathrm{o}}\}$ and define $w(\alpha)$ to be the corresponding coefficient in the identity (2.1). Thus, $w(v) = 1$, $w(r^{\mathrm{d}}) = w(s^{\mathrm{d}}) = e^{\beta\kappa} - 1$, and $w(r^{\mathrm{o}}) = w(s^{\mathrm{o}}) = e^{\beta\kappa}(e^{\beta J} - 1)$. Let Λ denote a graph with sites $\mathbb{S}_\Lambda$ and bonds $\mathbb{B}_\Lambda$. Let $\Omega_\Lambda = I^{\mathbb{B}_\Lambda}$ denote the set of bond configurations in Λ. As will become clear, not all Ω_Λ will be used; configurations in which blue and red bonds share an endpoint need not be considered. With the above notations, $e^{-\beta H_\Lambda}$ may be written as

$$e^{-\beta H_\Lambda(\boldsymbol{\sigma})} = \sum_{\boldsymbol{\omega}\in\Omega_\Lambda} \boldsymbol{D}(\boldsymbol{\omega}) \prod_{b\in\mathbb{B}_\Lambda} w(\omega_b)\chi_b(\boldsymbol{\omega},\boldsymbol{\sigma}) \prod_{x\in\mathbb{S}_\Lambda} e^{\beta h(\delta^S_{\sigma_x}-\delta^R_{\sigma_x})}, \quad (2.2)$$

where $\chi_b(\boldsymbol{\omega},\boldsymbol{\sigma})$ is one of the functions 1, $\delta^S_{\sigma_x}\delta^S_{\sigma_y}$, $\delta^R_{\sigma_x}\delta^R_{\sigma_y}$, $\delta^S_{\sigma_x\sigma_y}$, $\delta^R_{\sigma_x\sigma_y}$, according to the label ω_b of the bond $b = \langle x, y\rangle$, and $\boldsymbol{D}(\boldsymbol{\omega})$ is the indicator that $\boldsymbol{\omega}$ fulfils the aforementioned restriction.

For each such $\boldsymbol{\omega}$, we must now perform the trace (sum over spin configurations) to arrive at the weights, $W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\omega})$, for the graphical representation. The result is not particularly difficult; but only after the introduction of some additional notation. For each $\boldsymbol{\omega}$, let $\Lambda_R(\boldsymbol{\omega})$ denote the red portion of the graph: those bonds that are R-ordered or R-disordered, and all sites that are endpoints of such bonds. Let $N_R(\boldsymbol{\omega})$ denote the number of R-sites in $\Lambda_R(\boldsymbol{\omega})$. Some of the bonds in $\Lambda_R(\boldsymbol{\omega})$ are R-ordered bonds. These divide $\Lambda_R(\boldsymbol{\omega})$ into 'R-connected' components; let $C_R(\boldsymbol{\omega})$ denote the number of such components. Similar notation applies to the blue portion of the configuration. Finally, let $N_\emptyset(\boldsymbol{\omega}) \equiv |\mathbb{S}_\Lambda| - [N_R(\boldsymbol{\omega}) + N_S(\boldsymbol{\omega})]$ denote the number of sites that do not fall into either category.

It is not hard to see that

$$W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\omega}) = \Big[\prod_{b\in\mathbb{B}_\Lambda} w(\omega_b)\Big](se^{\beta h} + re^{-\beta h})^{N_\emptyset(\boldsymbol{\omega})} e^{\beta h N_S(\boldsymbol{\omega})} e^{-\beta h N_R(\boldsymbol{\omega})} s^{C_S(\boldsymbol{\omega})} r^{C_R(\boldsymbol{\omega})}. \quad (2.3)$$

Indeed, each unmatched site just contributes the factor $se^{\beta h}+re^{-\beta h}$, each S-site a factor $e^{\beta h}$, and similarly for the R-sites. Finally, the ordered bonds dictate which fraction of the s^{N_S} (or r^{N_R}) spin states are actually allowed on each connected component, resulting thus in the factors $s^{C_S(\boldsymbol{\omega})}$ (and $r^{C_R(\boldsymbol{\omega})}$).

FKG monotonicity.

From the perspective of 'S above R' (or blue above red), there is a natural ordering for the bond variables $s^{\mathrm{o}} \succ s^{\mathrm{d}} \succ v \succ r^{\mathrm{d}} \succ r^{\mathrm{o}}$, which induces a partial ordering on the configurations. We show that the graphical representation is monotone with respect to this ordering.

Proposition II.1. *Let $\nu(\,\cdot\,) = \nu_{s,r;\Lambda}^{(\beta,h)}(\,\cdot\,)$ denote the random-cluster measure on a finite graph Λ, defined according to the weights (2.3). Then ν is strong FKG w.r.t. the ordering $s^{\mathrm{o}} \succ s^{\mathrm{d}} \succ v \succ r^{\mathrm{d}} \succ r^{\mathrm{o}}$.*

Proof. We must verify the FKG lattice condition. To facilitate matters, let us exchange 'components for loops'. Indeed, if $B_R(\boldsymbol{\omega})$ is the number of dark-red bonds and $\ell_R(\boldsymbol{\omega})$ is the number of independent loops formed by them, we may write

$$C_R(\boldsymbol{\omega}) = \ell_R(\boldsymbol{\omega}) - B_R(\boldsymbol{\omega}) + N_R(\boldsymbol{\omega}). \quad (2.4)$$

Thus, the weights may be written as

$$W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\omega}) = \Big[\prod_{b\in\mathbb{B}_\Lambda} w'(\omega_b)\Big]\,(se^{\beta h} + re^{-\beta h})^{|\mathbb{S}_\Lambda|}$$
$$\times \Big[\frac{se^{\beta h}}{se^{\beta h} + re^{-\beta h}}\Big]^{N_S(\boldsymbol{\omega})} \Big[\frac{re^{-\beta h}}{se^{\beta h} + re^{-\beta h}}\Big]^{N_R(\boldsymbol{\omega})} s^{\ell_S(\boldsymbol{\omega})} r^{\ell_R(\boldsymbol{\omega})}. \quad (2.5)$$

Here $w'(\omega_b)$ are changed *a priori* factors whose value will not play any role in the following.

As is well known, the verification of the FKG lattice condition may be done inductively by comparing configurations that disagree on at most two places. Thus, let $b_1, b_2 \in \mathbb{B}_\Lambda$, let $\boldsymbol{\theta}$ denote a bond configuration in $\mathbb{B}_\Lambda \setminus \{b_1, b_2\}$ and let η_1, η_2, ζ_1, and ζ_2 denote bond variables on b_1 and b_2 with $\eta_1 \succ \zeta_1$ and $\eta_2 \succ \zeta_2$. We use $(\boldsymbol{\theta}, \eta_1, \eta_2)$ to denote the configuration equal to η_1 (η_2) on b_1 (b_2) and $\boldsymbol{\theta}$ elsewhere on $\mathbb{B}_\Lambda$, and similarly for $(\boldsymbol{\theta}, \zeta_1, \zeta_2)$ etc. We must show

$$W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\theta}, \eta_1, \eta_2) W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\theta}, \zeta_1, \zeta_2) \geq W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\theta}, \zeta_1, \eta_2) W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\theta}, \eta_1, \zeta_2) \tag{2.6}$$

It is clear that the *a priori* factors (i.e., the bond weights w') cancel exactly. Further, it is observed that $\boldsymbol{D}(\boldsymbol{\omega}_1), \boldsymbol{D}(\boldsymbol{\omega}_2) = 1$ imply $\boldsymbol{D}(\boldsymbol{\omega}_1 \vee \boldsymbol{\omega}_2), \boldsymbol{D}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = 1$, where $\boldsymbol{\omega}_1 \vee \boldsymbol{\omega}_2$ denotes the maximum and $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2$ the minimum of the two configurations. Thus, we may omit any discussing of constraints.

We now claim that $N_S(\boldsymbol{\theta}, \eta_1, \eta_2) + N_S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \leq N_S(\boldsymbol{\theta}, \zeta_1, \eta_2) + N_S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Let $S(\boldsymbol{\omega}) \subset \mathbb{S}_\Lambda$ denote the set of sites that touch at least one blue bond (i.e., the site set of $\Lambda_S(\boldsymbol{\omega})$). It is not hard to see that $S(\boldsymbol{\theta}, \eta_1, \eta_2) = S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cup S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Indeed, $S(\boldsymbol{\theta}, \eta_1, \eta_2) \supset S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ (because the former contains more S-bonds), and similarly with $S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. So $S(\boldsymbol{\theta}, \eta_1, \eta_2)$ contains the union of both. Now suppose that $x \in S(\boldsymbol{\theta}, \eta_1, \eta_2)$. If this is caused by an S-bond in $\boldsymbol{\theta}$, then $x \in S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ *and* $x \in S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. If not, then x is either the endpoint of one (or both) of η_1 or η_2—say η_1, in which case x belongs to $S(\boldsymbol{\theta}, \eta_1, \zeta_2)$—or the end-point of ξ_1 or ξ_2, and then it also pertains to both sets. On the other hand, we only claim that $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cap S(\boldsymbol{\theta}, \eta_1, \zeta_2)$ (with the inclusion sometimes being strict). This is clear since $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ and $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Hence we have

$$\begin{aligned} N_S(\boldsymbol{\theta}, \eta_1, \eta_2) + N_S(\boldsymbol{\theta}, \zeta_1, \zeta_2) &= \bigl|S(\boldsymbol{\theta}, \eta_1, \eta_2)\bigr| + \bigl|S(\boldsymbol{\theta}, \zeta_1, \zeta_2)\bigr| \\ &\leq \bigl|S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cup S(\boldsymbol{\theta}, \eta_1, \zeta_2)\bigr| + \bigl|S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cap S(\boldsymbol{\theta}, \eta_1, \zeta_2)\bigr| \\ &= \bigl|S(\boldsymbol{\theta}, \zeta_1, \eta_2)\bigr| + \bigl|S(\boldsymbol{\theta}, \eta_1, \zeta_2)\bigr| = N_S(\boldsymbol{\theta}, \zeta_1, \eta_2) + N_S(\boldsymbol{\theta}, \eta_1, \zeta_2), \end{aligned} \tag{2.7}$$

with the third line by inclusion-exclusion.

Thus we are down to showing the necessary inequalities among the loop counting functions. However, most of these are actually equalities; the only exception is when η_1, η_2, ζ_1, and ζ_2 are all bonds of the same family in which case the desired inequality reduces to the usual argument for the random-cluster model. $\square$

As an immediate consequence we obtain a domination comparison:

Corollary. *If* $h^{(1)} > h^{(2)}$, *then*

$$\nu_{s,r;\Lambda}^{(\beta,h^{(1)})}(\,\cdot\,) \underset{\text{FKG}}{\geq} \nu_{s,r;\Lambda}^{(\beta,h^{(2)})}(\,\cdot\,).$$

Proof. It suffices to show that $W_{s,r;\Lambda}^{(\beta,h^{(1)})}(\boldsymbol{\omega})/W_{s,r;\Lambda}^{(\beta,h^{(2)})}(\boldsymbol{\omega})$ is an increasing function (assuming that both quantities are nonzero). Using formula (2.5) and defining

$$e^{-\alpha_S(h)} \equiv \frac{se^{\beta h}}{se^{\beta h} + re^{-\beta h}} \quad \text{and} \quad e^{-\alpha_R(h)} \equiv \frac{re^{-\beta h}}{se^{\beta h} + re^{-\beta h}}, \tag{2.8}$$

we have

$$\frac{W_{s,r;\Lambda}^{(\beta,h^{(1)})}(\boldsymbol{\omega})}{W_{s,r;\Lambda}^{(\beta,h^{(2)})}(\boldsymbol{\omega})} = \varPhi_\Lambda\big(h^{(1)},h^{(2)}\big)\left[\frac{e^{-\alpha_R(h^{(1)})}}{e^{-\alpha_R(h^{(2)})}}\right]^{N_R(\boldsymbol{\omega})}\left[\frac{e^{-\alpha_S(h^{(1)})}}{e^{-\alpha_S(h^{(2)})}}\right]^{N_S(\boldsymbol{\omega})}, \tag{2.9}$$

where $\varPhi_\Lambda\big(h^{(1)},h^{(2)}\big)$ is a number independent of $\boldsymbol{\omega}$ (note that the modified weights w' are independent of the external field). It is thus sufficient to establish that $e^{-\alpha_R(h)}$ is monotone decreasing and $e^{-\alpha_S(h)}$ is monotone increasing as functions of h. This is easily checked. □

Remark. It is noted that the measures are perfectly well defined for non-integer r and s and that the above monotonicities hold for all r and s with $r,s \geq 1$. However, in this paper, we will need to make explicit use of the underlying spin-system; hence, we will not discuss these more general cases. Finally, we remark that all of the results of this subsection hold for an arbitrary (finite) graph with arbitrary fields h_i and arbitrary (non-negative) couplings $J_{x,y}$ and $\kappa_{x,y}$. In particular, FKG dominations are established under the condition that $h_x^{(1)} \geq h_x^{(2)}$ for all x.

Reflection positivity.

We begin with some preliminary notations. Let $\mathcal{T}_L$ denote a d-dimensional (lattice) torus, assumed for convenience to have an even number L of sites in all directions, and let P denote the intersection of $\mathcal{T}_L$ with a hyperplane orthogonal to one of the coordinate directions. (Thus P consists of two disconnected 'planes of sites'.) The set P divides $\mathcal{T}_L$ into left ($\mathcal{T}_L^{\mathcal{L}}$) and right ($\mathcal{T}_L^{\mathcal{R}}$) halves—*both* halves defined to include P.

Let $\boldsymbol{\omega} \in \Omega_{\Lambda_L}$ denote bond configurations (such as the ones described above) on $\mathcal{T}_L$ and let $\mathcal{F}_{\mathcal{L}}$ ($\mathcal{F}_{\mathcal{R}}$) be the set of functions of bond configurations restricted to $\mathcal{T}_L^{\mathcal{L}}$ ($\mathcal{T}_L^{\mathcal{R}}$). Finally, let ϑ_P be the reflection operator that maps bond configuration on the left to configurations on the right and vice versa (i.e., $\boldsymbol{\omega}_{\mathcal{L}} \leftrightarrow \vartheta_P\boldsymbol{\omega}_{\mathcal{L}}$). For $f \in \mathcal{F}_{\mathcal{R}}$, define $\vartheta_P f \in \mathcal{F}_{\mathcal{L}}$ by $[\vartheta_P f](\boldsymbol{\omega}_{\mathcal{R}}) = f(\vartheta_P\boldsymbol{\omega}_{\mathcal{L}})$. Let $\mathbb{P}(\,\cdot\,)$ denote a probability measure on bond configurations and $\mathbb{E}$ the expectation with respect to this measure. Then $\mathbb{P}$ is said to be *reflection positive* if for every $g, f \in \mathcal{F}_{\mathcal{R}}$ one has

$$\mathbb{E}(f\,\vartheta_P g) = \mathbb{E}(g\,\vartheta_P f) \tag{2.10}$$

and

$$\mathbb{E}(f\,\vartheta_P f) \geq 0. \tag{2.11}$$

Proposition II.2. *Let $\mathcal{T}$ denote the d-dimensional torus and let $\nu(\,\cdot\,) \equiv \nu_{s,r;\mathcal{T}}^{(\beta,h)}(\,\cdot\,)$ denote the random-cluster measures as defined by the weights (2.3). Then ν is reflection positive with respect to reflections through all planes P containing sites.*

Proof. The proof follows from the reflection positivity of a certain joint measure on bond-site configurations—the Edwards-Sokal measure—that contains the random-cluster measure and the Gibbs measure of the considered spin system as marginals. This measure is essentially defined by the r.h.s. of (2.2). Namely, let us write

$$\begin{aligned} W_{\mathrm{ES},s,r;\mathcal{T}}^{(\beta,h)}(\boldsymbol{\omega},\boldsymbol{\sigma}) &= \Bigl[\boldsymbol{D}(\boldsymbol{\omega}) \prod_{b\in\mathbb{B}_{\mathcal{T}}} \chi_b(\boldsymbol{\omega},\boldsymbol{\sigma})\Bigr] \prod_{b\in\mathbb{B}_{\mathcal{T}}} w(\omega_b) \prod_{x\in\mathbb{S}_{\mathcal{T}}} e^{\beta h(\delta_{\sigma_x}^S - \delta_{\sigma_x}^R)} \\ &\equiv \boldsymbol{D}(\boldsymbol{\omega},\boldsymbol{\sigma}) \prod_{b\in\mathbb{B}_{\mathcal{T}}} w(\omega_b) \prod_{x\in\mathbb{S}_{\mathcal{T}}} e^{\beta h(\delta_{\sigma_x}^S - \delta_{\sigma_x}^R)}. \end{aligned} \tag{2.12}$$

The weights $W_{\mathrm{ES},s,r;\mathcal{T}}^{\beta,h}$ define the Edwards-Sokal measure $\nu_{\mathrm{ES},s,r;\mathcal{T}}^{(\beta,h)}$ abbreviated hereafter as ν_{ES}. In the above language, equations (2.2) and (2.3) read

$$e^{-\beta H_{\mathcal{T}}(\boldsymbol{\sigma})} = \sum_{\boldsymbol{\omega}} W_{\mathrm{ES},s,r;\mathcal{T}}^{(\beta,h)}(\boldsymbol{\omega},\boldsymbol{\sigma}), \tag{2.13a}$$

and

$$W_{\mathrm{ES},s,r;\mathcal{T}}^{(\beta,h)}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\sigma}} W_{\mathrm{ES},s,r;\mathcal{T}}^{(\beta,h)}(\boldsymbol{\omega},\boldsymbol{\sigma}), \tag{2.13b}$$

which verifies the preceding claim concerning the marginals. Our proof amounts to showing that the full ES-measure is reflection positive; here, of course, with respect to reflection operators acting on the larger space of bond-spin configurations and functions thereof. (Notwithstanding, we will make no notational distinctions.) Thus, let f denote a function that is determined by the bond-spin configurations on the 'right' and consider $\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f)$, where $\mathbb{E}_{\mathrm{ES}}(\,\cdot\,)$ denotes expectation with respect to the measure ν_{ES}. Let $(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ denote a bond-spin configuration in the plane P; that is to say a spin value on each $x \in P$ and a bond value on each $\langle x,y\rangle$ with $x,y \in P$. We may write

$$\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f) = \sum_{(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)} \nu_{\mathrm{ES}}(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\, \mathbb{E}_{\mathrm{ES}}(f\vartheta_P f\,|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P). \tag{2.14}$$

The conclusion now follows from the observation that the restrictions of the conditional measures $\nu_{\mathrm{ES}}(\,\cdot\,|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ to the left and right half of the torus are independent and identical under reflections. Indeed, the 'identical under reflections' property is an obvious consequence of the underlying symmetry of the model. Independence is established as follows:

Let us write $(\boldsymbol{\omega},\boldsymbol{\sigma})=(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P;\boldsymbol{\omega}_{\mathcal{R}},\boldsymbol{\sigma}_{\mathcal{R}};\boldsymbol{\omega}_{\mathcal{L}},\boldsymbol{\sigma}_{\mathcal{L}})$ corresponding to the configurations in P, $\mathcal{R}$, and $\mathcal{L}$, respectively. Let $\boldsymbol{D}_P(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ denote the analogue of the function $\boldsymbol{D}(\boldsymbol{\omega},\boldsymbol{\sigma})$, defined in (2.14), that here checks the consistency of the configuration only in P. Further, let $\boldsymbol{D}_{\mathcal{R}}\bigl(\boldsymbol{\omega}_{\mathcal{R}},\boldsymbol{\sigma}_{\mathcal{R}}|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P\bigr)$ be the function that indicates consistency only for the right half of the configuration given $(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ and similarly for $\boldsymbol{D}_{\mathcal{L}}\bigl(\boldsymbol{\omega}_{\mathcal{L}},\boldsymbol{\sigma}_{\mathcal{L}}|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P\bigr)$. It is not hard to check that

$$\boldsymbol{D}\bigl(\boldsymbol{\omega},\boldsymbol{\sigma}\bigr)=\boldsymbol{D}_P\bigl(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P\bigr)\boldsymbol{D}_{\mathcal{R}}\bigl(\boldsymbol{\omega}_{\mathcal{R}},\boldsymbol{\sigma}_{\mathcal{R}}|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P\bigr)\boldsymbol{D}_{\mathcal{L}}\bigl(\boldsymbol{\omega}_{\mathcal{L}},\boldsymbol{\sigma}_{\mathcal{L}}|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P\bigr) \tag{2.15}$$

for every $(\boldsymbol{\omega},\boldsymbol{\sigma})$. By checking the terms in (2.6), we easily see that the weights factorize, which is equivalent to independence. Thus we have

$$\mathbb{E}_{\mathrm{ES}}(f\,\vartheta_P f)=\sum_{(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)}\nu_{\mathrm{ES}}(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\bigl[\mathbb{E}_{\mathrm{ES}}(f\,|\,\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\bigr]^2 \tag{2.16}$$

which is manifestly non-negative. Similarly one establishes

$$\mathbb{E}_{\mathrm{ES}}(f\,\vartheta_P g)=\mathbb{E}_{\mathrm{ES}}(g\,\vartheta_P f) \tag{2.17}$$

and the proof is complete. $\square$

The use of RP for establishing the existence of discontinuous transitions is based on two standard Lemmas [FL,KS]. Here 'behavioral pattern' refers to a particular set of configurations (typically on a box). Let $\mathcal{T}_L$ denote a torus of size L and let $\langle\,\cdot\,\rangle_L\equiv\langle\,\cdot\,\rangle_{s,r;\mathcal{T}_L}^{(\beta,h)}$ be a state on the configurations in $\mathbb{B}_{\mathcal{T}_L}$.

Lemma II.3. *Let $\{c_\ell\}$ be a collection (possibly overlapping) cubes of size one, and consider a behavioral pattern b_ℓ associated with each cube c_ℓ. Let $\chi_{b_\ell}(c_\ell)$ indicate the occurence of b_ℓ on c_ℓ. If $\langle\,\cdot\,\rangle_L$ is reflection positive and L is even, then*

$$\Bigl\langle\prod_\ell\chi_{b_\ell}(c_\ell)\Bigr\rangle_L\le\prod_\ell\bigl(\bigl\langle\chi_{b_\ell}(\mathcal{T}_L)\bigr\rangle_L\bigr)^{\frac{1}{|L^d|}}.$$

where $\chi_{b_\ell}(\mathcal{T}_L)$ enforces the pattern b_ℓ to all even translates of c_ℓ and its mirror image to all odd translates of c_ℓ.

Lemma II.4. *Let a and b denote two distinct patterns on a cube $c \in \mathcal{T}_L$. Let H be a Hamiltonian that depends on the parameter α that varies in the range $[\alpha_a, \alpha_b]$, and let $\langle \cdot \rangle_{L,\alpha}$ denote the Gibbs state on $\mathcal{T}_L$ induced by the Hamiltonian H at the parameter value α. Let $A \in (\frac{1}{2}, 1]$ and $B \in [0, \frac{1}{4}]$ be such that $B \le \left[\frac{1}{2} - \sqrt{\frac{1}{2} - \frac{A}{2}}\right]^2$, and let ϵ_a, ϵ_b $\in (0, \frac{1}{2})$. Suppose that for all $\alpha \in [\alpha_a, \alpha_b]$, all $c, \tilde{c} \in \mathcal{T}_L$, and all L large enough*

(0) $\chi_a(c)\chi_b(c) = 0,$

(i) $\langle \chi_a(c) + \chi_b(c) \rangle_{L,\alpha} \ge A,$

(ii) $\langle \chi_a(c)\chi_b(\tilde{c}) \rangle_{L,\alpha} \le B,$

and

(iiia) $\langle \chi_a(c) \rangle_{L,\alpha_a} > 1 - \epsilon_a$

(iiib) $\langle \chi_b(c) \rangle_{L,\alpha_b} > 1 - \epsilon_b.$

Then there is an $\alpha_{\rm c} \in [\alpha_a, \alpha_b]$ and two distinct translation invariant Gibbs states $\langle \cdot \rangle^a_{\alpha_{\rm c}}$ and $\langle \cdot \rangle^b_{\alpha_{\rm c}}$ such that

$$\langle \chi_a(c) \rangle^a_{\alpha_{\rm c}} \ge 1 - \bar{\epsilon} \qquad \textit{and} \qquad \langle \chi_b(c) \rangle^b_{\alpha_{\rm c}} \ge 1 - \bar{\epsilon},$$

where $\bar{\epsilon} = \bar{\epsilon}(A, B)$ is such that $\bar{\epsilon} \to 0$ as $A \to 1$ and $B \to 0$.

Boundary conditions.

As already stated, graphical representation (2.3) will be our major tool of study of the Gibbs phases associated with the Hamiltonian (1.1). However, we first have to take properly into account the effect of boundary conditions. In particular, we have to clarify to what extent one can generalize the FKG domination arguments from the previous subsections. There are two ways that a random-cluster measure can be associated with a boundary condition. These lead eventually to two classes of random-cluster measures: $\mathcal{S}$-measures and $\mathcal{G}$-measures, with the former defined by prescribing a *spin* boundary condition, whereas the latter is defined by prescribing a *graphical* boundary condition.

Given a finite set $\Lambda \subset \mathbb{Z}^d$, let $\partial\Lambda$ be its boundary, i.e., the set of sites in $\mathbb{Z}^d$ whose distance from Λ equals one. To implement the first possibility, we take a spin configuration $\tilde{\boldsymbol{\sigma}}$ and define the Edwards-Sokal measure on $\boldsymbol{\sigma}$'s in $\Lambda \cup \partial\Lambda$ and $\boldsymbol{\omega}$'s on the bonds thereof, however, with the spins at $\partial\Lambda$ fixed to $\tilde{\boldsymbol{\sigma}}$. Let $\nu_{\Lambda,\tilde{\boldsymbol{\sigma}}}$ denote the $\boldsymbol{\omega}$-marginal of this measure. Of particular interest are the limits of these measures as $\Lambda \nearrow \mathbb{Z}^d$: we denote by $\mathcal{S}$ the set of all possible accumulation points, closed under convex combinations and closed in the weak topology on probability measures. We call these $\mathcal{S}$-measures (they were called equilibrium random-cluster measures in [ACCN]).

Our major object of interest is the original *spin* system (1.1). Even though all properties of the graphical marginals are derived starting from the spin system, not all information is retained by the marginals. This may actually cast doubts whether the representation (i.e., the $\mathcal{S}$-measures) is still capable of capturing the important features of the spin system, e.g., the non-uniqueness of Gibbs states. This is (partially) answered in the following.

Lemma II.5. *The existence of two distinct $\mathcal{S}$-measures μ_1 and μ_2 implies the existence of two distinct Gibbs measures ν_1 and ν_2 for the Hamiltonian (1.1). Moreover, if μ_1 and μ_2 can be obtained for (expanding) sequences $\{\Lambda_n^{(1)}\}$, $\{\Lambda_n^{(2)}\}$, and boundary conditions $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, respectively, then ν_1 and ν_2 can be generated by $(\{\Lambda_n^{(1)}\}, \boldsymbol{\sigma}_1)$ and $(\{\Lambda_n^{(2)}\}, \boldsymbol{\sigma}_2)$, respectively, as (possibly subsequential) limits of finite-volume states.*

Proof. See Appendix.

Let $\mathcal{S}^\star$ be the translation invariant measures in $\mathcal{S}$. Note that any accumulation point μ of the torus states is an $\mathcal{S}^\star$-measure. Namely, if we go to a subsequence for which also the distribution functions for the *spins* at the exterior boundary of any finite volume converge, we see that the expectation of any cylinder function of bonds in the set A can be written as the combination of ($\mathcal{S}$-class) states in any Λ encompassing A, where the spin boundary condition is now the subject of average. Moreover, the expectaion is independent of Λ and thus only the tail of the boundary condition matters, but in that case the average runs over infinite-volume $\mathcal{S}$-class measures, so μ is indeed an $\mathcal{S}$-measure. The translation invariance of μ is trivial.

To implement the other possibility, i.e., to define boundary conditions directly in the graphical representation, one may explicitly consider a 'graphical' boundary configuration $\tilde{\boldsymbol{\omega}}$, modify appropriately the weights in (2.3) for clusters that stick out of the finite volume so that a consistent family of measures is recovered, and then study the DLR measures associated therewith. While the techniques based on the DLR condition are well developed in the spin language (see, e.g., [Ge]), for the random-cluster measures the theory is still rather 'weak in the knees', mainly for the lack of quasilocality (see [vEFS] Section 4.5.3 for a concrete problem of the latter kind, whose solution was given in [Gr] and [PVdV]). Therefore, we refrain from discussing it in its full generality.[10] On the other hand, we would like to have some way of incorporating explicit 'graphical' boundary conditions into our considerations, because only then the full power of FKG-ordering can be employed. To this end, we will consider a restricted class of boundary conditions.

Let $\partial\mathbb{B}(\Lambda)$ to denote the (exterior) bond-boundary of Λ, i.e., the set of bonds with one end in $\partial\Lambda$ and the other in Λ. A boundary condition (i.e., a 'graphical' configuration on $\partial\mathbb{B}(\Lambda)$) for the random-cluster problem is said to be of the $\mathcal{G}$-class[11] if

[10] See [Gr] and [BC] for statements employing the DLR structure of the random-cluster measures. An alternative is to rely on DLR structure of Edwards-Sokal states—this is the approach used in [BBCK].

[11] A related notion in [BC] is the wiring diagram.

$\partial\Lambda$ is divided into disjoint components $C_1^B, \dots, C_k^B; C_1^R, \dots, C_\ell^R$ and $C_1^f, \dots, C_t^f$. Each of the components $C_1^R, \dots, C_\ell^R$ act as a single site but always in a red state and similarly $C_1^B, \dots, C_k^B$ act as single blue sites and, finally, the components $C_1^f, \dots, C_t^f$ act as 'free' sites with some *a priori* weights for red or blue.

Given a boundary condition from $\mathcal{G}$-class, the weight (2.3) is modified to the effect that $C_S(\boldsymbol{\omega})$ and $C_R(\boldsymbol{\omega})$ are now counting the connected components *including* the boundary components. The infinite-volume states generated by a $\mathcal{G}$-boundary condition from thus modified finite-volume measures will be denoted (with a slight abuse of notation) also by $\mathcal{G}$.

It is clear from the definition of the $\mathcal{G}$ measures that they can all be generated by convex combinations of measures with a spin boundary condition. Hence, $\mathcal{G} \subset \mathcal{S}$. On the other hand, whereas *constant* red (blue) boundary condition gives rise to red- (blue-)wired $\mathcal{G}$-states $\nu_{\Lambda,\text{red-w}}^{(\beta,h)}$ ($\nu_{\Lambda,\text{blue-w}}^{(\beta,h)}$) (the 'wiring' refers to connecting all boundary sites by bonds of the respective dark color to an auxiliary site), other spin boundary conditions do not necessarily yield a measure in the $\mathcal{G}$-class. Indeed, if in the spin system one sets half of the boundary spins to one of the red-type spin-states and the other half to another type of red-state, the resulting random-cluster measure is the red-wired measure conditioned on having no dark-red connection between the two halves. The conditioning here is crucial and one cannot specify this measure using only $\mathcal{G}$-boundary conditions.

The fact that $\mathcal{G}$ does not embody all Gibbs measures of interest is an unpleasant problem for techniques based on graphical representations. These difficulties (which have been encountered in other systems, c.f. [C]) can, to some extent, be circumvented but only after a certain amount of work. For our purposes, the key practical implication here is the lack of FKG (or at least of its proof) outside the $\mathcal{G}$-class. (In the $\mathcal{G}$-class, wherein each of the boundary conditions can be simulated on an extended graph, FKG follows from the observation that the FKG lattice condition is in power even after conditioning on a set of bonds taking a definite value.)

We close this section with the statement of two lemmas that concern ordering and uniqueness of the $\mathcal{S}$-measures. In addition to the æsthetic appeal, the following will be crucial in our subsequent analysis of the phase diagram. Both proofs are, for brevity of exposition, relegated to the Appendix. Note that the limits

$$\nu_{\text{blue-w}}^{(\beta,h)} = \lim_{\Lambda\nearrow\mathbb{Z}^d} \nu_{\Lambda,\text{blue-w}}^{(\beta,h)} \qquad \text{and} \qquad \nu_{\text{red-w}}^{(\beta,h)} = \lim_{\Lambda\nearrow\mathbb{Z}^d} \nu_{\Lambda,\text{red-w}}^{(\beta,h)} \tag{2.18}$$

exist by FKG and are measures both of $\mathcal{G}$ and $\mathcal{S}^\star$-class.

Lemma II.6. *Consider the system defined by the Hamiltonian (1.1) with β fixed and h varying in $(-\infty, +\infty)$. Then for Lebesgue-a.e. h, both the sets of $\mathfrak{G}$-measures and $\mathfrak{S}^\star$-measures are singletons (and then necessarily $\mathfrak{G} = \mathfrak{S}^\star$).*

Lemma II.7. *Let $\nu^{(\beta,h)}$ be either a $\mathfrak{G}$-measure or an $\mathfrak{S}^\star$-measure. Then*

$$\nu^{(\beta,h)}_{\text{blue-w}}(\,\cdot\,) \underset{\text{FKG}}{\geq} \nu^{(\beta,h)}(\,\cdot\,) \underset{\text{FKG}}{\geq} \nu^{(\beta,h)}_{\text{red-w}}(\,\cdot\,),$$

Moreover, for $h^{(1)} > h^{(2)}$, any $\mathfrak{S}^\star$-measure at $h^{(1)}$ FKG dominates every $\mathfrak{S}^\star$-measure at $h^{(2)}$, and similarly for $\mathfrak{G}$.

Percolation and magnetization.

The phases that we will study are all characterized by 'percolation' of one sort or another. *A priori*, there are five distinct situations: Percolation of dark-blue bonds, percolation of blue bonds without percolation of dark-blue bonds, similarly for the reds, and no percolation at all. However, some caution is needed: the partially ordered phases (here defined by the intermediate sort of percolations) certainly do not represent genuine thermodynamic phases, except perhaps when they coexist. For this reason, it is mainly the dark-bond percolation that plays the role of an order parameter.

Given β and h, let $\Pi^B_\infty(\beta,h)$ ($\Pi^R_\infty(\beta,h)$) be the probabilities that dark-blue (dark-red) bonds percolate under the measures $\nu^{(\beta,h)}_{\text{blue-w}}$ ($\nu^{(\beta,h)}_{\text{red-w}}$, respectively). It is an easy corollary to Lemma II.7 that Π^B_∞ is actually the maximal probability at which dark-blue bonds can percolate under any $\mathfrak{G}\cup\mathfrak{S}^\star$-measure, and similarly for Π^R_∞. As is standard, Π^B_∞ and Π^R_∞ are easily related to the spin order parameters, namely, to the blue and red magnetization. Here the red magnetization is defined by adding a symmetry breaking term of the type $-\mathfrak{h}^R\sum_x[\delta_{\sigma_x j} - (1/r)\delta^R_{\sigma_x}]$ (with $j \in R$) to the Hamiltonian and calculating the derivative of the free energy evaluated at $\mathfrak{h}^R = 0^+$. Similarly for the blue magnetization.

Proposition II.8. *Consider the system on $\mathbb{Z}^d$ described by the Hamiltonian (1.1) and let $M^R(\beta,h)$ and $M^B(\beta,h)$ denote the red and blue spontaneous magnetizations. Then*

$$M^R(\beta,h) = \frac{r-1}{r}\,\Pi^R_\infty(\beta,h)$$

and

$$M^B(\beta,h) = \frac{s-1}{s}\,\Pi^B_\infty(\beta,h).$$

Proof. As follows from convexity of the free energy in the parameter $\mathfrak{h}^R$, the red magnetization can alternatively be defined by optimizing $\delta_{\sigma_0 j} - (1/r)\delta^R_{\sigma_0}$ over all possible translation-invariant Gibbs states (for an argument proving an analogous assertion see the proof of Lemma II.7 in the Appendix). Given β and h, all Gibbs states are parametrized by the spin boundary conditions. Fixing the boundary condition to $\tilde{\boldsymbol{\sigma}}$, it is straightforward to establish that the expected value of $\delta_{\sigma_0 j} - (1/r)\delta^R_{\sigma_0}$ under the Potts measure exactly equals $(1-\frac{1}{r})$-times the probability under the corresponding graphical marginal that x is connected by a path of dark-red bonds to an j component of $\tilde{\boldsymbol{\sigma}}$, i.e.,

$$\bigl\langle \delta_{\sigma_0 r_1} - (1/r)\delta^R_{\sigma_0} \bigr\rangle^{\tilde{\boldsymbol{\sigma}}}_\Lambda = \Bigl(1-\frac{1}{r}\Bigr)\nu^{(\beta,h)}_{\tilde{\boldsymbol{\sigma}},\Lambda}\bigl(\{\, 0 \underset{\text{dark-red}}{\longrightarrow} \partial\Lambda_j(\tilde{\boldsymbol{\sigma}})\}\bigr). \tag{2.19}$$

Here $\partial\Lambda_j(\tilde{\boldsymbol{\sigma}}) = \{y \in \partial\Lambda\colon \tilde{\sigma}_y = j\}$, $\langle\,\cdot\,\rangle^{\tilde{\boldsymbol{\sigma}}}_\Lambda$ is the spin state in Λ with boundary condition $\tilde{\boldsymbol{\sigma}}$ and $\nu^{(\beta,h)}_{\tilde{\boldsymbol{\sigma}},\Lambda}$ is the corresponding $\boldsymbol{\mathcal{S}}$-measure.

The r.h.s. of (2.19) is dominated by the expectation of the local event that there is a dark-red path running from 0 outside a fixed volume $\Delta \subset \Lambda$. By taking expectation of both sides w.r.t. a translation invariant spin Gibbs measure and by taking the limit $\Lambda \nearrow \mathbb{Z}^d$ (possibly along a subsequence), we recover an $\boldsymbol{\mathcal{S}}^\star$-measure on the r.h.s. which is FKG dominated by the red-wired measure from Lemma II.7. Since $\{\, 0 \underset{\text{dark-red}}{\longrightarrow} \Delta^{\mathrm{c}}\}$ is an increasing event, the limit $\Delta \nearrow \mathbb{Z}^d$ then shows that

$$\bigl\langle \delta_{\sigma_0 r_1} - (1/r)\delta^R_{\sigma_0} \bigr\rangle \le \Bigl(1-\frac{1}{r}\Bigr)\Pi^R_\infty(\beta,h), \tag{2.20}$$

for any translation invariant spin Gibbs state $\langle\,\cdot\,\rangle$. However, this inequality is clearly saturated for the state $\langle\,\cdot\,\rangle^j$ generated by a constant configuration $\tilde{\boldsymbol{\sigma}} \equiv j$, which is translation invariant. Hence, $M^R(\beta,h) = \frac{r-1}{r}\,\Pi^R_\infty(\beta,h)$ as we were to prove. The case of blue magnetization is completely analogous. $\square$

3. THE PROOF OF THE MAIN THEOREM

Preliminaries.

The proof of the Main Theorem hinges on RP techniques as applied to graphical representations. In order to apply Lemma II.3 and II.4, we first need some definitions.

We will consider various behavioral patterns which are either *good* or *bad*. The good behavioral patterns will be denoted by $\mathcal{O}_\mathcal{R}$, $\mathcal{O}_\mathcal{S}$, $\mathcal{D}_\mathcal{R}$, $\mathcal{D}_\mathcal{S}$; these are defined by the property that every bond in a unit cube is of the corresponding type: r-ordered, s-ordered, r-disordered and s-disordered, respectively. Any other configuration on a cube is deemed to be bad. We recall that two bonds of differing color cannot share a vertex and hence the presence of two colors on a cube necessitates the intervention of a vacant bond. As a consequence, badness can occur for only one of two reasons, the occurrence of a vacant bond or a pair of adjacent bonds, one ordered, the other disordered, that are both of the *same* color. (In the latter case, we call such pairs *mismatched*.)

The good patterns are expected to dominate typical configurations in the low and intermediate temperature regimes. However, as required by Lemma II.4(ii), simultaneous occurrence of different good patterns should be (sufficiently) improbable. The proof of the latter invokes the observation that a 'barrier' of bad cubes necessarily separates any pair of different good cubes. Let us aggregate bad cubes that are joined through at least one edge into connected components that are called, traditionally, *contours*. Item (ii) of Lemma II.4 then boils down to showing that the probability that a contour 'encircles' the origin is small. It is this step where the chessboard estimates (Lemma II.3) provide an effective service. In particular, all we need to show is that the *estimate* on the probability of the various contour elements is small—the rest is easily reduced to a counting argument which is identical to the one for the Ising contours.

As seen on the right hand side of the display in Lemma II.3, the relevant objects are the partition functions constrained so that the stated pattern repeats periodically. Of fundamental importance in the present analysis are the partition functions associated with the (purported) favored patterns. These will be denoted by $\mathcal{Z}_{\mathcal{O}_\mathcal{R}}$, $\mathcal{Z}_{\mathcal{O}_\mathcal{S}}$, $\mathcal{Z}_{\mathcal{D}_\mathcal{R}}$, and $\mathcal{Z}_{\mathcal{D}_\mathcal{S}}$, respectively (the dependence on the scale of the torus will always be understood from the context and will be supressed notationally). These objects are readily computed:

$$\mathcal{Z}_{\mathcal{O}_\mathcal{R}} = [e^{-\beta h} e^{d\beta\kappa}(e^{\beta J} - 1)^d]^{L^d} r \tag{3.1i}$$

$$\mathcal{Z}_{\mathcal{O}_\mathcal{S}} = [e^{\beta h} e^{d\beta\kappa}(e^{\beta J} - 1)^d]^{L^d} s \tag{3.1ii}$$

$$\mathcal{Z}_{\mathcal{D}_\mathcal{R}} = [re^{-\beta h}(e^{\beta\kappa} - 1)^d]^{L^d} \tag{3.1iii}$$

and

$$\mathcal{Z}_{\mathcal{D}_\mathcal{S}} = [se^{\beta h}(e^{\beta\kappa} - 1)^d]^{L^d}. \tag{3.1iv}$$

Of further interest, there is the 'vacant' partition function given by

$$\mathcal{Z}_v = [re^{-\beta h} + se^{\beta h}]^{L^d}. \tag{3.1v}$$

The subject of our next lemma is that for all β sufficiently large (and for r and s large) the probability of any bad cube is small. Even though the general proof is not too involved, we relegate it to the Appendix. Here we provide the proof for $d = 2$ that is particularly simple due to the employment of the diagonal torus which reduces the problem to estimating a single-bond events.

Lemma III.1. *Let $\delta > 0$. Then there exist $\bar{r} = \bar{r}(d, \delta)$, $\bar{s} = \bar{s}(d, \delta)$, and $\bar{\beta} = \bar{\beta}(\kappa, d, \delta)$ so that for any $\kappa > 0$, $r \geq \bar{r}$, $s \geq \bar{s}$, and $\beta \geq \bar{\beta}$ such that the probability of a bad cube is less than δ. Namely, $\lim_{L\to\infty}\langle \chi_{\mathcal{O}_\mathcal{R}} + \chi_{\mathcal{O}_\mathcal{S}} + \chi_{\mathcal{D}_\mathcal{R}} + \chi_{\mathcal{D}_\mathcal{S}}\rangle_L > 1 - \delta$.*

Proof ($d = 2$). By the discussion in the second paragraph of the present section, it is only necessary to show that the probability of a vacant bond or a mismatched pair is small.

Let us start with a vacant bond. We use standard reflection positivity arguments for the case of two-dimensional diagonal torus (see [S] for a full discussion or [CM, Lemma 4.3] for a proof along these lines complete with pictures). Let $\langle i,j\rangle$ denote any bond in $\mathcal{T}_L$ (the diagonal torus of size L). Reflecting the bond $n+1$ times where $2^n = L^2$ we obtain the estimate for the vacant bond:

$$\big\langle \boldsymbol{D}(\omega_{\langle i,j\rangle} = v)\big\rangle_L \le \left(\frac{\mathcal{Z}_v}{\mathcal{Z}}\right)^{\frac{1}{2L^2}} = \frac{(se^{\beta h} + re^{-\beta h})^{\frac{1}{2}}}{\mathcal{Z}^{\frac{1}{2L^2}}}. \tag{3.2}$$

An estimate for $\mathcal{Z}$ is provided by the r and/or s disordered partition functions, $\mathcal{Z} \ge \mathcal{Z}_{\mathcal{D}_\mathcal{R}} + \mathcal{Z}_{\mathcal{D}_\mathcal{S}}$. Since $[(se^h + re^{-h})^{L^2}]/[(se^h)^{L^2} + (re^{-h})^{L^2}] \le 2^{L^2}$, we have

$$\big\langle \boldsymbol{D}(\omega_{\langle i,j\rangle} = v)\big\rangle_L \le \frac{\sqrt{2}}{e^{\beta\kappa} - 1}. \tag{3.3}$$

Next we consider the mismatched pairs. Focusing attention, say, on the s-type, let $\langle i,j\rangle$ and $\langle i,j'\rangle$ denote an adjacent pair of bonds. We will consider the event $\{\omega_{\langle i,j\rangle} = s^{\mathrm{d}}, \omega_{\langle i,j'\rangle} = s^{\mathrm{o}}\}$. If we agree to use only $\mathcal{Z}_{\mathcal{D}_\mathcal{S}}$ and $\mathcal{Z}_{\mathcal{O}_\mathcal{S}}$ in the lower bound of the partition function, a moments thought shows that the calculation is identical to the s-state Potts model. Thus we get

$$\lim_{L\to\infty} \big\langle \boldsymbol{D}(\omega_{\langle i,j\rangle} = s^{\mathrm{d}}, \omega_{\langle i,j'\rangle} = s^{\mathrm{o}})\big\rangle_L \le s^{-\frac{1}{4}}. \tag{3.4}$$

The argument is similar for the r-mismatched pairs. $\square$

A second ingredient needed to set Lemma II.4 in motion is the bound (ii).

Lemma III.2. *There exists a function $\epsilon(d,\delta)$, such that $\epsilon(d,\delta) \to 0$ as $\delta \to 0$, and a constant $\bar{\delta}(d)$ such that for any two distinct patterns $\mathcal{P}_1, \mathcal{P}_2 \in \{\mathcal{O}_\mathcal{R}, \mathcal{O}_\mathcal{S}, \mathcal{D}_\mathcal{R}, \mathcal{D}_\mathcal{S}\}$ one has*

$$\Big\langle \chi_{\mathcal{P}_1}(c)\chi_{\mathcal{P}_2}(\tilde{c})\Big\rangle_L \le \epsilon(d,\delta)$$

whenever $0 < \delta < \bar{\delta}$, $\kappa > 0$, $r \ge \bar{r}$, $s \ge \bar{s}$, and $\beta \ge \bar{\beta}$ (with $\bar{r}$, $\bar{s}$, and $\bar{\beta}$ as in Lemma III.1).

Proof. As already noted before, by going along any connected path from c to $\tilde{c}$, one eventually bumps into a bad cube. Hence, on torus of size L, if the two patterns are to occur simultaneously, the cubes c and $\tilde{c}$ have to be separated by a contour consisting of bad cubes (that either surrounds one of the cubes or winds around the torus). The probability of such a contour is directly estimated by chessboard estimates: for a contour composed of $|\gamma|$ bad cubes we get the bound $\delta^{|\gamma|}$.

To get an explicit expression of the function $\epsilon(d,\delta)$, it is actually convenient to consider just the surface of the above union of bad cubes, which is an Ising contour attached to the faces of the bad cubes in γ. Since there are at most $2d$ plaquettes per each cube in γ, each plaquette carries at most the weight $\delta^{\frac{1}{2d}}$. By employing the recent Lebowitz-Mazel [LM] estimate on the number of Ising contours (which is asymptotically optimal as $d \to \infty$), we arrive at the expression

$$\epsilon(d,\delta) = \big(2 + o(C(d)\delta^{\frac{1}{2d}})\big)\,\delta^{2d} + \big(1 + o(C(d)\delta^{\frac{1}{2d}})\big)\,\frac{1}{2}d(d-1)L^d\delta^{L^{d-1}}. \qquad (3.5)$$

Here $C(d) = \exp\big(O(\frac{\log d}{d})\big)$ is the connectivity constant for the Ising contours.

Indeed, by Corollary 1.2 of [LM], the contribution to $\langle \chi_{\mathcal{P}_1}(c)\chi_{\mathcal{P}_2}(\tilde{c})\rangle_L$ corresponding to a contour surrounding either c or $\tilde{c}$ is dominated by the lowest-order term, provided $C(d)\delta^{\frac{1}{2d}} \ll 1$. Hence, one gets $2\delta^{2d}$, where the '2' accounts for the uncertainty whether the contour runs around c or $\tilde{c}$. The case of a contour wrapped around the torus is fairly analogous; one only needs to observe that the Lebowitz-Mazel counting argument requires the contours neither to be closed nor to be encircling a given point—it is only required that the contour contains a given plaquette. Hence, also in this case the lowest-order term dominates, yielding $\frac{1}{2}d(d-1)L^d\delta^{L^{d-1}}$, where the prefactor counts the possible positions of the plaquette. $\square$

Proof of (I) and (II).

We will start by setting the parameters. It will be assumed that κ and J are fixed and strictly positive, whereas r and s are to be adjusted such that the technical ingredients (i.e., Lemmas III.1 and III.2) are in power. The quantity δ will be our generic 'small parameter', i.e., a number chosen small enough so that Lemma II.4 yields, in conjuction with Lemmas III.1 and III.2, the needed bound. In particular, $\delta \ll C(d)^{-2d}$ must be assumed.

For a fixed δ, let the numbers $r \geq s$ be such that $r \geq \bar{r}(d,\delta)$ and $s \geq \bar{s}(d,\delta)$, respectively, and the assumptions

(1) $\bar{\beta}(\kappa,d,\delta) \ll \frac{1}{Jd}\log s$
(2) $\bar{\epsilon}(1-\delta, 3\delta^{2d})\max\{1,\frac{\kappa}{J}\} \ll 1$

hold (with $\bar{\beta}(\kappa,d,\delta)$ from Lemma III.1 and $\bar{\epsilon}(1-\delta,3\delta^{2d})$ from Lemma II.4).

Remark. Since we are clearly about to use Lemma II.4 in the context of graphical representations, the reader may have doubts whether this result is really applicable, when the Gibbsianness (i.e., the property *defined* by stipulating the DLR condition) of the limiting states is questionable. But this is only due to the wording of the statement of Lemma II.4—as the proof shows, actually the existence of two distinct *limiting* states is established, obtained by conditioning from torus states. In particular, the states emerging from Lemma II.4 are of the $\mathfrak{S}^\star$-class, because they are limits of the torus states conditioned on densities of the respective patterns having large enough value. Since any finite volume can be omitted while evaluating the latter densities, the same argument we used to prove that torus states are $\mathfrak{S}^{\star}$-class applies also to these conditional measures.

The proof now comes in two stages.

(1) $\mathcal{R}$–$\mathcal{S}$ transition. Let $\chi_{\mathcal{R}} = \chi_{\mathcal{D}_{\mathcal{R}}} + \chi_{\mathcal{O}_{\mathcal{R}}}$ be the indicator for a red cube and similarly for $\chi_{\mathcal{S}}$. Since $\beta \geq \bar{\beta}$, we have by Lemma III.1 that $\langle \chi_{\mathcal{R}} + \chi_{\mathcal{S}} \rangle_L^{\beta,h}$ is close to one for any h. Clearly, as $h \to \infty$, the mean value $\langle \chi_{\mathcal{R}} \rangle_L^{\beta,h}$ is supressed and as $h \to -\infty$, the mean value $\langle \chi_{\mathcal{S}} \rangle_L^{\beta,h}$ vanishes—both of these uniformly in L. The conditions of Lemma II.4 are met; for each $\beta \geq \bar{\beta}$, there is an h_β with $|h_\beta| < \infty$ at which R and S-type phases coexist: there exist two $\mathfrak{S}^{\star}$-states $\langle \cdot \rangle_{\mathcal{R}}^{\beta,h_\beta}$ and $\langle \cdot \rangle_{\mathcal{S}}^{\beta,h_\beta}$ satisfying

$$\langle \chi_{\mathcal{R}} \rangle_{\mathcal{R}}^{\beta,h_\beta} \geq 1 - \epsilon' \tag{3.6r}$$

and

$$\langle \chi_{\mathcal{S}} \rangle_{\mathcal{S}}^{\beta,h_\beta} \geq 1 - \epsilon'. \tag{3.6s}$$

with $\epsilon' = \bar{\epsilon}(1 - \delta, 3\delta^{2d})$.

Although the conditions of Lemma II.4 do not rule out the existence, for a given β, of several such points (i.e., 'reentrance'), the FKG monotonicity (as proved in Lemma II.7) resoundingly does. Indeed, any $\mathfrak{S}^{\star}$-state at parameters (β, h) with $h > h_\beta$ will FKG dominate the state $\langle \cdot \rangle_{\mathcal{S}}^{\beta,h_\beta}$ and similarly for $h < h_\beta$.

We now claim that the function h_β is continuous. Indeed let $\beta_* > \bar{\beta}$ and consider a sequence $\{\beta_k\}$ with $\beta_k \to \beta_*$. Crude estimates show that the $|h_{\beta_k}|$ are uniformly bounded so let $-\infty < h_* < \infty$ denote an accumulation point of $\{h_{\beta_k}\}$. Now at the points (β_k, h_{β_k}), we have the $\mathcal{S}$-state where the inequality (3.6s) holds. It follows by a compactness argument that there exists an $\mathfrak{S}^{\star}$-state at (β_*, h_*), where the same inequality holds. Thus $h_* \geq h_{\beta_*}$. Using the same argument with the r's, we conclude that $h_* \leq h_{\beta_*}$ and continuity is established. The curve given by the function h_β will constitute our $\mathcal{C}_{RS}$.

(2) Order-disorder transition. The situation with the order-disorder transition is similar; the disadvantage is that we do not have an FKG monotonicity as the temperature varies. This is remedied by convexity arguments, based on the fact that β couples almost directly to the relevant observables.

It will be useful to consider the events $\mathfrak{b}_{\mathcal{O}}$ and $\mathfrak{b}_{\mathcal{D}}$ that a given bond is ordered/disordered respectively: $\mathfrak{b}_{\mathcal{O}} = \mathfrak{b}_{\mathcal{O}_{\mathcal{S}}} \cup \mathfrak{b}_{\mathcal{O}_{\mathcal{R}}}$ and similarly for $\mathfrak{b}_{\mathcal{D}}$. As above, we may also define the indicators $\chi_{\mathcal{O}}$ and $\chi_{\mathcal{D}}$ for events on cubes. However we claim that for $\beta \geq \bar{\beta}$ the quantities $\langle \chi_{\mathcal{O}} \rangle_L^{\beta,h}$ and $\langle \chi_{\mathfrak{b}_{\mathcal{O}}} \rangle_L^{\beta,h}$ are essentially interchangable. Indeed, first $\langle \chi_{\mathcal{O}} \rangle_L^{\beta,h} \leq \langle \chi_{\mathfrak{b}_{\mathcal{O}}} \rangle_L^{\beta,h}$. On the other hand, writing $\langle \chi_{\mathcal{O}} \rangle_L^{\beta,h} = \langle \chi_{\mathfrak{b}_{\mathcal{O}}} \rangle_L^{\beta,h} \langle \chi_{\mathcal{O}} | \, \mathfrak{b}_{\mathcal{O}} \rangle_L^{\beta,h}$ and using FKG and Lemma III.2 we get $\langle \chi_{\mathcal{O}} \rangle_L^{\beta,h} \geq \langle \chi_{\mathfrak{b}_{\mathcal{O}}} \rangle_L^{\beta,h} (1 - \delta)$.

For this portion of the proof, we will keep $\tilde{h} = \beta h$ fixed and allow β to vary in $[\bar{\beta}, \infty)$. It follows by inspection of (3.1i–v) that, provided the assumption (1) above holds, the variables $\chi_{\mathfrak{b}_\mathcal{O}}$ and $\chi_{\mathfrak{b}_\mathcal{D}}$ satisfy the conditions of Lemma II.4 and thus there is a $\beta_{\tilde{h}}$ at which two $\mathbf{\mathfrak{S}}^\star$-states $\langle\,\cdot\,\rangle_\mathcal{D}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}$ and $\langle\,\cdot\,\rangle_\mathcal{O}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}$ coexist satisfying $\langle\chi_{\mathfrak{b}_\mathcal{O}}\rangle_\mathcal{O}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}} \geq 1-\epsilon'$ and $\langle\chi_{\mathfrak{b}_\mathcal{D}}\rangle_\mathcal{D}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}} \geq 1-\epsilon'$.

Consider the 'bond densities' $\rho_\mathcal{O}(\beta,h)$ and $\rho_\mathcal{D}(\beta,h)$ representing the thermodynamic density of bonds of the two types. These objects are well defined in most states—certainly in translation invariant states where they equal to the expectations of $\mathfrak{b}_\mathcal{O}$ and $\mathfrak{b}_\mathcal{D}$—but their value may depend on the state. To account notationally for such a case, we will indicate different states by a superscript, e.g., $\rho_\mathcal{O}^*(\beta,h)$. However, the densities for different states are not completely uncorrelated: an examination of the weights in (2.1) shows that the quantity

$$\begin{aligned}\mathcal{Q}(\beta,h) &= \left(\kappa + J\frac{e^{\beta J}}{e^{\beta J}-1}\right)\rho_\mathcal{O}(\beta,h) + \kappa\frac{e^{\beta\kappa}}{e^{\beta\kappa}-1}\rho_\mathcal{D}(\beta,h)\\ &\equiv \mathcal{A}_\mathcal{O}(\beta)\rho_\mathcal{O}(\beta,h) + \mathcal{A}_\mathcal{D}(\beta)\rho_\mathcal{D}\end{aligned} \tag{3.7}$$

has the property that if $\beta_1 > \beta_2$, then for any translation-invariant states $*$ and $\#$,

$$\mathcal{Q}^*(\beta_1, \tilde{h}/\beta_1) \geq \mathcal{Q}^\#(\beta_2, \tilde{h}/\beta_2). \tag{3.8}$$

This follows because $\mathcal{Q}$ represents the derivative of the free energy with respect to β with $\tilde{h} = \beta h$ held fixed. Note that $*, \# \in \mathbf{\mathfrak{S}}$ is not required.

In what follows, the treatment is slightly simplified by assuming that $\kappa \leq J$; we will proceed under this assumption and, at the end, discuss briefly the complementary case. Let $\beta > \beta_{\tilde{h}}$ with $\tilde{h} = \beta h$ and let $*$ denote the translation-invariant state at (β, h) designed to minimize $\rho_\mathcal{O}$. Then

$$\mathcal{A}_\mathcal{O}(\beta)\rho_\mathcal{O}^*(\beta) + \mathcal{A}_\mathcal{D}(\beta)\rho_\mathcal{D}^*(\beta) \geq \mathcal{Q}^\mathcal{O}(\beta_{\tilde{h}}) \geq \mathcal{A}_\mathcal{O}(\beta_{\tilde{h}})(1-\epsilon') \tag{3.9}$$

where the suppressed h's in all arguments are evaluated at $h = \tilde{h}/\beta_{\tilde{h}}$. Since both $\mathcal{A}_\mathcal{O}$ and $\mathcal{A}_\mathcal{D}$ decrease with β we may replace on the left side β by $\beta_{\tilde{h}}$:

$$[\mathcal{A}_\mathcal{O}(\beta_{\tilde{h}}) - \mathcal{A}_\mathcal{D}(\beta_{\tilde{h}})]\rho_\mathcal{O}^*(\beta) + \mathcal{A}_\mathcal{D}(\beta_{\tilde{h}}) \geq \mathcal{A}_\mathcal{O}(\beta_{\tilde{h}})(1-\epsilon'), \tag{3.10}$$

where we have further used that $\rho_\mathcal{O}^*(\beta) + \rho_\mathcal{D}^*(\beta) \leq 1$. If $J \geq \kappa$ then all the quantities $(e^{\beta J}-1)^{-1}$, $(e^{\beta\kappa}-1)^{-1}$ are uniformly small by the condition that all inverse temperatures are larger than $\bar{\beta}$. Since $\mathcal{A}_\mathcal{O}(\beta) - \mathcal{A}_\mathcal{D}(\beta) = J(1-e^{-\beta J})^{-1} - \kappa(e^{\beta\kappa}-1)^{-1} \approx J$, we arrive at $\rho_\mathcal{O}^*(\beta) \geq 1-\epsilon$ with $\epsilon \approx \epsilon'[J+\kappa]/J$.

The argument for $\beta < \beta_{\tilde{h}}$ is similar but requires the additional ingredient that for $\beta > \bar{\beta}$, vacant bonds are rare in *any* $\mathbf{S}^\star$-state. (Of course we already know this for the limit of any torus state.) We will be content to prove this statement away from the coexistence line $\mathcal{C}_{RS}$. Consider the point $(\beta, \tilde{h}/\beta)$ with $\beta < \tilde{h}/h_\beta$ (i.e., 'above' $\mathcal{C}_{RS}$). Now, at the point (β, h_β) the $\mathcal{S}$-state has a blue bond density in excess of $1-\epsilon'$. Since the (blue-disordered$\cup$blue-disordered)-bond event is clearly increasing, this probability is larger in any $\mathbf{S}^\star$-state at $(\beta, \tilde{h}/\beta)$, by the assumption upon β. Thus the vacant bond density is less than ϵ'. A similar argument using in turn red-bond density shows that vacant bonds are uniformly rare in all $\mathbf{S}^\star$-states at $(\beta, \tilde{h}/\beta)$ with $\beta > \tilde{h}/h_\beta$ (i.e., 'below' $\mathcal{C}_{RS}$).

Now consider $\beta < \beta_{\tilde{h}}$ and assume that $(\beta, \tilde{h}/\beta) \notin \mathcal{C}_{RS}$. Let $\#$ denote the state at $(\beta, \tilde{h}/\beta)$ designed to maximize $\rho_{\mathcal{O}}$ within the $\mathbf{S}^\star$-class. We have

$$\begin{aligned}\epsilon' \mathcal{A}_{\mathcal{O}}(\beta_{\tilde{h}}) + \mathcal{A}_{\mathcal{D}}(\beta_{\tilde{h}}) \geq \mathcal{Q}^{\mathcal{D}}(\beta_{\tilde{h}}) &\geq \mathcal{A}_{\mathcal{O}}(\beta)\rho_{\mathcal{O}}^{\#}(\beta) + \mathcal{A}_{\mathcal{D}}(\beta)\rho_{\mathcal{D}}^{\#}(\beta) \\ &= \big(\mathcal{A}_{\mathcal{O}}(\beta) - \mathcal{A}_{\mathcal{D}}(\beta)\big)\rho_{\mathcal{O}}^{\#}(\beta) + \big(\rho_{\mathcal{O}}^{\#}(\beta) + \rho_{\mathcal{D}}^{\#}(\beta)\big)\mathcal{A}_{\mathcal{D}}(\beta)\end{aligned} \tag{3.11}$$

where we have again supressed the h dependence in our arguments. Using the fact that the density of vacants is always less than ϵ', we have $\rho_{\mathcal{O}}^{\#}(\beta) + \rho_{\mathcal{D}}^{\#}(\beta) \geq 1 - \epsilon'$ and thus

$$\epsilon'\big(\mathcal{A}_{\mathcal{O}}(\beta_{\tilde{h}}) + \mathcal{A}_{\mathcal{D}}(\beta)\big) + \mathcal{A}_{\mathcal{D}}(\beta_{\tilde{h}}) - \mathcal{A}_{\mathcal{D}}(\beta) \geq \big(\mathcal{A}_{\mathcal{O}}(\beta) - \mathcal{A}_{\mathcal{D}}(\beta)\big)\rho_{\mathcal{O}}^{\#}(\beta). \tag{3.12}$$

Notice that $\mathcal{A}_{\mathcal{D}}(\beta_{\tilde{h}}) - \mathcal{A}_{\mathcal{D}}(\beta)$ is negative (and anyway small). We obtain $\rho_{\mathcal{O}}^{\#}(\beta) \leq \epsilon$, with $\epsilon \approx \epsilon'(J + 2\kappa)/J$.

If $\kappa/J > 1$ the argument is pretty much the same only we cannot immediately discard terms like $e^{-\beta J}$ on the grounds that $\beta > \bar{\beta}$. Two ingredients are required: First, the δ parameter must be made small enough (i.e., s must be large enough) so that when the ϵ' term emerges from the Lemma II.4, the quantity $\epsilon'\kappa/J$ is still small. Second, the value of $\bar{\beta}$ must be trimmed so that it can be stipulated that for $\beta > \bar{\beta}$ the quantity $(e^{\beta J} - 1)$ is not small, say larger than unity. Under these conditions, the proof follows *mutatis mutandis*.

The proof of the continuity of $\beta_{\tilde{h}}$ is the same as for h_β; thus we have our curve $\mathcal{C}_{OD}$. These two curves define our four regions: O_S, O_R, D_R and D_S. In the interior of these regions, the characterizations corresponding to the bounds in (II) of the Main Theorem are clearly satisfied. Namely, if $\beta < \beta_{\tilde{h}}$ and $h > h_\beta$, then just about all of the bonds are disordered and blue in any $\mathbf{S}^\star$-state. This means that in all translation invariant Gibbs spin states, each of them being paired with a translation-invariant $\mathbf{S}$-state in an Edwards-Sokal measure, satisfy $\langle \delta_{\sigma_x\sigma_y}\delta_{\sigma_x}^{S}\rangle \geq 1-\epsilon$. The proof of the other inequalities is similar. The existence of r and s separate magnetized states in their respective ordered regions is obvious: All of these emerge from the torus states. (These will be discussed further below in the *Proof of (IV) and (V).*). □

Proof of (III).

On the basis of crude estimates, both functions $\tilde{h} \mapsto \beta_{\tilde{h}}$ and $\beta \mapsto h_\beta$ are bounded away from the boundary of $\mathcal{J}$, hence, it is clear that they must coincide at least at one point. When $r = s$, both $\mathcal{C}_{OD}$ and $\mathcal{C}_{RS}$ enjoy the flip symmetry $h \to -h$, which entails that $\mathcal{C}_{RS}$ lies on $h = 0$ and is intersected by $\mathcal{C}_{OD}$ at *exactly* one (quadruple-coexistence) point—this establishes the phase diagram in the symmetric case. Insofar we have yet not made use of the condition $r/s \gg 1$. It is this condition that forces non-trivial coincidence of the curves, and, consequently, the two triple points. The existence of triple points is a consequence of the following claim.

Lemma III.3. *Suppose that $r \gg s$ and consider the quantities $\langle \chi_{\mathcal{O}_\mathcal{R}} \rangle_L$, $\langle \chi_{\mathcal{O}_\mathcal{S}} \rangle_L$, $\langle \chi_{\mathcal{D}_\mathcal{R}} \rangle_L$ and $\langle \chi_{\mathcal{D}_\mathcal{S}} \rangle_L$. Then for all β and h, at least one of these objects is small. In particular, let Θ_L denote the chessboard estimate for $\min\{\langle \chi_{\mathcal{O}_\mathcal{S}} \rangle_L, \dots, \langle \chi_{\mathcal{D}_\mathcal{S}} \rangle_L\}$:*

$$\Theta_L = \left[\frac{\min\{\mathcal{Z}_{\mathcal{O}_\mathcal{R}}, \mathcal{Z}_{\mathcal{O}_\mathcal{S}}, \mathcal{Z}_{\mathcal{D}_\mathcal{R}}, \mathcal{Z}_{\mathcal{D}_\mathcal{S}}\}}{\mathcal{Z}_{\mathcal{O}_\mathcal{R}} + \mathcal{Z}_{\mathcal{O}_\mathcal{S}} + \mathcal{Z}_{\mathcal{D}_\mathcal{R}} + \mathcal{Z}_{\mathcal{D}_\mathcal{S}}} \right]^{\frac{1}{L^d}}.$$

Then there is $L_0 = L_0(r/s)$ such that

$$\Theta_L \le 2\sqrt{\frac{s}{r}}$$

for all $L \ge L_0$.

Proof. We first invite the reader to reexamine the weights for $\mathcal{Z}_{\mathcal{O}_\mathcal{S}}, \dots, \mathcal{Z}_{\mathcal{D}_\mathcal{S}}$ in (3.1). If $h \le 0$, then $\mathcal{Z}_{\mathcal{D}_\mathcal{S}}/\mathcal{Z}_{\mathcal{D}_\mathcal{R}} \le (s/r)^{L^d}$ and we are done. If $h > 0$, we may have $se^{\beta h} \ge re^{-\beta h}$. But then $\mathcal{Z}_{\mathcal{O}_\mathcal{R}}/\mathcal{Z}_{\mathcal{O}_\mathcal{S}} = r(e^{-2\beta h})^{L^d}/s \le (s/r)^{L^d-1}$ and we are again done, provided L is large enough. Thus suppose $re^{-\beta h} > se^{\beta h}$. We have

$$\begin{aligned} \Theta_L^{L^d} \le \frac{[\mathcal{Z}_{\mathcal{O}_\mathcal{R}} \mathcal{Z}_{\mathcal{D}_\mathcal{S}}]^{\frac{1}{2}}}{\mathcal{Z}_{\mathcal{O}_\mathcal{S}} + \mathcal{Z}_{\mathcal{D}_\mathcal{R}}} &= \frac{r^{\frac{1}{2}}([ABs]^{\frac{1}{2}})^{L^d}}{s[Ae^{\beta h}]^{L^d} + [Bre^{-\beta h}]^{L^d}} \\ &= \left(\frac{s}{r}\right)^{\frac{1}{2}(L^d-1)} \frac{([ABr]^{\frac{1}{2}})^{L^d}}{s^{\frac{1}{2}}[Ae^{\beta h}]^{L^d} + s^{-\frac{1}{2}}[Bre^{-\beta h}]^{L^d}}, \end{aligned} \tag{3.13}$$

where we have abbreviated $A = [e^{\beta\kappa}(e^{\beta J} - 1)]^d$ and $B = [e^{\beta\kappa} - 1]^d$. By noting that the ratio of the partition sums at the extreme right is less than a half, the proof is over once L is large enough. □

For the duration of this portion of the proof, it is more convenient to use the parametrization by β and $\tilde{h}$, where $\tilde{h} = \beta h$. The argument comes again in two steps.

(1) Lower triple point. In the region $\beta \gg 1$ the curve $\mathcal{C}_{RS}$ separates O_R from O_S phases and in the region $\tilde{h} \ll -1$, the curve $\mathcal{C}_{OD}$ separates O_R from D_R. Let us follow the curves from these extreme ranges of parameters until they first touch. This point will be denoted by $\odot \equiv (\beta^{\odot}, \tilde{h}^{\odot})$. At $\odot$, we have coexistence of O_R, O_S and D_R phases. Note, as is thus clear by Lemma III.3, that in a neighborhood of $\odot$, the *chessboard estimate* for $\langle \chi_{\mathcal{D}_S} \rangle_L$ is small. (In fact, chessboard estimates are continuous in all parameters. The existence of nearby regions of states of the other types forces the small one to be D_S by Lemma III.3) We will successively establish three claims:

(A) The 'unused' portions of $\mathcal{C}_{RS}$ and $\mathcal{C}_{OD}$ lie in the quadrant $Q = \{\tilde{h} > \tilde{h}^{\odot},\, \beta < \beta^{\odot}\}$.
(B) In this quadrant, in a neighborhood of $\odot$, the two curves coincide.
(C) The point $\odot$ is the unique triple point where O_R, O_S and D_R coexist.

Let us start on (A). Consider the line segment $\{\tilde{h} = \tilde{h}^{\odot},\, \bar{\beta} \le \beta < \beta^{\odot}\}$. This is a line where *only* disordered states can exist because one exists at $\odot$ and hence for all higher temperatures with the same value of $h^{\odot}$. Similarly, consider the line segment $\{\beta = \beta^{\odot},\, \tilde{h} > \tilde{h}^{\odot}\}$. Here only O_S states can exist by FKG-domination from the previous subsection (note that $h > h^{\odot}/\beta^{\odot}$ on this segment). Consequently, the unused parts of both $\mathcal{C}_{OD}$ and $\mathcal{C}_{RS}$ must avoid these segments. Since $\mathcal{C}_{OD}$ is parametrizable by $\tilde{h}$ and $\mathcal{C}_{RS}$ by β, these parts have to lie inside the quadrant defined by these segments.

We now claim that in this quadrant, in a neighborhood of $\odot$, the curve $\mathcal{C}_{OD}$ cannot rise above $\mathcal{C}_{RS}$. Indeed, let

$$Q^{+} = \big\{(\beta, \tilde{h}) \in Q \colon \tilde{h} > \beta h_{\beta}\big\}. \tag{3.14}$$

By definition, everything in Q^{+} is $\mathcal{S}$-type. If $\mathcal{C}_{OD}$ has a point inside Q^{+} then in a neighborhood of this point, there is a region of (exclusive) D_S states. Here we have invoked the continuity of $\tilde{h} \mapsto \beta_{\tilde{h}}$, i.e., the function that determines $\mathcal{C}_{OD}$, and the fact that Q^{+} is an open set. However, if we are sufficiently close to $\odot$, continuity of the chessboard estimate forbids at least the torus state from being a D_S state. By exactly the same argument as the rareness of vacant bonds in any $\mathbb{S}^{\star}$-state was established in the paragraph right after (3.9), this implies that the probability of $\mathcal{D}_S$ patterns is uniformly low near $\odot$ in Q^{+}. Hence, $\mathcal{C}_{OD}$ stays below $\mathcal{C}_{RS}$ in the vicinity of $\odot$ in Q.

Finally we will show that in Q, the curve $\mathcal{C}_{OD}$ is *never* below $\mathcal{C}_{RS}$. This will finish off (B) and prove (C) as well; however this time the result is global. Indeed, setting

$$Q^{-} = \big\{(\beta, \tilde{h}) \in Q \colon \tilde{h} < \beta h_{\beta}\big\}, \tag{3.15}$$

let us suppose there is a point of $\mathcal{C}_{OD}$ in Q^-. It follows that there is a whole *open* region of O_R-states in Q. By FKG, this implies that all $\mathcal{S}^\star$-states below this region are O_R and hence ordered. On the other hand, the lower boundary of the quadrant has already been determined to consist exclusively of disordered $\mathcal{S}^\star$-states. Thus the two curves coincide for a while and, when they eventually split, they must do so in such a way that $\mathcal{C}_{OD}$ does not dip below $\mathcal{C}_{RS}$ ever more. In other words, the O_R states are gone for good.

(2) Upper triple point. The argument for the other triple point is similar but hindered by the absence of FKG-monotonicity. As the previous discussion was to rule out the 'bubbles' *below* $\mathcal{C}_{RS}$, now the key point will be to deal with the 'bubbles' that can appear *above* $\mathcal{C}_{RS}$.

Following the curves from the high-temperature/high-field side, let $\otimes = (\beta^\otimes, \tilde{h}^\otimes)$ denote the first point that these curves coincide. The quadrant below and to the left of this point will be denoted by K, i.e., $K = \{(\beta, \tilde{h})\colon \beta > \beta^\otimes,\ \tilde{h} < \tilde{h}^\otimes\}$. Again, 'beyond' the point $\otimes$, both curves are locked into the quadrant. Indeed, $\{\beta > \beta^\otimes,\ \tilde{h} = \tilde{h}^\otimes\}$ is a line that consists exclusively of ordered states and $\{\beta = \beta^\otimes,\ \tilde{h} < \tilde{h}^\otimes\}$ is a line that consists exclusively of $\mathcal{R}$-states. (As a matter of fact, D_R-states.) We already know that $\mathcal{C}_{OD}$ lies above or on $\mathcal{C}_{RS}$ all the way down to $\odot$. We must rule out the possibility that it lies strictly above. Explicitly, we will rule out the existence of D_S-states in K. To that end, we note that K can be reparametrized as

$$K = \bigl\{(\beta, \tilde{h})\colon \beta = \beta^\otimes + \alpha,\ h = \tilde{h}^\otimes - \alpha\Delta,\ 0 < \alpha \le \infty,\ \Delta > 0\bigr\}. \tag{3.16}$$

The argument we will use is similar to the one in (3.7–11), which is just the case $\Delta = 0$.

Let $\eta_{\mathcal{R}}$ denote the density of R-sites. As differentiating of the free energy with respect to α (with Δ fixed) reveals, in any translation-invariant state at $(\beta, \tilde{h}) = \bigl(\beta^\otimes + \alpha,\ \tilde{h}^\otimes - \alpha\Delta\bigr)$, the quantity $\mathcal{A}_{\mathcal{O}}(\beta)\rho_{\mathcal{O}} + \mathcal{A}_{\mathcal{D}}(\beta)\rho_{\mathcal{D}} + 2\Delta\eta_{\mathcal{R}}$ must be larger than its value in any translation-invariant state at $(\beta^\otimes, \tilde{h}^\otimes)$. Let Δ, α be fixed and let $\star$ denote any such state. By comparison to the O_S-state at $\otimes$ we have

$$\mathcal{A}_{\mathcal{O}}(\beta)\rho_{\mathcal{O}}^\star + \mathcal{A}_{\mathcal{D}}(\beta)\rho_{\mathcal{D}}^\star + 2\Delta\eta_{\mathcal{R}}^\star \ge \mathcal{A}_{\mathcal{O}}(\beta^\otimes)(1 - \epsilon') \tag{3.17i}$$

and similarly, by comparison with the D_R state at $\otimes$,

$$\mathcal{A}_{\mathcal{O}}(\beta)\rho_{\mathcal{O}}^\star + \mathcal{A}_{\mathcal{D}}(\beta)\rho_{\mathcal{D}}^\star + 2\Delta\eta_{\mathcal{R}}^\star \ge \mathcal{A}_{\mathcal{D}}(\beta^\otimes)(1 - \epsilon') + 2\Delta(1 - \epsilon') \tag{3.17ii}$$

This easily finishes the proof. Namely, let $2\Delta \le \mathcal{A}_{\mathcal{O}}(\beta^\otimes) - \mathcal{A}_{\mathcal{D}}(\beta^\otimes)$. Then by the same arguments as in (3.8–9) we get

$$\bigl[\mathcal{A}_{\mathcal{O}}(\beta^\otimes) - \mathcal{A}_{\mathcal{D}}(\beta^\otimes)\bigr](\rho_{\mathcal{O}}^\star + \eta_{\mathcal{R}}^\star) + \mathcal{A}_{\mathcal{D}}(\beta^\otimes) \ge \mathcal{A}_{\mathcal{O}}(\beta^\otimes)(1 - \epsilon'). \tag{3.18i}$$

However, this entails that the bigger of the quantities $\rho_{\mathcal{O}}^\star$, $\eta_{\mathcal{R}}^\star$ is close to a half, which rules out that $\star$ is any D_S state. If $2\Delta > \mathcal{A}_{\mathcal{O}}(\beta^\otimes) - \mathcal{A}_{\mathcal{D}}(\beta^\otimes)$, then (assuming without loss of generality $\eta_{\mathcal{R}}^\star \le 1-\epsilon'$) we are led to

$$\bigl[\mathcal{A}_{\mathcal{O}}(\beta^\otimes) - \mathcal{A}_{\mathcal{D}}(\beta^\otimes)\bigr]\rho_{\mathcal{O}}^\star + \epsilon'\mathcal{A}_{\mathcal{D}}(\beta^\otimes) \ge \bigl[\mathcal{A}_{\mathcal{O}}(\beta^\otimes) - \mathcal{A}_{\mathcal{D}}(\beta^\otimes)\bigr](1-\epsilon'-\eta_{\mathcal{R}}^\star), \tag{3.18ii}$$

which again cannot be satisfied if both $\rho_{\mathcal{O}}^\star$ and $\eta_{\mathcal{R}}^\star$ are small.

Since Δ was arbitrary, there is no alternative to the scenario that the curves coincide down to $\odot$, where the 'new' O_R states enter the play. But this proves that there are just two triple points and that they are indeed connected by a single line of $O_S + D_R$ coexistence. $\square$

Proof of (IV) and (V).

Item (IV) is a standard high temperature result that is particularly easy to justify in the graphical representation. Consider any infinite-volume spin Gibbs measure ν and its (arbitrary) Edwards-Sokal coupling ν^{ES}. Then we claim that under the $\boldsymbol{\omega}$ marginal of ν^{ES} (and β small), most of the configuration is actually in the vacant state. Namely, if two neighboring sites are of the same spin-type but not the same spin-state, then the only other possibility is a disordered bond-state, whose relative weigth is then $1-e^{-\beta\kappa}$. Similarly, if two neighboring sites are in the same state, the relative weight of an ordered bond is $1-e^{-\beta J}$ and of a disordered bond is $e^{-\beta J}(1-e^{-\beta\kappa})$.

Let us call non-vacant bonds 'open' and the vacant ones 'closed'. Since the bond states are independent under the measure $\nu^{\mathrm{ES}}(\,\cdot\,|\,\boldsymbol{\sigma})$, the 'open/closed' marginal of $\nu^{\mathrm{ES}}(\,\cdot\,|\,\boldsymbol{\sigma})$ is FKG-dominated by the ordinary bond percolation on $\mathbb{Z}^d$ with parameter $p = 1-e^{-\beta(J+\kappa)}$, where $J,\kappa>0$ is used to bound the other possibilities. Let now $p < p_c(d)$, where $p_c(d)$ marks the onset of bond percolation on $\mathbb{Z}^d$. Then a classical result [MMS][AB] yields that the size of the maximal cluster intersecting a fixed finite volume Λ has an exponentially small tail, with a Λ-independent rate (if we are content to restrict p and, consequently, β only to extremely small values, then this follows already from a Peierls argument). The FKG domination says the same applies to the graphical marginal of $\nu^{\mathrm{ES}}(\,\cdot\,|\,\boldsymbol{\sigma})$ for ν-a.s. $\boldsymbol{\sigma}$.

It is not difficult to observe that this actually implies the exponential decay of all truncated correlations with a uniform decay rate, because disconnected regions act independently of each other. Namely, let $(A_i)_{i=1}^N$ be disjoint finite sets of sites, ψ_i be cylindric function in A_i, let $\chi = \chi(A_1,\dots,A_N)$ indicate that all A_i's are connected and let $\ell(A_1,\dots,A_N)$ denote the minimal number of bonds needed for this connection. Let $\langle\,\cdot\,\rangle_\Lambda$ be any state in $\Lambda \supset \cup_i A_i$ with some fixed boundary condition. Then the logarithmic generating function

$$\mathcal{H}_\Lambda(z_1,\dots,z_N) = \log\left[\frac{\bigl\langle e^{\sum_{i=1}^N z_i\psi_i}\bigr\rangle_\Lambda}{\bigl\langle e^{\sum_{i=1}^N z_i\psi_i}|\{\chi=0\}\bigr\rangle_\Lambda}\right] \tag{3.19}$$

of the functions f_i (note that the denominator plays no role for truncated correlators involving *all* functions f_i) exists for any $z_i \in \mathbb{C}$ such that $|z_i|$ is small enough, and it satisfies the inequality

$$\bigl|\mathcal{H}_\Lambda(z_1,\dots,z_N)\bigr| \le C\biggl(\prod_{i=1}^N e^{2|z_i|\,\|\psi_i\|}\biggr)e^{-\delta\ell(A_1,\dots,A_N)}, \tag{3.20}$$

as is verified by splitting the numerator in (3.18) depending whether $\chi=0$ or 1 and then using that $\log(1-x)\le x_0^{-1}\log(1-x_0)x$ for $0\le x\le x_0<1$ and that $\langle\chi\rangle_\Lambda \le \mathcal{O}(1)e^{-\delta\ell(A_1,\dots,A_N)}$, where δ is the rate of the connectivity function in the percolation model above. Note that C is finite and independent of Λ whenever $\max_i\{|z_i|\,\|\psi_i\|\}$ is small enough. The multidimensional Cauchy theorem then implies that condition IIc stating that the truncated correlator

$$\bigl|\langle\psi_1^{k_1};\dots;\psi_N^{k_N}\rangle_\Lambda\bigr| \le k_1!\dots k_N!\;\widetilde{C}^{k_1+\dots+k_N}\,e^{-\delta\ell(A_1,\dots,A_N)}, \tag{3.20}$$

and consequently all the other conditions in [DS] (or, alternatively, the condition in [vdBM]) for complete analyticity are satisfied.

In order to prove item (V), note that in the interior of O_R, O_S, D_R and D_S there is percolation of the appropriate type. Indeed, in any $\mathcal{S}^\star$-state, we know that the probability of the dominant type of bond is close to one in any $\mathcal{S}^\star$-state and on the torus, we know that the probability of contours is rare. Thence the event that two bonds at the opposite ends of the torus are connected by a path of relevant bonds is close to one uniformly in the size of the torus. Thus, in any torus state, percolation of the appropriate bond-type is inevitable.

We will finish by showing that, for the interior points, in every $\mathcal{S}^\star$-state there is no percolation of any of the sub-dominant types. First note that, throughout the interior of the regions, one can produce $\mathcal{S}^\star$-states where no other than the relevant bonds percolate. Namely, in the torus states, the probability that the non-appropriate bonds form a contour running around the torus tends to 0 in the thermodynamic limit, hence, conditioning on the complement event yields an $\mathcal{S}^\star$-state in this limit, assigning a uniformly positive probability to the event that any two sites are connected by the relevant bonds. With this in the hand, percolation of non-relevant bonds can be immediately ruled out inside the O-regions, using FKG domination from Lemma II.7, because 'non-appropriate percolation' is a monotone event in this case. In the D-regions, one first rules on the percolation of O-bonds of the respective color (again by FKG) and then the percolation of the bonds of the complementary color. $\square$

4. APPENDIX

Proof of Lemma II.5.

The measures μ_1 and μ_2 are distinct, hence they are distinguished by the expectation of a local function g. Since both have been obtained essentially as limits of finite-volume states, it is enough to show that the expectation of the bond configuration function g under the corresponding Edwards-Sokal measure can be expressed as an expectation, under the same measure, of a local function f depending only on spin configuration $\boldsymbol{\sigma}$. The compactness of the space of all measures (in the weak-* topology) then proves the existence of the two desired distinct spin-states in the respective sets of cluster points.

The function g can be rewritten as $g(\,\cdot\,) = \sum_{\boldsymbol{\omega}} g(\boldsymbol{\omega})\delta_{\{\boldsymbol{\omega}\}}(\,\cdot\,)$. Thus, relying also on the inclusion-exclusion principle, without loss of generality it suffices to construct the function f only for g that indicates a fixed configuration $\bar{\boldsymbol{\omega}}$ on a finite set of bonds Σ, with *no* vacant bonds. Then, Σ decomposes into the disjoint union of the sets $\Sigma_{\mathcal{O}_\mathcal{R}}$, $\Sigma_{\mathcal{D}_\mathcal{R}}$, $\Sigma_{\mathcal{O}_\mathcal{S}}$, and $\Sigma_{\mathcal{D}_\mathcal{S}}$ of those bonds where $\bar{\boldsymbol{\omega}}$ is dark-red, light-red, dark-blue, and light-blue, respectively. Now, the corresponding $f_{\bar{\boldsymbol{\omega}}}$ will be the product $f_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\sigma}) = \prod_{b=\langle x,y\rangle\in\Sigma} f_{\bar{\omega}_b}(\sigma_x,\sigma_y)$, where

$$f_{\bar{\omega}_b}(\sigma_x,\sigma_y) = \begin{cases} \frac{e^{\beta J}-1}{e^{\beta J}}\,\delta^S_{\sigma_x\sigma_y} & b=\langle x,y\rangle\in\Sigma_{\mathcal{O}_\mathcal{S}} \\ \frac{e^{\beta J}-1}{e^{\beta J}}\,\delta^R_{\sigma_x\sigma_y} & b=\langle x,y\rangle\in\Sigma_{\mathcal{O}_\mathcal{R}} \\ \frac{e^{\beta\kappa}-1}{e^{\beta\kappa}}\left(1-\frac{e^{\beta J}-1}{e^{\beta J}}\,\delta^S_{\sigma_x\sigma_y}\right)\delta^S_{\sigma_x}\delta^S_{\sigma_y} & b=\langle x,y\rangle\in\Sigma_{\mathcal{D}_\mathcal{R}} \\ \frac{e^{\beta\kappa}-1}{e^{\beta\kappa}}\left(1-\frac{e^{\beta J}-1}{e^{\beta J}}\,\delta^R_{\sigma_x\sigma_y}\right)\delta^R_{\sigma_x}\delta^R_{\sigma_y} & b=\langle x,y\rangle\in\Sigma_{\mathcal{D}_\mathcal{S}} \end{cases} \tag{A.1}$$

Observe that for every $b\in\Sigma$ we have (note that $\bar{\omega}_b\neq v$)

$$f_{\bar{\omega}_b}(\sigma_x,\sigma_y)\sum_{\omega_b} w(\omega_b)\chi_b(\omega_b,\sigma_x,\sigma_y) = w(\bar{\omega}_b)\chi_b(\bar{\omega}_b,\sigma_x,\sigma_y) \tag{A.2}$$

(see (2.2) to recall the meaning of w's and χ's), where in the arguments of χ we have retained only the relevant terms. By noting that the external-field terms have not been tampered with at all during this operation we can easily convince ourselves that

$$\sum_{\boldsymbol{\omega},\boldsymbol{\sigma}} f_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\sigma})W^{\beta,h}_{\mathrm{ES},s,r;\Lambda}(\boldsymbol{\omega},\boldsymbol{\sigma}) = \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}} \delta_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\omega})W^{\beta,h}_{\mathrm{ES},s,r;\Lambda}(\boldsymbol{\omega},\boldsymbol{\sigma}), \tag{A.3}$$

where $W^{\beta,h}_{\mathrm{ES},s,r;\Lambda}$ is the corresponding Edwards-Sokal weight in Λ (see (2.14)), with the dependence on the boundary condition $\tilde{\boldsymbol{\sigma}}$ being only implicit. Namely, fix a configuration $\boldsymbol{\sigma}$ and carry out the summation over $\boldsymbol{\omega}_\Sigma$ on the l.h.s. Then (A.2) asserts that this is replacable by putting the indicator $\delta_{\{\bar{\boldsymbol{\omega}}\}}$, independently of $\boldsymbol{\sigma}$ to have been chosen. The equality then follows by summing over the rest of the variables. $\square$

Proof of Lemmas II.6 and II.7.

We first prove Lemma II.6 for any $\mathcal{G}$-measure. The claim for $\mathcal{S}^{\star}$-measures will be then an easy corollary of the FKG domination in Lemma II.7. At several points in the forthcoming derivation, we will have occasion to use the Strassen theorem [S]—or more precisely the corollary to Strassen's theorem. This result may be stated as follows:

Theorem A.1. *Let $X_1, \dots X_N$ denote a collection of real-valued random variables (for simplicity each assumed to take on only a finite number of values) and let μ_1 and μ_2 denote measures on the configurations of these variables. Suppose that a priori $\mu_1 \underset{\mathrm{FKG}}{\geq} \mu_2$. Then if for each i, the distributions of X_i are identical it follows that $\mu_1 = \mu_2$.*

Proof. See, e.g. [L] page 75.

Given a spin configuration $\boldsymbol{\sigma}$, let $\boldsymbol{\xi}$ denote its red/blue marginal, i.e., $\xi_x = R$ if $\sigma_x \in R$ and $\xi_x = B$ if $\sigma_x \in B$. We start with a somewhat weaker version of Lemma II.6.

Lemma A.2. *Let $\tilde{\mu}^{(\beta,h),*}_{r,s,\Lambda}(\,\cdot\,)$ denote the $\boldsymbol{\xi}$-marginal of the Edwards-Sokal measure with a $\mathcal{G}$ boundary condition. Then for Lebesgue-a.e. h, there is a unique infinite-volume measure for this marginal.*

Proof. We first claim that $\tilde{\mu}^{(\beta,h),*}_{s,r;\Lambda}(\,\cdot\,)$ is FKG w.r.t. the order $R \prec B$. Indeed, first let us observe that cylinder functions of these site variables may be evaluated via conditional expectations given a bond configuration $\boldsymbol{\omega}$. In particular, each site that is the endpoint of a blue bond *is* blue, similarly for the reds, and vacant sites are independently red or blue with probability $re^{-\beta h}/(se^{\beta h}+re^{-\beta h})$ and $se^{\beta h}/(se^{\beta h}+re^{-\beta h})$, respectively. Thus, if $\mathfrak{A}$ is an increasing event determined by the site variables we may write

$$\tilde{\mu}^{(\beta,h),*}_{s,r;\Lambda}(\mathfrak{A}) = \sum_{\boldsymbol{\omega}} \nu^{(\beta,h)}_{s,r;\Lambda}(\boldsymbol{\omega}) A(\boldsymbol{\omega}). \tag{A.4}$$

Here $A(\boldsymbol{\omega}) = \mathbb{E}_{\mathrm{ES}}(\mathbb{I}_{\mathfrak{A}} \,|\, \boldsymbol{\omega})$, with $\mathbb{I}_{\mathfrak{A}}$ being the indicator function, and $\nu^{(\beta,h)}_{s,r;\Lambda}$ is the random-cluster measure in Λ (with the boundary-condition dependence being only implicit).

It is not hard to see that $A(\boldsymbol{\omega})$ is an increasing function. Furthermore, if $\mathfrak{A}$ and $\mathfrak{B}$ are both increasing events, it is easy to see that

$$\mathbb{E}_{\mathrm{ES}}(\mathbb{I}_{\mathfrak{A}}\mathbb{I}_{\mathfrak{B}} \,|\, \boldsymbol{\omega}) \geq \mathbb{E}_{\mathrm{ES}}(\mathbb{I}_{\mathfrak{A}} \,|\, \boldsymbol{\omega}) \mathbb{E}_{\mathrm{ES}}(\mathbb{I}_{\mathfrak{B}} \,|\, \boldsymbol{\omega}) \equiv A(\boldsymbol{\omega}) B(\boldsymbol{\omega}). \tag{A.5}$$

Indeed the only randomness in the above conditional expectations come from the 'vacant' sites which are independently assigned their colours; thus we use the FKG property for Bernoulli measures. Hence we have

$$\tilde{\mu}^{(\beta,h),*}_{s,r;\Lambda}(\,\mathfrak{A} \cap \mathfrak{B}) \geq \sum_{\boldsymbol{\omega}} \nu^{(\beta,h)}_{s,r;\Lambda}(\boldsymbol{\omega}) A(\boldsymbol{\omega}) B(\boldsymbol{\omega}) \geq \tilde{\mu}^{(\beta,h),*}_{s,r;\Lambda}(\mathfrak{A}) \tilde{\mu}^{(\beta,h),*}_{s,r;\Lambda}(\mathfrak{B}) \tag{A.6}$$

where the last step follows from the FKG property of $\nu_{s,r;\Lambda}^{(\beta,h)}(\,\cdot\,)$ and the identity in (A.6). It is also evident that the $\mathfrak{G}$-type boundary conditions, the measure $\tilde{\mu}_{s,r;\Lambda}^{(\beta,h),*}(\,\cdot\,)$ enjoy the same FKG hierarchy as the corresponding $\nu_{s,r;\Lambda}^{(\beta,h)}(\,\cdot\,)$, e.g., the blue-wired is highest, red-wired is lowest, etc.

We now claim that for a.e. h, there is a single limiting $\tilde{\mu}$ measure in the $\mathfrak{G}$-class. Indeed, the free energy has a.e. a continuous derivative w.r.t. h and at the points of continuity, the fraction of blue sites (which is coupled to h in the Hamiltonian) exists and is independent of the state. Moreover, in both the red-wired and blue-wired states (which are both translation invariant) this fraction is exactly the probability of a blue at any fixed site. Hence, by the corollary to Strassen's theorem, these are the *same* state. Since all $\mathfrak{G}$-states lie in between these extremes, there is just one such state. □

Corollary. *Let $\tilde{\mu}_{s,r;\Lambda}^{(\beta,h),\dagger}(\,\cdot\,)$ denote the marginal of the random-cluster measure that counts only whether each bond is red, blue or vacant. Then for Lebesgue-a.e. h, the limiting red- and blue-wired measures coincide.*

Proof. Let x and y denote a neighboring pair of sites. Since by Lemma A.2 the limiting $\boldsymbol{\xi}$-marginals $\tilde{\mu}_{r,s,\text{blue-w}}^{(\beta,h),*}(\,\cdot\,)$ and $\tilde{\mu}_{r,s,\text{red-w}}^{(\beta,h),*}(\,\cdot\,)$ agree, it follows that the probability that both x and y are blue is the same in both systems. Let us denote this probability by $g_{x,y}^{b}$ and further let b_B and b_R denote the probabilities that the bond $\langle x,y\rangle$ is blue in the blue-wired and red-wired measures, and similarly v_B and v_R for the probabilities that the bonds is vacant. Finally, let $\lambda_{v,B}^{b}$ denote the conditional probability, in the limiting blue-wired measure that both endpoints of the bond $\langle x,y\rangle$ are blue given that this bond is vacant. Let $\lambda_{v,R}^{b}$ be the similar quantity for the red-wired boundary condition.

Since the underlying random-cluster measures are *strong* FKG, it is observed that

$$\lambda_{v,B}^{b} \geq \lambda_{v,R}^{b} \tag{A.7}$$

It is also observed that $\lambda_{v,R}^{b} < 1$. Clearly

$$b_B + v_B\lambda_{v,B}^{b} = g_{x,y}^{b} b_R + v_R\lambda_{v,R}^{b}, \tag{A.8}$$

i.e.,

$$0 = (b_B - b_R)(1 - \lambda_{v,R}^{b}) + v_B(\lambda_{v,B}^{b} - \lambda_{v,R}^{b}) + ([b_B + v_B] - [b_R + v_R])\lambda_{v,R}^{b}. \tag{A.9}$$

But *a priori* each term on the right hand side is non-negative so it follows that all three are zero. In particular, $b_B = b_R$ (implying that the blue-bond densities are equal) and thus also $v_B = v_R$. Using the corollary to Strassen's theorem, the desired conclusion is obtained. □

Proof of Lemma II.6—$\mathcal{G}$-measures. Let x and y denote a neighboring pair of sites and let $\alpha^B_{x,y}$ denote the conditional probability in the blue-wired measure that the sites x and y are connected in the complement of the bond $\langle x, y\rangle$ given that both the sites x and y are blue. Let $\alpha^R_{x,y}$ denote the corresponding probability in the red-wired measure. Now given that x and y are both blue, the only possibilities for the bond $\langle x, y\rangle$ is vacant, light blue or dark blue.

When these sites are externally connected by dark-blue bonds, the ratio of these probabilities is $1 : e^{\beta\kappa} - 1 : e^{\beta\kappa}(e^{\beta J} - 1)$. For notational clarity, let us temporarily denote these quantities by $1 : C : D$. On the other hand, if the two sites are disconnected, the ratios read $1 : C : s^{-1}D$. Now in the red- and blue-wired states, we have determined that the probability of a blue bond is the same and the probability a neighboring pair of blue sites is the same. The ratio of these probabilities is equal which gives us

$$\begin{aligned} \alpha^B_{x,y}\frac{C+D}{1+C+D} + (1-\alpha^B_{x,y})\frac{C+s^{-1}D}{1+C+s^{-1}D} \\ = \alpha^R_{x,y}\frac{C+D}{1+C+D} + (1-\alpha^R_{x,y})\frac{C+s^{-1}D}{1+C+s^{-1}D} \end{aligned} \tag{A.10}$$

The above is only possible if $\alpha^B_{x,y} = \alpha^R_{x,y}$ and from this it follows that the dark-blue bond density is the same in both measures. Thence, all bond densities are the same and, again using the corollary to the Strassen theorem the measures coincide $\square$

As a simple Corollary, we obtain a domination bound for the extreme $\mathcal{G}$-measures:

Corollary. *For any $h^{(1)} > h^{(2)}$,*

$$\nu^{(\beta,h^{(1)})}_{\text{red-w}}(\,\cdot\,) \underset{\text{FKG}}{\geq} \nu^{(\beta,h^{(2)})}_{\text{blue-w}}(\,\cdot\,).$$

Proof. Let g be a monotone increasing cylinder function. Let

$$\bar{h} = \inf\bigl\{\, h\colon \nu^{(\beta,h)}_{\text{blue-w}}(g) > \nu^{(\beta,h^{(1)})}_{\text{red-w}}(g)\bigr\} \tag{A.11}$$

and suppose $\bar{h} < h^{(1)}$. Since both $h \mapsto \nu^{(\beta,h)}_{\text{blue-w}}$ and $h \mapsto \nu^{(\beta,h)}_{\text{red-w}}$ are increasing, this makes inevitable that $\nu^{(\beta,h)}_{\text{blue-w}}(g) > \nu^{(\beta,h)}_{\text{red-w}}(g)$ for all $h \in (\bar{h}, h^{(1)})$. However, this is in contradiction with $\nu^{(\beta,h)}_{\text{blue-w}} = \nu^{(\beta,h)}_{\text{red-w}}$ for Lebesgue almost all h, hence $\bar{h} \geq h^{(1)}$ (in fact, the equality holds). Since g was arbitrary, the proof is over. $\square$

Proof of Lemma II.7. Let β, h be fixed and let us suppress them from the notation. For $\mathcal{G}$-measures the claim is a trivial corollary of Proposition II.1 and Corollary above, thus we shall concentrate on $\mathcal{S}^\star$-masures. We shall show that $\nu_{\Lambda,\text{blue-w}}(g) \geq \nu(g)$ for any increasing cylinder function g and any $\mathcal{S}^\star$-measure ν.

Let thus g be a cylinder function with support on finite set A of bonds. Consider its 'spread' over Λ, i.e., $\mathfrak{g}_\Lambda = \sum_{x:\, \tau^x(A)\subset \mathbb{B}_\Lambda} g\circ\tau^x$, with τ^x denoting the 'shift by x'. Let now $\langle\,\cdot\,\rangle^\alpha_{\Lambda,\tilde{\boldsymbol{\sigma}}}$ denote the $\boldsymbol{\omega}$-marginal of the Edwards-Sokal measure with the weight (2.12) (considering the volume Λ with the boundary condition $\tilde{\boldsymbol{\sigma}}$ instead of the torus $\mathcal{T}$) modified by the factor $e^{\alpha\mathfrak{g}_\Lambda(\boldsymbol{\omega})}$. The respective normalizing constant $Z^\alpha_{\Lambda,\tilde{\boldsymbol{\sigma}}}$ of this measure is then also α-dependent and is logarithmically convex in α. Observing that $F(\alpha) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\log Z^\alpha_{\Lambda,\tilde{\boldsymbol{\sigma}}}$ does not depend on the boundary condition and that it is a convex function in α, we can employ the standard convexity argument to infer that

$$\Bigl\langle \frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr\rangle^0_{\Lambda,\tilde{\boldsymbol{\sigma}}_1} - \epsilon_\Lambda \le \frac{\mathrm{d}F}{\mathrm{d}\alpha^+}\Big|_{\alpha_1} \le \frac{\mathrm{d}F}{\mathrm{d}\alpha^-}\Big|_{\alpha_2} \le \Bigl\langle \frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr\rangle^\alpha_{\Lambda,\tilde{\boldsymbol{\sigma}}_2} + \epsilon_\Lambda \tag{A.12}$$

for any $\alpha > \alpha_2 > \alpha_1 > 0$ and any two spin boundary conditions $\tilde{\boldsymbol{\sigma}}_1, \tilde{\boldsymbol{\sigma}}_2$. Here $\epsilon_\Lambda = O\bigl(\frac{|\partial\Lambda|}{|\Lambda|}\bigr)$ uniformly in the boundary condition.

For the interpretation of the l.h.s. it is important that any $\boldsymbol{\mathcal{S}}$-measure ν is the $\boldsymbol{\omega}$-marginal of some Edwards-Sokal measure whose $\boldsymbol{\sigma}$ marginal is a (conventional) Gibbs measure. By using the fact that every local cylinder function g of bonds can be, under expectation w.r.t. the Edwards-Sokal measure interchanged into a spin function f, as follows from the proof of Lemma II.5, we can view the l.h.s. of (A.12) as a spin Gibbs specification. By averaging over a translation-invariant spin Gibbs measure we arrive at $\langle \frac{\mathfrak{f}_\Lambda}{|\Lambda|}\rangle = \langle f\rangle = \nu(g)$, where we denoted by $\mathfrak{f}_\Lambda$ the 'spread' of f.

It remains to work out the r.h.s. of (A.12) into the desired form. Let us consider an auxiliary measure $\langle\,\cdot\,\rangle^\alpha_{\Lambda,*,\text{blue-w}}$, derived from $\nu_{\Lambda,\text{blue-w}}$ by modifying the *a priori* weights (see (2.1)) in the following manner: dark-red, light-red, light-blue, dark-blue will pick up additional factors $e^{-2\alpha\mathrm{var}(g)}$, $e^{-\alpha\mathrm{var}(g)}$, $e^{\alpha\mathrm{var}(g)}$, $e^{2\alpha\mathrm{var}(g)}$, respectively, where the variance $\mathrm{var}(g)$ of the function g is defined as

$$\mathrm{var}(g) = \sup_b \sup_{\substack{\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}:\omega_{b'}=\tilde{\omega}_{b'}\\ \forall\, b'\neq b}} \bigl|\, g(\boldsymbol{\omega}) - g(\tilde{\boldsymbol{\omega}})\bigr|. \tag{A.13}$$

By observing the proof of Proposition II.1, it is easily checked that $\langle\,\cdot\,\rangle^\alpha_{\Lambda,*,\text{blue-w}}$ satisfies the FKG lattice condition and, for $\alpha \ge 0$, a similar argument we used in (2.9) proves that it actually dominates the wired measure $\langle\,\cdot\,\rangle^\alpha_{\Lambda,\text{blue-w}}$.

Now we can use the fact that any constant blue boundary condition generates the blue-wired measure, so by choosing $\boldsymbol{\sigma}_2$ in (A.12) to be such *prior* to averaging over the state $\langle\,\cdot\,\rangle$, we arrive at the inequality

$$\nu(g) - 2\epsilon_\Lambda \le \langle g\rangle^\alpha_{\Lambda,\text{blue-w}} \le \langle g\rangle^\alpha_{\Lambda,*,\text{blue-w}} \tag{A.14}$$

for all $\alpha > 0$ and all finite Λ. By passing to the limits $\alpha \downarrow 0$ and $\Lambda \nearrow \mathbb{Z}^d$ we get the desired bound. Since g was arbitrary the upper bound in the display in Lemma II.7 is proved. The lower bound is completely analogous.

The FKG-domination for the $\mathcal{S}^\star$-measures is then the result of the following estimate

$$\nu^{(\beta,h^{(1)})}(\cdot) \underset{\text{FKG}}{\geq} \nu^{(\beta,h^{(1)})}_{\text{red-w}}(\cdot) \underset{\text{FKG}}{\geq} \nu^{(\beta,h^{(2)})}_{\text{blue-w}}(\cdot) \underset{\text{FKG}}{\geq} \nu^{(\beta,h^{(2)})}(\cdot) \tag{A.15}$$

for any $h^{(1)} > h^{(2)}$, and any $\mathcal{S}^\star$-measures $\nu^{(\beta,h^{(1)})}$ and $\nu^{(\beta,h^{(2)})}$. The middle inequality follows by Corollary above. $\square$

Proof of Lemma III.1—general case.

In dimensions $d > 2$, no diagonal torus is available. Hence we have to proceed by brute force in deriving the chessboard estimate. In particular, since the events we shall be studying (i.e., the various ways that a given cube is bad) do not, after dissemination, result in the probability of a *definite* configuration, no direct use of the formulas (3.1i–v) can be made for the numerator estimate (c.f. the proof of Lemma III.1 in the case of $d = 2$).

The way we estimate the r.h.s. of the formula in Lemma II.3 in the case of $d > 2$ is by redistributing the weights of the graphical representation: we define new *a priori* weights

$$\begin{aligned} \tilde{w}(s^{\mathrm{o}}) &= w(s^{\mathrm{o}})e^{\beta h/d} & \tilde{w}(r^{\mathrm{o}}) &= w(r^{\mathrm{o}})e^{-\beta h/d} \\ \tilde{w}(s^{\mathrm{d}}) &= w(s^{\mathrm{d}})(se^{\beta h})^{1/d} & \tilde{w}(r^{\mathrm{d}}) &= w(r^{\mathrm{d}})(re^{-\beta h})^{1/d} \end{aligned} \tag{A.16}$$
$$\tilde{w}(\,v\,) = w(v)(se^{\beta h} + re^{-\beta h})^{1/d}.$$

The weights $\tilde{w}$ have one significant advantage over w. Namely, the weight W (see (2.3)) corresponding to constant configurations (which appear, standardly, in the denominator estimate) is given *exactly* by taking the product of the respective $\tilde{w}$'s. For non-constant configurations we obtain the following estimate.

Lemma A.3. *Given a configuration $\boldsymbol{\omega}$, let $\tilde{C}_R(\boldsymbol{\omega})$ and $\tilde{C}_S(\boldsymbol{\omega})$ denote the number of connected r- and s-ordered components containing at least one bond. Further, let $\mathcal{P}_{\mathcal{R}}(\boldsymbol{\omega})$ and $\mathcal{P}_{\mathcal{S}}(\boldsymbol{\omega})$ denote the number of mismatched r- and s-pairs in $\boldsymbol{\omega}$. Then*

$$W^{(\beta,h)}_{s,r;\Lambda}(\boldsymbol{\omega}) \leq r^{\tilde{C}_R(\boldsymbol{\omega})} s^{\tilde{C}_S(\boldsymbol{\omega})} \Big[\prod_{b \in \mathbb{B}(\mathcal{T}_L)} \tilde{w}(\boldsymbol{\omega}) \Big] r^{-\mathcal{P}_{\mathcal{R}}(\boldsymbol{\omega})/2d} s^{-\mathcal{P}_{\mathcal{S}}(\boldsymbol{\omega})/2d}.$$

Proof. Since the *a priori* factors of W (i.e., the square bracket in (2.3)) are trivially reproduced, we just have to show that neither the site-terms nor the terms counting the connected components have decreased.

To see the former observe that the vertex terms $e^{\pm\beta h}$ and $se^{\beta h}+re^{-\beta h}$ can be split equally over the neighboring bonds, giving rise to powers $1/d$ in (3.1), where for the vacant bond adjacent to a non-vacant one we used $se^{\beta h}, re^{-\beta h} \le se^{\beta h}+re^{-\beta h}$. The same holds for numbers s, r inherent to isolated vertices of the s- (r-)disorder (note that the additional negative powers $r^{-\mathcal{P}_{\mathcal{R}}(\boldsymbol{\omega})/2d}$ and $s^{-\mathcal{P}_{\mathcal{R}}(\boldsymbol{\omega})/2d}$ partly compensate for the mismatched pairs where neither s nor r are needed). Finally, the non-trivial connected components are dominated by $r^{\tilde{C}_R(\boldsymbol{\omega})}$ and $s^{\tilde{C}_S(\boldsymbol{\omega})}$, respectively. $\square$

Proof of Lemma III.1. As follows from the chessboard estimates (Lemma II.3), it suffices to prove that a each particular bad pattern a (i.e., a 'graphical' configuration on a cube c) has small probability. We use $\mathcal{Z}_a$ for the partition function constrained on the disseminated pattern a, i.e., $\mathcal{Z}_a=\sum_{\boldsymbol{\omega}} W^{(\beta,h)}_{s,r;\mathcal{T}}(\boldsymbol{\omega})\bigl[\chi_a(\mathcal{T}_L)\bigr](\boldsymbol{\omega})$.

It will be important to know, for counting the connecting components under the indicator $\chi_a(\mathcal{T}_L)$, how many connected components are there within the pattern a itself. Let us use $\tilde{C}_R(a)$ and $\tilde{C}_S(a)$ to denote the number of nontrivial (i.e., containing at least one bond) r and s-ordered components of the configuration a on the cube c. It is an elementary observation that each such component gives rise to at most $(L/2)^{d-1}$ connected components in the disseminated configuration. Indeed, a single bond is disseminated just into $(L/2)^{d-1}$ parallel lines through the torus. Similarly, each mismatched pair yields in total $L^2(L/2)^{d-2}=4(L/2)^d$ clones in the dissemination; there are L^2 clones in the plane containing the initial pair and the plane itself is replicated into $(L/2)^{d-2}$ parallel planes.

Note that every bond is shared by a total of $d2^{d-1}$ elementary cubes. Then, under the condition that $s\le r$, Lemma A.3 implies

$$\mathcal{Z}_a \le r^{[\tilde{C}_R(a)+\tilde{C}_S(a)](L/2)^{d-1}}\Bigl[\prod_{b\in c}\tilde{w}(a_b)\Bigr]^{L^d/2^{d-1}} s^{-\frac{2}{d}[\mathcal{P}_{\mathcal{S}}(a)+\mathcal{P}_{\mathcal{R}}(a)](L/2)^d}, \tag{A.17}$$

with $\mathcal{P}_{\mathcal{R}}(a)$ and $\mathcal{P}_{\mathcal{S}}(a)$ denoting the number of mismatched pairs in a. Since, trivially, the full partition sum $\mathcal{Z}\ge\sum_{\alpha\in I}\tilde{w}(\alpha)^{dL^d}$, by Lemma II.3 we get

$$\langle\,\chi_a(c)\,\rangle_L \le \Bigl(\frac{\mathcal{Z}_a}{\mathcal{Z}}\Bigr)^{\frac{1}{|\mathcal{T}_L|}} \le r^{\frac{1}{L}}\Biggl[\prod_{b\in c}\frac{\tilde{w}(a_b)}{\max_{\alpha\in I}\tilde{w}(\alpha)}\Biggr]^{\frac{1}{2^{d-1}}} s^{-\frac{1}{d2^{d-1}}[\mathcal{P}_{\mathcal{S}}(a)+\mathcal{P}_{\mathcal{R}}(a)]}, \tag{A.18}$$

where we used that $\tilde{C}_R+\tilde{C}_S\le 2^{d-1}$, with the r.h.s. corresponding to a 'dimer' covering of the elementary cube. Now, the pattern a is bad and thus it contains either a vacant bond or a mismatched pair. Since

$$\frac{\tilde{w}(v)}{\max_\alpha\tilde{w}(\alpha)} \le \frac{2^{\frac{1}{d}}}{e^{\beta\kappa}-1}, \tag{A.19}$$

(compare (3.3)) the total probability that a bad pattern occurs is simply computed by enumerating all possible arrangements of the pattern a. In this way we get the bound bounded as

$$\sum_{\substack{a \\ \text{bad pattern}}} \langle \chi_a(c) \rangle_L \le r^{\frac{1}{L}} \left(\frac{d2^d}{2} \left(\frac{2^{\frac{1}{d}}}{e^{\beta\kappa}-1} \right)^{-\frac{1}{2^{d-1}}} + 2^d d(d-1)\, s^{-\frac{1}{d2^{d-1}}} \right) \tag{A.20}$$

once L is large enough. To count the possible configurations, we used that there are $d(d-1)2^{d-1}$ places to put a mismatched pair on the cube, and that each has two possible colors. By setting the quantity on the r.h.s. equal to δ the proof is finished. $\square$

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REFLECTION POSITIVITY OF THE RANDOM-CLUSTER MEASURE INVALIDATED FOR NONINTEGER q

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Abstract. We consider the random-cluster Potts measure on lattice torus that weights each connected component by a positive number q. We show, by constructing a counterexample, that this measure is not reflection-positive, unless q is integer.

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1. INTRODUCTION

Reflection positivity (RP) has long been known to set up a framework for establishing discontinuous phase transitions in lattice systems [FSS][FL][FILS][KS][S][CKS]. As has been shown recently, in the context of models allowing for a graphical representation, its combination with the latter can substantially reduce the length of the proofs of phase coexistence [BCK][CM]. In the two-dimensional q-state Potts model, this technique also offers an easy way to establish that the transition occurs exactly at the self-dual point [CM] and to improve the method-required bound on q (L. Chayes, private communication).

The graphical equivalent of the q-state Potts model is the random-cluster measure (RCM) [FK]. If we refrain from discussing boundary conditions, RCM is a Bernoulli bond-process modified by assigning the number q to each connected component. Since q is not stipulated to be integer in this definition, RCM enables one to think of 'extending' the q-state Potts model to non-integer spin-numbers. Similarly, such 'extensions' turn out to exist also for various alterations of the Potts model, see e.g. [BCK].

It would be plausible to expect that the above technology based on merging RP with graphical representations encompasses also the continuous 'extensions' of the Potts models. Apparently, this is not the case, mainly for the lack of RP—the existing proofs go through the Edwards-Sokal coupling [ES] back to the q-state Potts model and, consequently, demand that q be integer. For continuous q, one has been left only with speculations (e.g., about a mapping onto the six-vertex model (L. Chayes, private communication)), so far, that might yield a direct proof of RP at least in some limited domains of q.

In this paper we demonstrate that, in fact, RP does not hold in the random-cluster measure when q is not integer. It should be emphasized that this does not disqualify the order/disorder phase transition in RCM, since the latter is easily inferred from the discrete case by monotonicity in q. The counterexample we construct involves functions that are highly non-local. Therefore, neither the possibility that reflection positivity can be recovered in infinite volume for, e.g., local functions is entirely ruled out.

The rest of the paper is organized as follows. In the next section we give a definition of reflection positivity and state the main result (Theorem 1). The proof comes in the third section. The main tool is a suitable representation derived in Proposition 2, that we believe is of some own interest. In fact, its form suggest a closed link with the 'convexity-violation' arguments of [GG], to which the failure of reflection positivity proved here adds another 'canonical' example. In the proof, we are predominantly concerned with RP w.r.t. hyperplanes containing sites. However, as is commented on in the end, the case of reflection positivity w.r.t. hyperplanes intersecting bonds is fairly analogous.

2. DEFINITIONS AND MAIN RESULT

Let $\mathcal{T}_N$ be a lattice torus of linear size N, with N being an even integer. We use $\mathbb{B}(\mathcal{T}_N)$ to denote the set of bonds of $\mathcal{T}_N$. Let each bond $b \in \mathbb{B}(\mathcal{T}_N)$ be assigned a variable ω_b taking on values 0 or 1. Let $p \in [0,1]$ and $q > 0$. Then the *random-cluster measure* (RCM) on torus $\mathcal{T}_N$ with parameters p and q is a probability measure on $\{0,1\}^{\mathbb{B}(\mathcal{T}_N)}$ weighting the configuration $\boldsymbol{\omega} = (\omega_b)_{b\in\mathbb{B}(\mathcal{T}_N)}$ by the expression

$$\mathbb{P}_{p,q}(\boldsymbol{\omega}) = \frac{1}{Z_N}\, p^{|N_1(\boldsymbol{\omega})|}(1-p)^{|N_0(\boldsymbol{\omega})|} q^{C(\boldsymbol{\omega})}. \tag{1}$$

Here $N_1(\boldsymbol{\omega}) = \{b : \omega_b = 1\}$, $N_0(\boldsymbol{\omega}) = \mathbb{B}(\mathcal{T}_N) \smallsetminus N_1(\boldsymbol{\omega})$, and $C(\boldsymbol{\omega})$ is the number of all connected components that arise from $\mathcal{T}_N$ after cutting all bonds from $N_0(\boldsymbol{\omega})$ (thus, $C(\boldsymbol{\omega})$ includes also the isolated sites). The partition function Z_N provides the appropriate normalization. The expectation w.r.t. $\mathbb{P}_{p,q}$ we denote by $\mathbb{E}$.

Let $P \subset \mathcal{T}_N$ be a hyperplane containing sites, orthogonal to one of the coordinate directions (we think of P as composed of two antipodal components). We use the symbol ϑ_P to denote the reflection w.r.t. P. The components of P divide $\mathcal{T}_N$ into two connected parts $\mathcal{T}_N^{\mathcal{L}}$ and $\mathcal{T}_N^{\mathcal{R}}$, such that $P = \mathcal{T}_N^{\mathcal{L}} \cap \mathcal{T}_N^{\mathcal{R}}$ and $\vartheta_P \mathcal{T}_N^{\mathcal{L}} = \mathcal{T}_N^{\mathcal{R}}$. We use $\mathcal{F}_{\mathcal{L}}$ ($\mathcal{F}_{\mathcal{R}}$) to denote the set of observables (functions) depending only on $\boldsymbol{\omega}|_{\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})}$ ($\boldsymbol{\omega}|_{\mathbb{B}(\mathcal{T}_N^{\mathcal{R}})}$, respectively), where $\mathbb{B}(A)$ stands for the set of bonds whose both ends lie in $A \subset \mathcal{T}_N$.

Definition. *Suppose a measure $\tilde{\mathbb{P}}$ acting on $\boldsymbol{\omega}$ be given, with the expectation $\tilde{\mathbb{E}}$. We say that $\tilde{\mathbb{P}}$ is reflection-positive if for every $f, g \in \mathcal{F}_{\mathcal{L}}$*

$$\tilde{\mathbb{E}}(f\vartheta_P g) = \tilde{\mathbb{E}}(g\vartheta_P f), \tag{2}$$

$$\tilde{\mathbb{E}}(f\vartheta_P f) \geq 0. \tag{3}$$

The condition (2) typically follows directly from the symmetry of $\tilde{\mathbb{P}}$ w.r.t. the reflection ϑ_P, so the difficult part to check is (3) (hence also the name). Now we can state our result:

Theorem 1. *Let $q \leq N^{d-1}$ and $p \in (0,1)$. Then the random-cluster measure $\mathbb{P}_{p,q}$ on $\mathcal{T}_N$ is reflection-positive if and only if q is a positive integer.*

Remark. The singular case of $p = 0$ or 1 leads to $\mathbb{P}_{p,q}$ supported on a single configuration. It follows that $\mathbb{P}_{0,q}$ and $\mathbb{P}_{1,q}$ are reflection-positive for all $q > 0$.

3. PROOF

Fix a hyperplane $P \subset \mathcal{T}_N$. The main idea of the proof is to classify the configurations in $\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})$ according to the partitions of P into connected sets induced thereby. In this way a representation of the l.h.s. of (3) can be derived that bids an opportunity to choose f so that reflection positivity of RCM is violated.

Let $\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P$ be a configuration in $\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})$. Then there is a one-to-one correspondence between the partitions of P into sets whose elements are mutually connected via $N_1(\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P)$, and the graphs G on P whose components are complete graphs. Namely, an edge $(i,j) \in G$ iff i and j are connected (i.e., absence of the edge means that the sites are disconnected). We use χ_G to indicate configurations giving rise to G. Let

$$\tilde{W}_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P) = \frac{1}{\sqrt{Z_N}}\, p^{|N_1(\boldsymbol{\omega}_{\mathcal{L}})|+|N_1(\boldsymbol{\omega}_P)|/2}\,(1-p)^{|N_0(\boldsymbol{\omega}_{\mathcal{L}})|+|N_0(\boldsymbol{\omega}_P)|/2}\, q^{C_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}})}, \tag{4}$$

where $C_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}})$ denotes the number of the connected components of $\boldsymbol{\omega}_{\mathcal{L}}$ disconnected from the set $N_1(\boldsymbol{\omega}_P)$. Then we have the following representation of (3):

Proposition 2. *For each $f \in \mathcal{F}_{\mathcal{L}}$, G, and $\boldsymbol{\omega}_P$ let $f_G(\boldsymbol{\omega}_P) = \sum_{\boldsymbol{\omega}_{\mathcal{L}}}[f\chi_G\tilde{W}_{\mathcal{L}}](\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P)$. Then*

$$\mathbb{E}(f\vartheta_P f) = \sum_{\boldsymbol{\omega}_P}\sum_G q(q-1)\dots(q-\iota(G)+1)\Big[\sum_{G'\subseteq G} f_{G'}(\boldsymbol{\omega}_P)\Big]^2, \tag{5}$$

where $\iota(G)$ denotes the number of components of G.

Proof. We begin with q integer and then use analytic continuation. For integer q, one can devise a coupling [ES] between RCM and the Potts model with q spins and $e^{-\beta J} = 1-p$. Namely, the measure

$$\mathbb{P}_{\mathrm{ES}}(\boldsymbol{\omega},\boldsymbol{\sigma}) = \frac{1}{Z_N}\, p^{|N_1(\boldsymbol{\omega})|}(1-p)^{|N_0(\boldsymbol{\omega})|}\prod_{\langle i,j\rangle\in N_1(\boldsymbol{\omega})}\delta_{\sigma_i,\sigma_j} \tag{6}$$

(where $\langle\,\cdot\,,\cdot\,\rangle$ denotes nearest-neighbor sites) has RCM of (1) as its $\boldsymbol{\omega}$-marginal, whereas the $\boldsymbol{\sigma}$-marginal is easily identified with the Potts model at the above inverse temperature. If we set

$$W_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P, \boldsymbol{\sigma}_{\mathcal{L}} \vee \boldsymbol{\sigma}_P) = \frac{1}{\sqrt{Z_N}}\, p^{|N_1(\boldsymbol{\omega}_{\mathcal{L}})|}(1-p)^{|N_0(\boldsymbol{\omega}_{\mathcal{L}})|}\prod_{\langle i,j\rangle\in N_1(\boldsymbol{\omega}_{\mathcal{L}})}\delta_{\sigma_i,\sigma_j} \times p^{|N_1(\boldsymbol{\omega}_P)|/2}(1-p)^{|N_0(\boldsymbol{\omega}_P)|/2}, \tag{7}$$

then the Edwards-Sokal measure allows us to represent $\mathbb{E}(f\vartheta_P f)$ as follows

$$\mathbb{E}(f\vartheta_P f) = \sum_{\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P} \Delta(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\Big[\sum_{\boldsymbol{\omega}_{\mathcal{L}}} f(\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P)\sum_{\boldsymbol{\sigma}_{\mathcal{L}}} W_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P,\boldsymbol{\sigma}_{\mathcal{L}}\vee\boldsymbol{\sigma}_P)\Big]^2, \qquad (8)$$

where the function

$$\Delta(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P) = \prod_{\langle i,j\rangle\in N_1(\boldsymbol{\omega}_P)} \delta_{\sigma_i,\sigma_j} \qquad (9)$$

guards that $\boldsymbol{\sigma}_P$ stays consistent with $\boldsymbol{\omega}_P$.

Now we would like to get the above graphs into play. This is done by observing that unity can be written as

$$\prod_{\substack{(i,j)\\ i,j\in P}} \left[\delta_{\sigma_i,\sigma_j} + (1-\delta_{\sigma_i,\sigma_j})\right] = \sum_G \prod_{(i,j)\in G}\delta_{\sigma_i,\sigma_j}\prod_{(i,j)\notin G}(1-\delta_{\sigma_i,\sigma_j}). \qquad (10)$$

We insert this expression in (8), just before the square bracket. If $\boldsymbol{\omega}_P$ and $\boldsymbol{\sigma}_P$ are such that the product of Δ- and δ-factors equals one, then two important consequences can be drawn for the summations inside the brackets: first, the graph corresponding to $\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P$ must be a subgraph of G (sites of P with different spins must not be connected). Second, if also the latter holds, then the summation over $\boldsymbol{\sigma}_{\mathcal{L}}$ yields a number independent of $\boldsymbol{\sigma}_P$. This number is easily identified with $\tilde{W}_{\mathcal{L}}$ from (4).

With these findings, (8) can be rewritten as

$$\begin{aligned}\mathbb{E}(f\vartheta_P f) = \sum_{\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P}\sum_G \Delta(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\prod_{(i,j)\in G}\delta_{\sigma_i,\sigma_j}\prod_{(i,j)\notin G}(1-\delta_{\sigma_i,\sigma_j})&\\ \times\Big[\sum_{\boldsymbol{\omega}_{\mathcal{L}}} f(\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P)\tilde{W}_{\mathcal{L}}(\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P)\sum_{G'\subseteq G}\chi_{G'}(\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P)\Big]^2. &\qquad (11)\end{aligned}$$

Note that the very last sume gives exactly one whenever $\boldsymbol{\omega}_{\mathcal{L}}\vee\boldsymbol{\omega}_P$ is consistent with G. Now there is nothing to constrain the summation over $\boldsymbol{\sigma}_P$ any more (note that also $\Delta(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)=1$ automatically for $\boldsymbol{\omega}_P$, $\boldsymbol{\sigma}_P$ consistent with G), which gives us the desired claim for all q integer.

For q non-integer, we use continuation in q. First observe that (5) is expressed purely in terms of RCM. Then multiplying both sides by Z_N, we recover an equality between polynomials in q. Since (5) holds for all positive integers, we conclude that it holds for all q real (in fact, even q complex, by continuity). $\square$

Proof of Theorem 1.

The integer case is a direct consequence of (5). For q non-integer, we describe a counterexample. Let us choose $\boldsymbol{\omega}_P$ such that the set of its connected components $C(\boldsymbol{\omega}_P)$ is large enough. Let us set

$$f = \delta_{\boldsymbol{\omega}_P} \sum_G a_G \chi_G, \tag{12}$$

where $\delta_{\boldsymbol{\omega}_P}$ induces $\boldsymbol{\omega}_P$ on $\mathbb{B}(P)$ and the sum restricts to such G that $b_G = \sum_{\boldsymbol{\omega}_{\mathcal{L}}} [\chi_G \tilde{W}_{\mathcal{L}}](\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P) > 0$. We gather the latter graphs in the set $\mathcal{G}$. Note that $\mathcal{G} \neq \emptyset$, since the *minimal* graph $\bar{G}$, exhibiting only the connections within $\boldsymbol{\omega}_P$, always belongs to $\mathcal{G}$ (as $p \neq 0, 1$, the configuration $\boldsymbol{\omega}_{\mathcal{L}} \vee \boldsymbol{\omega}_P$ with $N_1(\boldsymbol{\omega}_{\mathcal{L}}) = \emptyset$ gets always some non-zero weight from $\tilde{W}_{\mathcal{L}}$).

We shall show that the numbers a_G can be chosen so that the system of $|\mathcal{G}|$ linear equations

$$\sum_{\substack{G, G' \in \mathcal{G} \\ G' \subseteq G}} a_{G'} b_{G'} = \Delta_{\bar{G}}(G), \tag{13}$$

where $\Delta_{\bar{G}}(\,\cdot\,)$ indicates $\bar{G}$, are satisfied. For that let us introduce a *linear* order $\prec$ on $\mathcal{G}$, respecting inclusions (i.e., $G \subseteq G'$ implies $G \prec G'$). Such a linear order always exists, as can be easily proved by induction. In the basis labeled according to $\prec$, the l.h.s. of (13) is clearly represented by the *lower-triangular* matrix $B_{G,G'} = b_{G'} 1_{\{G' \subseteq G\}}$, with all the diagonal entries non-vanishing. Consequently, $B_{G,G'}$ is invertible and (13) can be solved in favor of a non-trivial $(a_G)_{G \in \mathcal{G}}$.

Now it remains to convince oneself that the formulas (12,13) imply that

$$\mathbb{E}(f \vartheta_P f) = q(q-1)\dots(q - |C(\boldsymbol{\omega}_P)| + 1), \tag{14}$$

which boils down to checking that only the term with $G = \bar{G}$ contributes to the sum over G in (5). If we now choose $\boldsymbol{\omega}_P$ such that $q + 2 > |C(\boldsymbol{\omega}_P)| > q + 1$, then the expression (14) is blatantly negative, since only the last term $(q - |C(\boldsymbol{\omega}_P)| + 1) < 0$. Hence, (3) does not hold in general, unless q is integer. □

Remark. The proof we just gave deals with RP w.r.t. hyperplanes on sites. The argument is readily adapted also to the case of hyperplanes intersecting bonds. Namely, in order to derive (5) one has to mind only the following: first, $\boldsymbol{\omega}_P$ is the restriction of $\boldsymbol{\omega}$ to the bonds that intersect P, second, the connectedness issues handled above by χ_G concern now the bonds from $N_1(\boldsymbol{\omega}_P)$, and third, it is the spins congruent with these bonds that are responsible for the crucial prefactor in (5). Once the relation (5) is established, the formulas (12-14) can be taken over almost literally to obtain the desired result.

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GIBBS STRUCTURE AND PHASE COEXISTENCE IN THE POTTS MODEL WITH EXTERNAL FIELDS

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Abstract. We study the $(q+2)$-state Potts model with each of the spin values coupled to a distinct external field. Our results are twofold: First, for q large and only two of the fields non-vanishing we establish, by adapting the Pirogov-Sinai method to an appropriate graphical representation, the phase diagram of the model except for a neighborhood of certain (presumably second-order) transition lines. Second, we develop the Gibbs formalism for the coupled measures on Potts spins and Fortuin-Kasteleyn bonds and use it to derive both general structure theorems as well as specific percolation characterizations of the phase coexistence. By combining these techniques, some of the transition lines not covered by the perturbation analysis can further be studied.

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1. INTRODUCTION

In this paper we study the ferromagnetic (q+2)-state Potts model with each value of the spin being affected by a separate external field. The (formal) Hamiltonian of the model is

$$H(\boldsymbol{\sigma}) = -J \sum_{\langle x,y\rangle} \delta_{\sigma_x,\sigma_y} - \sum_{m=1}^{q+2} \sum_x h_m \delta_{\sigma_x,m}. \tag{1.1}$$

Here J is a positive coupling constant, $(h_m)_{m=1}^{q+2}$ are real numbers standing for the external fields, and $\langle \cdot\,,\cdot\rangle$ denotes a nearest-neighbor pair on $\mathbb{Z}^d$.

The q-state Potts model is a system well known for the presence of an order/disorder transition established rigorously in [KS]. This transition is essentially entropy driven, marking the coexistence of q low-energy ordered states and a high-entropy disordered state. Moreover, the distinction between the ordered and disordered states is the sharper the larger the single-spin space is. Unlike the Potts interaction, the second term in (1.1) gives rise to symmetry-breaking within the family of q Potts spins, with the associated transitions driven thus by a pure energy-effect. The reduction of symmetry effectively diminishes the size of the single-spin state space; it is thus of interest to analyze how these transition mechanisms compete (or cooperate) when the fields are switched on.

The second interesting aspect of the Potts system is that it allows for a reformulation in terms of a graphical representation, the so-called Fortuin-Kasteleyn representation [FK]. This representation provides a natural interpolation between the Potts, the Ising and the percolation models, and it features a couple of very useful FKG properties (see [H] and [CM] for an overview and related representations). Unfortunately, the nearest-neighbor interaction (1.1) translates into long-range one in this representation—an effect that hinders the application of the standard Gibbs formalism. Progress in this direction has been made (see [vEFS, Section 4.5.3], [BoC] for a partial solution and [Gr], [PVdV] for a complete solution in the case of the free and wired measures); nevertheless, the state of matters is still rather unsatisfactory.

The results we present in this paper can be split into two independent parts according to which of the two aforementioned subjects of interest they appeal. First we demonstrate, how an explicit control over the phase diagram can be obtained via the expansion techniques based on Pirogov-Sinai theory ([PSa], [PSb], [Z], [BI]). In particular, we compute explicitly the phase diagrams for the special cases when only h_1 is non-zero (one-field case), $h_1 = h_2$ and the others vanish (equal-field case), and finally the case when h_1, h_2 are arbitrary and the rest of the fields are zero. Unlike the original spin-based expansions [BKL], [M] of the Potts model, our expansion is based on a 'colored' random-cluster representation, going very much in the spirit of the previous studies [LMS], [LMMRS], [BKM], and also [vEFK] where even some particular form of an external field was considered. A detailed discussion of our results in this direction is given in Section 2.

The second part of our results addresses the general theory of Gibbs measures in systems with a graphical representation. In particular, by elaborating the classical Gibbs theory in the context of the Edwards-Sokal coupling measures [ES], we state and prove various structure theorems on the corresponding sets of spin, Edwards-Sokal and 'graphical' Gibbs measures (for a definition of the latter see [Gr]). Using the developed machinery, we also give a couple of percolation characterizations addressing the issues of both phase coexistence and Gibbs uniqueness. These are mainly generalizations of the corresponding results for the 'pure' q-state Potts model to be found, e.g., in [ACCN].

The organization of this paper is as follows. In the next section, we state precisely our result on the phase structure in the one-field, equal-field and two-field cases, and show the relevant phase diagrams in all these cases. In Section 3, the concept of Edwards-Sokal Gibbs measures is developed and the aforementioned structure theorems are stated. Section 4 gives detailed proofs of the latter. In Section 5 contour expansion techniques are set up on the basis of the 'colored' graphical representation. In Section 6, the theorem on the phase structure is proved.

2. THE TWO-FIELD MODEL, RESULTS AND PHASE DIAGRAMS

Let $\boldsymbol{\sigma} = (\sigma_x)_{x\in\mathbb{Z}^d}$ denote the configurations of the $q+2$–state (ferromagnetic) Potts model on the $\mathbb{Z}^d$ lattice. Consider the Hamiltonian (1.1), with $h_3 = \cdots = h_{q+2} = 0$. The fields h_1 and h_2 can take on arbitrary real values; the interesting particular cases are either when one of them vanishes or when both are equal. We shall treat the latter cases in a unified way: let us introduce two sets of spins $\mathcal{Q}$ and $\mathcal{K}$, where $|\mathcal{Q}| = q+1$, $|\mathcal{K}| = 1$ for the one-field case and $|\mathcal{Q}| = q$, $|\mathcal{K}| = 2$ for the equal-field case. The phase diagrams are as depicted in Fig. 1, with ordered phases—concentrated mostly on $\mathcal{Q}$ or $\mathcal{K}$ depending on the values of h_1 and h_2—dominating low temperatures, and a disordered state at higher temperatures.

One-field case. If $h_1 \le 0$ with all h_i such that $2 \le i \le q+2$ set to zero, the phase diagram looks very much like its standard-Potts counterpart. However, when $h_1 > 0$, only two phases appear in the phase diagram: the $\mathcal{K}$-ordered and the disordered phase. As h_1 increases, both coexisting states are more and more dominated by a single spin, which eventually forces them to become one. And, indeed, we establish, by means of the high-fugacity expansion, analyticity of the free energy once h_1 is large enough. The same thing follows for low enough β by the standard high-temperature expansion. It turns out that only a *bounded* region is left outside the scope of the perturbation methods—hence, there is no other way than that the coexistence line *terminates* somewhere inside the shaded region as is depicted in Fig. 1.

In general, it is by no means clear how the points of coexistence are arranged inside the shaded region. In particular, it is not *a priori* excluded that the line smears out into an open set of points in the parameter plane. However, in $d = 2$, the result of [GKR] asserting that the percolating cluster contains an infinite set of nested circles, can be used to establish that the

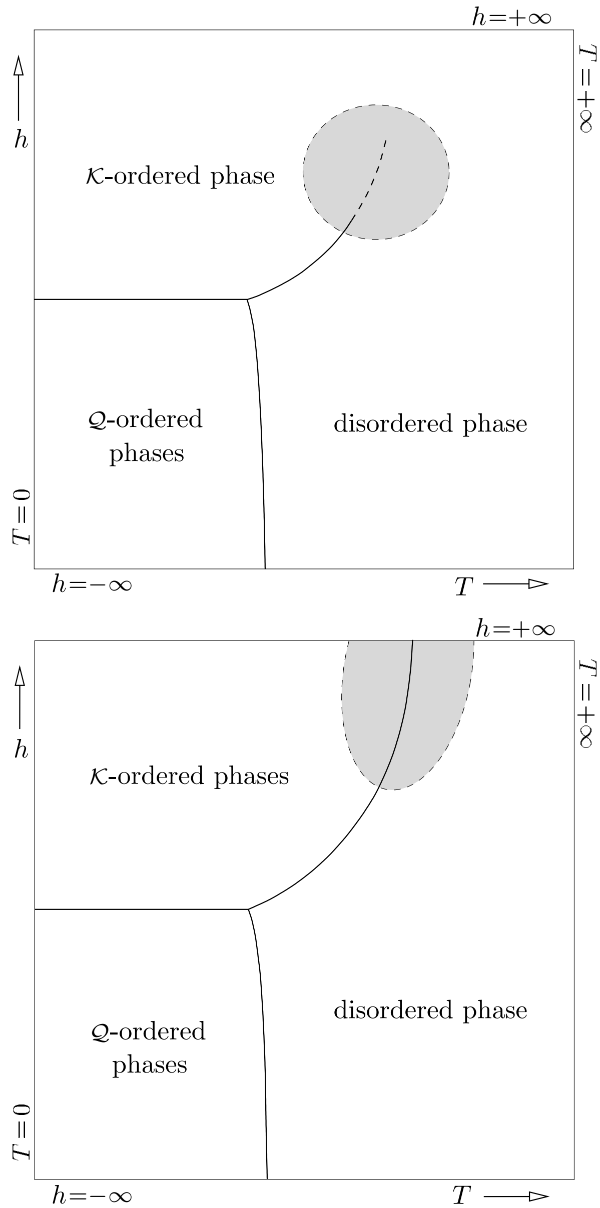

FIG. 1. Phase diagram of the Potts model with one or two equal external fields (i.e., $h_1 = h_2 = h$ in the latter case). The shaded region denotes the range of parameters, where the perturbative methods used in this paper fail to converge. Notwithstanding, the transition lines in the second diagram and, for $d = 2$, also in the first diagram, are everywhere well defined by percolation arguments.

Gibbs measure is unique everywhere off the percolation threshold of the corresponding bonds in the graphical representation (see Theorem III.4(iv)). Consequently, if there is a coexistence of *distinct* phases, it is locked onto the percolation line. Notably, this result holds for all $q \geq 1$; in particular, it can be regarded as an extension of [BaC]. Unfortunately, even for $d = 2$, the uniqueness of the end-point remains an open question.

Equal-field case. A similar order/disorder transition appears in the case when $|\mathcal{K}| = 2$ and the common value of the fields $h_1 = h_2 = h > 0$. Since the system effectivelly approaches the Ising model as $h_1 = h_2 \to +\infty$, for which the transition in temperature is conjectured to be of second order, a question arises whether also in this case the phase coexistence might end before $h = \infty$ is reached. Unfortunately, we are unable to draw any rigorous conclusion in this case, and that not even in the case $d = 2$, where the above conjecture on the Ising transition is proved rigorously [AB] (see [BL] for interesting conclusions if the contrary occurs in $d > 2$). In fact, neither are we able to tell whether the existence of states of broken symmetry in $\mathcal{K}$-family on the percolation line implies also the order/disorder coexistence (the reverse implication is true by a simple percolation characterization). Thus, even the possibility of having *two* critical points (i.e., one for the termination of the order/disorder coexistence and the other for the termination of the symmetry-breaking coexistence line) remains unresolved.

Two-field case. The case of both fields h_1 and h_2 taking non-degenerate values is also of some own interest. Namely, the phase diagram for constant inverse temperature β undergoes several interesting changes as β varies. Let β_c^q denote the critical temperature of the q-state Potts model. We know, e.g., from [ACCN], that β_c^q is strictly increasing in q.

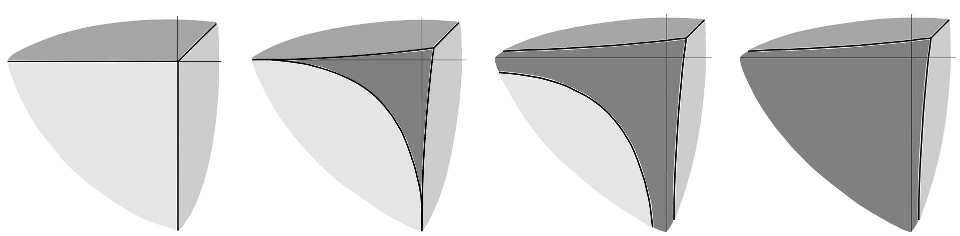

Fig. 2. Phase diagram (in external fields) at four different fixed temperatures. The boundary values for the depicted generic behaviors are $\beta = \beta_c^{q+2}$, β_c^{q+1}, and β_c^q.

There are four temperature regions yielding four different phase diagrams as shown in Fig. 2. Starting from low temperatures, $\beta > \beta_c^{q+2}$, only the ordered states (i.e., the 1-ordered, 2-ordered and q-ordered states, depending on the values of the fields) appear in the phase diagram, occupying *exactly* the same regions as at infinite β. Once β decreases below the value β_c^{q+2}, a 'triangle' of the disordered phase begins to grow in the center of the diagram (see Fig. 3 for an explicit computation of the phase diagram in this parameter range as drawn by Maple using the Pirogov-Sinai expansion), whose 'hypothenuse' increases in lenght as β decreases. At β_c^{q+1}, this triangle becomes infinite (the disordered phase then occupies an unbounded region), nevertheless, the q-ordered phases keep lingering at the lower-left corner of the phase

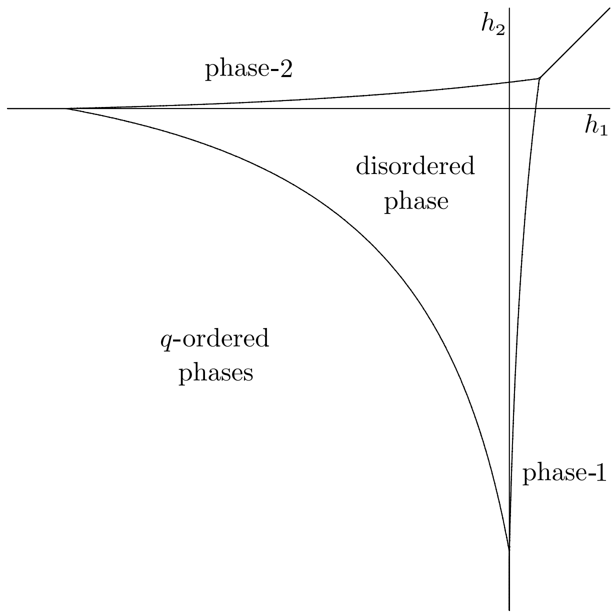

FIG. 3. The most interesting section of constant β through the phase diagram of the Potts model with two fields, as computed using the Pirogov-Sinai theory. For $\beta_c^{q+2} < \beta < \beta_c^{q+1}$, the disordered phase occupies a bounded region in the (h_1, h_2)–plane.

diagram. By futher decreasing β, also these recede towards $h_1, h_2 = -\infty$ to disappear, at β_c^q, for good from the phase diagram. Finally, for very small β (not shown in Fig. 2) there is only one phase. Here is a concise account of our main results:

Theorem II.1. *Consider the $q+2$-state Potts model with interaction (1.1) with $h_2 = \cdots = h_{q+2} = 0$. Let $q \gg 1$.*

(1) If $h_2 = 0$, then there is an $\bar{h} > 0$ such that the set $\mathfrak{R} = \{(\beta, h_1)\colon h_1 \le \bar{h}\}$ decomposes into three closed (non-disjoint) regions: $\mathcal{Q}$-ordered, $\mathcal{K}$-ordered and disordered region (see Fig. 1), wherein the following translation-invariant states exist: $q+1$ distinct ordered states $\langle\,\cdot\,\rangle_{\mathcal{Q}}^l$ ($l = 2, 3, \ldots, q+2$), a 1-ordered state $\langle\,\cdot\,\rangle_{\mathcal{I}}$, and a disordered state $\langle\,\cdot\,\rangle_{\mathcal{D}}$, respectively. The latter are characterized by the estimates

$$\begin{aligned} &\langle \delta_{\sigma_x \sigma_y} \delta_{\sigma_x, l} \rangle_{\mathcal{Q}}^l \ge 1 - \epsilon \qquad (l = 2, 3, \ldots, q+2) \\ &\langle \delta_{\sigma_x \sigma_y} \delta_{\sigma_x, 1} \rangle_{\mathcal{I}} \ge 1 - \epsilon \\ &\langle 1 - \delta_{\sigma_x \sigma_y} \rangle_{\mathcal{D}} \ge 1 - \epsilon, \end{aligned} \tag{2.1}$$

valid for any pair x and y of nearest neighbors and an $\epsilon = \epsilon(q)$ such that $\epsilon(q) \to 0$ as $q \to \infty$. Moreover, in $\mathfrak{R}$, the phases as well as the coexistence lines can be characterized by percolation *of the corresponding bonds in the graphical representation. If, furthermore, $d = 2$, then the coexistence points of the $\mathcal{K}$-ordered and the disordered phases lie on (a finite segment of) the percolation threshold for the $\mathcal{K}$-ordered bonds. In $\{(\beta, h): h_1 > 0\}$, the thermodynamic phase is unique outside this curve.*

(2) If $h_1 = h_2 = h$, then the parameter space $\mathfrak{S} = \{(\beta, h): 0 \le \beta \le \infty, -\infty \le h \le \infty\}$ splits, as before, into the regions where q distinct $\mathcal{Q}$-ordered states, two $\mathcal{K}$-ordered and a single disordered translation-invariant states exist. These are again characterized by relations (2.1). The coexistence curves are of first-order except possibly inside the shaded region as depicted in Fig. 1, in the vicinity of the point $(\beta, h) = (\beta_c^2, +\infty)$. In both (1) and (2), the Gibbs measure is unique inside the region where the ordered bonds (in the graphical representation) do not percolate.

(3) If h_1, h_2 are general, subject to the condition $h_1, h_2 \le \bar{h}$ for some $\bar{h}$ small positive, and if β is large enough, then the phase diagram for constant β looks as depicted in Fig.$\tilde{2}$. Here the transition lines are obtained by equating the (metastable) free energies computed from the Pirogov-Sinai expansion.

3. EDWARDS-SOKAL AND RANDOM-CLUSTER REPRESENTATIONS

In this section we define the concept of Gibbs measures for joint probability spaces of spin and bond variables. The main results come as Theorems III.1–5, whose proofs are relegated to the next section.

We consider the $(q+2)$-state Potts model in a general field. The Hamiltonian in Λ is given by the formula

$$H(\boldsymbol{\sigma}_\Lambda) = -J \sum_{\langle x,y\rangle \in \mathbb{B}_0(\Lambda)} \delta_{\sigma_x,\sigma_y} - \sum_{x\in\Lambda} \sum_{m=1}^{q+2} h_m \delta_{\sigma_x,m}. \tag{3.1}$$

Here, Λ is a finite subset of $\mathbb{Z}^d$, $\mathbb{B}_0(\Lambda)$ is the set of all bonds $b = \langle x, y\rangle$ of nearest neighbors with both endpoints in Λ, and $(h_m)_{m=1}^{q+2} \in \mathbb{R}^{q+2}$ is a collection of arbitrary fields.

In order to derive the Edwards-Sokal (ES) and random-cluster (RC) representation, we introduce a shorthand

$$h(\sigma_x) = \sum_{m=1}^{q+2} h_m \delta_{\sigma_x,m}, \tag{3.2}$$

and rewrite the Gibbs factor,

$$e^{-\beta H(\boldsymbol{\sigma}_\Lambda)} = \prod_{\langle x,y\rangle \in \mathbb{B}_0(\Lambda)} e^{\beta J \delta_{\sigma_x,\sigma_y}} \prod_{x\in\Lambda} e^{\beta h(\sigma_x)}, \tag{3.3}$$

by expanding each term $e^{\beta J \delta_{\sigma_x,\sigma_y}}$ as $1 + (e^\beta - 1)\delta_{\sigma_x,\sigma_y}$. By introducing bond configurations

$\boldsymbol{\eta}_{\mathbb{B}_0(\Lambda)} = \{\eta_b\}_{b\in\mathbb{B}_0(\Lambda)}$ with $\eta_b \in \{0,1\}$, we can get the Gibbs factor (3.3) as the sum

$$e^{-\beta H(\boldsymbol{\sigma}_\Lambda)} = \sum_{\boldsymbol{\eta}_{\mathbb{B}_0(\Lambda)}} \prod_{\substack{\langle x,y\rangle\in\mathbb{B}_0(\Lambda)\\ \eta_{\langle x,y\rangle}=1}} (e^{\beta J}-1)\delta_{\sigma_x,\sigma_y} \prod_{x\in\Lambda} e^{\beta h(\sigma_x)}. \tag{3.4}$$

In this manner one obtains the finite volume Gibbs measure of the Potts model as the spin marginal of a measure on both spin and bond configurations—the Edwards-Sokal measure. The bond configuration marginal is then the random-cluster measure.

So far we have neglected boundary conditions. Instead of modifying the preceding argument, we rather directly introduce the notion of infinite-volume Gibbs measures on the joint space of spin and bond variables. To define the Gibbs ES states, let us introduce for any (not necessarily related) pair of finite sets $\Lambda \subset \mathbb{Z}^d$, $\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$, and any fixed configurations $\boldsymbol{\sigma}_{\Lambda^c}$, $\boldsymbol{\eta}_{\mathbb{B}^c}$ outside of them (of course, only the values on the boundary matter), the measure $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}(\,\cdot\mid\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c})$ by

$$\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}(\boldsymbol{\sigma}_\Lambda,\boldsymbol{\eta}_{\mathbb{B}}\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c}) = \frac{W(\boldsymbol{\sigma}_\Lambda,\boldsymbol{\eta}_{\mathbb{B}}\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c})}{\sum_{\bar{\boldsymbol{\sigma}}_\Lambda,\bar{\boldsymbol{\eta}}_{\mathbb{B}}} W(\bar{\boldsymbol{\sigma}}_\Lambda,\bar{\boldsymbol{\eta}}_{\mathbb{B}}\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c})}, \tag{3.5}$$

where the convention $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}(\boldsymbol{\sigma}_\Lambda,\boldsymbol{\eta}_{\mathbb{B}}\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c}) = 0$ is assumed for the case that the sum in the denominator vanishes, and where

$$W(\boldsymbol{\sigma}_\Lambda,\boldsymbol{\eta}_{\mathbb{B}}\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c}) = \prod_{\substack{\langle x,y\rangle\in\mathbb{B}\cup\mathbb{B}(\Lambda)\\ \eta_{\langle x,y\rangle}=1}} (e^{\beta J}-1)\delta_{\sigma_x,\sigma_y} \prod_{x\in\Lambda} e^{\beta h(\sigma_x)} \tag{3.6}$$

with $\mathbb{B}(\Lambda)$ denoting the set of all bonds with at least one endpoint in Λ. The dependence on parameters J and $\{h_m\}$ will be explicitly marked out only when a reference to them is needed.

Our first theorem concerns the relation between the ES and spin Gibbs measures. Let $\mathcal{G}^{\mathrm{ES}}$ be the set of all infinite volume Gibbs ES states defined by imposing the DLR equations with specification (3.5). Namely, $\nu \in \mathcal{G}^{\mathrm{ES}}$ iff

$$\nu(f) = \int \nu(\mathrm{d}\boldsymbol{\sigma},\mathrm{d}\boldsymbol{\eta})\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}(f\,|\,\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}^c}) \tag{3.7}$$

for all pairs of finite sets Λ and $\mathbb{B}$ and any cylinder function f depending only on $\boldsymbol{\sigma}_\Lambda$ and $\boldsymbol{\eta}_{\mathbb{B}}$. Note that the fact that the underlying 'set of sites' contains both, the set $\mathbb{Z}^d$ and the set $\mathbb{B}(\mathbb{Z}^d) \equiv \mathbb{B}_0(\mathbb{Z}^d)$, does not prevent the abstract theory of Gibbs states in the version that allows for 'hard-core interactions' (c.f. [Ru], [Ge]) from being applied. The important condition, quasilocality of the specification $\{\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}\}$, is clearly satisfied.

Let $\mathcal{G}^{\mathrm{SPIN}}$ denote the set of all spin Gibbs states, defined by means of the DLR condition and the Hamiltonian (2.1), appropriately modified for incorporating the boundary condition. Let Π denote the mapping that assigns the spin marginal to any infinite volume ES measure. It is not *a priori* obvious that the spin marginal of any infinite-volume Gibbs ES state is an infinite-volume Gibbs spin state. However, it turns out that even a little more is true.

Theorem III.1. *The mapping* Π *is a linear isomorphism between simplices* $\mathcal{G}^{\mathrm{ES}}$ *and* $\mathcal{G}^{\mathrm{SPIN}}$*. When restricted to translation invariant measures,* Π *is an isomorphism between the simplex of all translation invariant Gibbs ES states and the simplex of all translation invariant Gibbs spin states. In particular,* $|\mathcal{G}^{\mathrm{ES}}| = 1$ *iff* $|\mathcal{G}^{\mathrm{SPIN}}| = 1$*.*

Remark. The last statement is false for the correspondence between ES Gibbs states and their random-cluster marginals. For instance, for $d = 2$ it is known that there are exactly two extremal Ising Gibbs states at low temperatures [A,Hi] and, therefore, two extremal ES Gibbs states, while the corresponding random-cluster marginals are identical.

Observing that the state $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c})$ does not depend on $\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c}$, we introduce the measure

$$\mu^{\mathrm{ES}}_{\Lambda,m}(\,\cdot\,) = \mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot\,|\,\boldsymbol{\sigma}^m_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c}), \tag{3.8}$$

where $\boldsymbol{\sigma}^m$ is the constant configuration, $\sigma^m_x = m$ for all $x \in \mathbb{Z}^d$, with $m \in \{1, \dots, q+2\}$. In a similar way, the measure $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}_0(\Lambda)}(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}_0(\Lambda)^c})$ does not depend on $\boldsymbol{\sigma}_{\Lambda^c}$, provided that the $\boldsymbol{\eta}$-boundary condition is chosen as $\boldsymbol{\eta}_{\mathbb{B}_0(\Lambda)^c} = \boldsymbol{\eta}^0_{\mathbb{B}_0(\Lambda)^c}$, where $\boldsymbol{\eta}^0$ denotes the configuration with $\eta^0_b = 0$ for all $b \in \mathbb{B}(\mathbb{Z}^d)$. In this case we introduce the measure

$$\mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}(\,\cdot\,) = \mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}_0(\Lambda)}(\,\cdot\;|\;\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}^0_{\mathbb{B}_0(\Lambda)^c}). \tag{3.9}$$

The $\boldsymbol{\eta}$-marginals of the measures $\mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}(\,\cdot\,)$ and $\mu^{\mathrm{ES}}_{\Lambda,m}$ are the random-cluster measures $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\,\cdot\,)$ and $\mu^{\mathrm{RC}}_{\Lambda,m}$ with *free* and m*-wired* boundary conditions, respectively. A particular role will be played by the RC measures with m-wired boundary conditions such that $h_m = h_{\max} := \max_i h_i$. Note that the measures $\mu^{\mathrm{RC}}_{\Lambda,m}$ are identical for all values m such that $h_m = h_{\max}$; we will use $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$ to denote any of them.

In order to state the next result, we introduce a partial order $\prec$ on $\{0,1\}^{\mathbb{B}(\mathbb{Z}^d)}$ by setting $\boldsymbol{\eta} \prec \boldsymbol{\eta}'$ whenever $\eta_b \le \eta'_b$ for every $b \in \mathbb{B}(\mathbb{Z}^d)$. Recall the following definition:

Definition. Let Ω be a measurable space endowed with a partial order $\prec$. Then the measure μ on Ω is said to be *FKG* if $\mu(FG) \ge \mu(F)\mu(G)$ for all measurable functions $F, G : \Omega \to \mathbb{R}$ that are increasing with respect to $\prec$. Moreover, if Ω is of the form $\Omega = \times_{b\in\mathbb{B}} \Omega_b$, then μ is said to be *strong FKG* if $\mu(\,\cdot\,|A)$ is FKG for all cylinder events of the form $A = \{\boldsymbol{\eta}\colon \eta_b = \alpha_b \;\forall b \in \tilde{\mathbb{B}}\}$, where $\tilde{\mathbb{B}} \subset \mathbb{B}$ is finite and $\alpha_b \in \Omega_b$ for all $b \in \tilde{\mathbb{B}}$.

Let us use $\mathbb{B}_1(\boldsymbol{\eta}) = \{b \in \mathbb{B}(\mathbb{Z}^d)\colon \eta_b = 1\}$ to denote the set of 1-bonds of the configuration $\boldsymbol{\eta}$ and $N_\infty(\boldsymbol{\eta})$ to denote the number of infinite components of $\mathbb{B}_1(\boldsymbol{\eta})$.

Theorem III.2. *Let* $\beta \ge 0$*,* $J \ge 0$*, and* $h_m \in \mathbb{R}$*,* $m = 1, \dots, q+2$*. Let* q_0 *denote the degree of the degeneracy of* $h_{\max}$*, i.e.,* $q_0 = \#\{m\colon h_m = h_{\max}\}$*.*

(i) For each finite $\Lambda \subset \mathbb{Z}^d$ *the measures* $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}$ *and* $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$ *are strong FKG.*

(ii) For each quasilocal function f, the limits

$$\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(f) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(f) \tag{3.10}$$

and

$$\mu^{\mathrm{RC}}_{\mathrm{free}}(f) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(f) \tag{3.11}$$

exist and are translation invariant.

(iii) Let $\nu \in \mathcal{G}^{\mathrm{ES}}$ be arbitrary and let μ denote its $\boldsymbol{\eta}$-marginal. Then

$$\mu(\,\cdot\,) \underset{\mathrm{FKG}}{\leq} \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(\,\cdot\,). \tag{3.12}$$

If, in addition, either $q_0 = 1$ or $\mu(N_\infty \leq 1) = 1$, then

$$\mu(\,\cdot\,) \underset{\mathrm{FKG}}{\geq} \mu^{\mathrm{RC}}_{\mathrm{free}}(\,\cdot\,). \tag{3.13}$$

(iv) Let $J_1 < J_2$ and let $\mu^{\mathrm{RC},J_1}_{\mathrm{maxwir}}$ denote the wired state at $J = J_1$ and let $\mu^{\mathrm{RC},J_2}_{\mathrm{free}}$ denote the free state at $J = J_2$. Then

$$\mu^{\mathrm{RC},J_1}_{\mathrm{maxwir}}(\,\cdot\,) \underset{\mathrm{FKG}}{\leq} \mu^{\mathrm{RC},J_2}_{\mathrm{free}}(\,\cdot\,). \tag{3.14}$$

Remark. The requirement that $h_m = h_{\max}$ is, in fact, crucial for the validity of (i) because $\mu^{\mathrm{RC}}_{\Lambda,m}$ is not strong FKG unless $h_m = h_{\max}$. In fact, for β large enough a contour argument shows that $\mu^{\mathrm{RC}}_{\Lambda,m}$ with $h_m < h_{\max}$ is not even FKG. Also note that (3.14) can be extended via (3.12) and (3.13) to any pair of RC measures at $J = J_1$ resp. $J = J_2$, provided the one at J_2 has at most one infinite cluster.

By using the general theorem on the uniqueness of the infinite cluster [BK], the conclusion about the existence of the limiting RC measures can be strengthened to their ES preimages:

Theorem III.3. *Let $\beta \geq 0$ and $h_m \in \mathbb{R}$, $m = 1, \dots, q+2$. Let $q_0 = \#\{m\colon h_m = h_{\max}\}$.*

If m is such that $h_m = h_{\max}$, then the weak limits

$$\mu^{\mathrm{ES}}_{m} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{ES}}_{\Lambda,m} \tag{3.15}$$

and

$$\mu^{\mathrm{ES}}_{\mathrm{free}} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}} \tag{3.16}$$

exist and are translation invariant.

We say that the ES (or RC) Gibbs measure ν exhibits *percolation* if $\nu(N_\infty > 0) > 0$. Let $\mathcal{G}^{\mathrm{ES}}_{\emptyset} = \{\nu \in \mathcal{G}^{\mathrm{ES}}\colon \nu(N_\infty = 0) = 1\}$ be the set of ES Gibbs measures not exhibiting percolation.

Since 'absence of percolation' is a tail event, the standard extremal decomposition properties of Gibbs states apply to $\mathcal{G}_\emptyset^{\mathrm{ES}}$. In particular, $\mathcal{G}_\emptyset^{\mathrm{ES}}$ is a Choquet simplex.

Denote by $\mu_{\mathrm{maxwir}}^{\mathrm{RC},J}$ the wired state corresponding to the coupling constant J. Given $(h_m)_{m=1}^{q+2}$ and β, the function $J \mapsto \mu_{\mathrm{maxwir}}^{\mathrm{RC},J}(0 \leftrightarrow \infty)$ is monotone increasing and right continuous in J, where $\{0 \leftrightarrow \infty\}$ is the event that the origin belongs to the set of sites of some infinite cluster. The value $J_c = \inf\{J\colon \mu_{\mathrm{maxwir}}^{\mathrm{RC},J}(0 \leftrightarrow \infty) > 0\}$ marks the edge of the region where there is percolation.

Theorem III.4. *Let $\beta \geq 0$ and $h_m \in \mathbb{R}$, $m = 1, \ldots, q+2$. Let $q_0 = \#\{m\colon h_m = h_{\max}\}$.*

(i) $|\mathcal{G}_\emptyset^{\mathrm{ES}}| \leq 1$.

(ii) If there is no percolation in the RC measure $\mu_{\mathrm{maxwir}}^{\mathrm{RC}}$, then there is a unique ES Gibbs state, i.e., $\mu_{\mathrm{maxwir}}^{\mathrm{ES}} \in \mathcal{G}_\emptyset^{\mathrm{ES}} \Rightarrow |\mathcal{G}^{\mathrm{ES}}| = 1$.

(iii) If $q_0 \geq 2$, then there is a unique ES Gibbs state if and only if *there is no percolation in the RC measure $\mu_{\mathrm{maxwir}}^{\mathrm{RC}}$.*

(iv) Let $d = 2$ and let $q_0 = 1$. Then $J \neq J_c$ implies $|\mathcal{G}^{\mathrm{ES}}| = 1$.

Up to now, we avoided using the notion of RC Gibbs measures, since the corresponding finite volume states with fixed boundary conditions are not quasi-local as functions of boundary conditions (see [vEFS,Gr,PVdV,BoC]). Instead, we used the ES Gibbs measures, whose finite volume specifications are local, and introduced RC measures only as their marginals. This approach significantly simplifies our proofs. Bearing in mind that the non-quasilocality of RC specifications prevents the straightforward application of the general theory of Gibbs states, we close this section with a theorem relating RC marginals of ES Gibbs meaasures to RC Gibbs states as introduced in [vEFS,Gr,PVdV,BoC].

We start with some notation: as in (3.5) we define for any finite set of bonds $\mathbb{B}$ and any configuration $\boldsymbol{\eta}_{\mathbb{B}^c}$, the measure

$$\mu_{\mathbb{B}}^{\mathrm{RC}}(\boldsymbol{\eta}_{\mathbb{B}} | \boldsymbol{\eta}_{\mathbb{B}^c}) = \frac{W_{\mathbb{B}}^{\mathrm{RC}}(\boldsymbol{\eta}_{\mathbb{B}} \mid \boldsymbol{\eta}_{\mathbb{B}^c})}{\sum_{\bar{\boldsymbol{\eta}}_{\mathbb{B}}} W_{\mathbb{B}}^{\mathrm{RC}}(\bar{\boldsymbol{\eta}}_{\mathbb{B}} \mid \boldsymbol{\eta}_{\mathbb{B}^c})} \tag{3.17}$$

with

$$W_{\mathbb{B}}^{\mathrm{RC}}(\boldsymbol{\eta}_{\mathbb{B}} \mid \boldsymbol{\eta}_{\mathbb{B}^c}) = (e^{\beta J} - 1)^{|\mathbb{B}_1(\boldsymbol{\eta}_{\mathbb{B}})|} \prod_{C(\boldsymbol{\eta}) : \mathbb{V}(C(\boldsymbol{\eta})) \cap \mathbb{V}(\mathbb{B}) \neq \emptyset} \sum_{m=1}^{q+2} e^{-\beta(h_{\max} - h_m)|\mathbb{V}(C(\boldsymbol{\eta}))|}. \tag{3.18}$$

Here the product runs over all connected components $C(\boldsymbol{\eta})$ of the set $\mathbb{B}_1(\boldsymbol{\eta})$ whose associated set of sites $\mathbb{V}(C(\boldsymbol{\eta}))$ intersects the vertices of $\mathbb{B}$. Interpreting $e^{-\infty} = 0$, any infinite cluster $C(\boldsymbol{\eta})$ intersecting $\mathbb{B}$ contributes just the factor q_0. As usual, one then introduces the set of Gibbs states $\mathcal{G}^{\mathrm{RC}}$ as the set of measures μ on $\{0,1\}^{\mathbb{B}(\mathbb{Z}^d)}$ that satisfy the DLR equation

$$\mu(f) = \int \mu(\mathrm{d}\boldsymbol{\eta})\, \mu_{\mathbb{B}}^{\mathrm{RC}}(f \,|\, \boldsymbol{\eta}_{\mathbb{B}^c}) \tag{3.19}$$

for any finite $\mathbb{B}$ and any cylinder function f with support in $\mathbb{B}$. As already observed in [BoC], this notion of RC Gibbs states does not accomodate all 'naturaly-arising' limiting states. When reformulated in terms of the ES measures, not every RC marginal of an ES Gibbs measure is a RC Gibbs state. To mention a concrete example, consider the ES Gibbs state corresponding to the standard Dobrushin state with a stable interface between two ordered phases.

The first two claims of the following theorem generalize the statements proven previously [Gr,PVdV,BoC] only for the case $h_1 = ... = h_{q+2} = 0$ (on the other hand, however, these authors consider the RC model defined for any (i.e., also noninteger) values of q).

Theorem III.5. *Let $\beta \geq 0$ and $h_m \in \mathbb{R}$, $m = 1, \dots, q+2$. Let $q_0 = \#\{m\colon h_m = h_{\max}\}$. Then*

(i) $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}, \mu^{\mathrm{RC}}_{\mathrm{free}} \in \mathcal{G}^{\mathrm{RC}}$

(ii) If $\mu \in \mathcal{G}^{\mathrm{RC}}$, then $\mu^{\mathrm{RC}}_{\mathrm{free}} \underset{\mathrm{FKG}}{\leq} \mu \underset{\mathrm{FKG}}{\leq} \mu^{\mathrm{RC}}_{\mathrm{maxwir}}$.

(iii) Let μ be the random-cluster marginal of some $\nu \in \mathcal{G}^{\mathrm{ES}}$ with the additional assumption that $\mu(N_\infty > 1) = 0$ whenever $q_0 > 1$. Then $\mu \in \mathcal{G}^{\mathrm{RC}}$.

(iv) Let $\mu \in \mathcal{G}^{\mathrm{RC}}$ with $\mu(N_\infty > 1) = 0$. Then there is some $\nu \in \mathcal{G}^{\mathrm{ES}}$ such that μ is the random cluster marginal of ν.

The results of Theorem III.4(i) and (iv) read in the context of RC Gibbs measures as follows.

Corollary.

(i) If $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(N_\infty = 0) = 1$, then $|\mathcal{G}^{\mathrm{RC}}| = 1$.

(ii) If $d = 2$ and $q_0 = 1$, then $J \neq J_c$ implies $|\mathcal{G}^{\mathrm{RC}}| = 1$.

4. PROOFS OF THEOREMS III.1–5

Theorems III.1–5 are proved in five successive subsections.

4.1. Relation between ES and spin Gibbs measures.

Proof of Theorem III.1. Let $\mu^{\mathrm{SPIN}}_\Lambda(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c})$ denote the Gibbs measure on spins in Λ with boundary condition $\boldsymbol{\sigma}_{\Lambda^c}$. The proof is based on the crucial observations that, for the special choice $\mathbb{B} = \mathbb{B}(\Lambda)$,

(A) $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c})$ does not depend on $\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c}$.

(B) The spin marginal of $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c})$ is precisely $\mu^{\mathrm{SPIN}}_\Lambda(\,\cdot\,|\,\boldsymbol{\sigma}_{\Lambda^c})$.

Let now $\nu \in \mathcal{G}^{\mathrm{ES}}$, $\Lambda \subset \mathbb{Z}^d$ be finite, and let f be a function depending only on the spin configuration in Λ. Then, by (3.7), (A), (B), and the definition of marginals, we have

$$(\Pi\nu)(f) = \nu(f) = \int \nu(\mathrm{d}\boldsymbol{\sigma}, \mathrm{d}\boldsymbol{\eta})\, \mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f|\,\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c}) =$$
$$= \int \nu(\mathrm{d}\boldsymbol{\sigma}, \mathrm{d}\boldsymbol{\eta})\, \mu^{\mathrm{SPIN}}_\Lambda(f|\,\boldsymbol{\sigma}_{\Lambda^c}) = \int (\Pi\nu)(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\mathrm{SPIN}}_\Lambda(f|\,\boldsymbol{\sigma}_{\Lambda^c}), \qquad (4.1)$$

proving that $\Pi\nu \in \mathcal{G}^{\text{SPIN}}$. Hence, indeed, Π is a map from $\mathcal{G}^{\text{ES}}$ to $\mathcal{G}^{\text{SPIN}}$.

To prove that Π is an isomorphism, let us first establish its surjectivity. We begin by noting that the set $\{(\Lambda, \mathbb{B}(\Lambda))\}$ is *cofinal* in the set of all pairs $\{(\Lambda, \mathbb{B})\}$, ordered by inclusion. (Namely, for any $(\Lambda, \mathbb{B})$ there exist $\bar{\Lambda}$ such that $\Lambda \subset \bar{\Lambda}$ and $\mathbb{B} \subset \mathbb{B}(\bar{\Lambda})$.) Then it is easy to see that the validity of (3.7) for the pairs $(\Lambda, \mathbb{B}(\Lambda))$ imply its validity for general $(\Lambda, \mathbb{B})$ (see [Ge, Remark 1.24]). Let now $\mu \in \mathcal{G}^{\text{SPIN}}$ and consider the following ES measure

$$\nu_\Lambda(\,\cdot\,) = \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot \,|\, \boldsymbol{\sigma}_{\Lambda^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^{\mathrm{c}}}) \tag{4.2}$$

on the set of on configurations in $\big(\Lambda, \mathbb{B}(\Lambda)\big)$. Here the configuration $\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^{\mathrm{c}}}$ is added only for the formal completeness since by (A) its value does not matter for ν_Λ. By taking into account the consistency of the finite-volume ES measures $\{\mu^{\text{ES}}_{\Lambda,\mathbb{B}}\}$, the measures $\nu_\Lambda(\,\cdot\,)$ satisfy the restricted DLR equations

$$\nu_\Lambda(f) = \int \nu_\Lambda(\mathrm{d}\boldsymbol{\sigma}, \mathrm{d}\boldsymbol{\eta})\, \mu^{\text{ES}}_{\tilde{\Lambda},\tilde{\mathbb{B}}}(f |\, \boldsymbol{\sigma}_{\tilde{\Lambda}^{\mathrm{c}}}, \boldsymbol{\eta}_{\tilde{\mathbb{B}}^{\mathrm{c}}}) \tag{4.3}$$

for any $\tilde{\Lambda} \subset \Lambda$, $\tilde{\mathbb{B}} \subset \mathbb{B}(\Lambda)$, and any $\tilde{\Lambda}, \tilde{\mathbb{B}}$-cylinder function f. Moreover, let $\Lambda_1 \supset \Lambda_2 \supset \tilde{\Lambda}$ be two sets. Then for any such function f (as before) we have

$$\begin{aligned}
\nu_{\Lambda_1}(f) &= \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{ES}}_{\Lambda_1,\mathbb{B}(\Lambda_1)}(f |\, \boldsymbol{\sigma}_{\Lambda_1^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda_1)^{\mathrm{c}}}) \\
&= \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{ES}}_{\Lambda_1,\mathbb{B}(\Lambda_1)}\big(\mu^{\text{ES}}_{\Lambda_2,\mathbb{B}(\Lambda_2)}(f|\,\cdot\,)\big|\, \boldsymbol{\sigma}_{\Lambda_1^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda_1)^{\mathrm{c}}}\big) \\
&= \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{SPIN}}_{\Lambda_1}\big(\mu^{\text{ES}}_{\Lambda_2,\mathbb{B}(\Lambda_2)}(f|\,\cdot\,)\big|\boldsymbol{\sigma}_{\Lambda_1^{\mathrm{c}}}\big) \\
&= \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{ES}}_{\Lambda_2,\mathbb{B}(\Lambda_2)}(f |\, \boldsymbol{\sigma}_{\Lambda_2^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda_2)^{\mathrm{c}}}) = \nu_{\Lambda_2}(f).
\end{aligned} \tag{4.4}$$

Here the first equality sign is due to (4.2), the second one is a result of Gibbsianness of $\mu^{\text{ES}}_{\Lambda_1,\mathbb{B}(\Lambda_1)}$, the third one is established by applying (A) to $\mu^{\text{ES}}_{\Lambda_2,\mathbb{B}(\Lambda_2)}(f|\,\cdot\,)$ and subsequently (B) to the expectation w.r.t. $\mu^{\text{ES}}_{\Lambda_1,\mathbb{B}(\Lambda_1)}$, and, finally, the fourth equality follows by the Gibbsianness of μ. Consequently, as $\Lambda \nearrow \mathbb{Z}^d$, $\nu_\Lambda(f)$ is eventually a constant for any cylinder function f. In particular, the weak limit $\nu = \lim_{\Lambda \nearrow \mathbb{Z}^d} \nu_\Lambda$ exists and, by (3.19), it satisfies (3.7), i.e., $\nu \in \mathcal{G}^{\text{ES}}$. Finally, $\Pi\nu = \mu$, since for any Λ-cylinder function f of spins

$$(\Pi\nu)(f) = \nu(f) = \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f |\, \boldsymbol{\sigma}_{\Lambda^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Lambda)^{\mathrm{c}}}) = \int \mu(\mathrm{d}\boldsymbol{\sigma})\, \mu^{\text{SPIN}}_{\Lambda}(f |\, \boldsymbol{\sigma}_{\Lambda^{\mathrm{c}}}) = \mu(f), \tag{4.5}$$

proving that Π is surjective.

In order to see that Π is also injective, we notice that if $\tilde{\nu} \in \mathcal{G}^{\text{ES}}$ is such that $\Pi\tilde{\nu} = \mu$, then

$$\tilde{\nu}(f) = \tilde{\nu}\big(\mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f \mid \cdot\,)\big) = (\Pi\tilde{\nu})\big(\mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f \mid \cdot\,)\big) = \mu\big(\mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f \mid \cdot\,)\big) \tag{4.6}$$

for any $(\Lambda, \mathbb{B}(\Lambda))$-cylinder function f. Here the first equation is the DLR equation for $\tilde{\nu}$, the second equation follows from (A), and the third equation is the assumption $\Pi\tilde{\nu} = \mu$. Now, the r.h.s.'s of (4.6) and (4.2) coincide, so $\tilde{\nu} = \nu$, with ν defined by taking the limit $\Lambda \nearrow \mathbb{Z}^d$ of ν_Λ in (4.2). In particular, all measures $\tilde{\nu}$ satisfying $\Pi\tilde{\nu} = \mu$ are equal, yielding thus injectivity of Π.

The part of the claim concerning translation invariant measures is proved in the same way, because both constructions (4.1) and (4.2) preserve translation invariance. □

4.2. FKG properties of the random-cluster measures.

Before beginning the proof of Theorem III.2, let us observe that the measure $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}$ can be explicitly written by normalizing the weights

$$W^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\boldsymbol{\eta}) = (e^{\beta J} - 1)^{|\boldsymbol{\eta}|} \prod_{C(\boldsymbol{\eta})} \Theta_{\mathrm{free}}\bigl(C(\boldsymbol{\eta})\bigr), \tag{4.7}$$

for any $\boldsymbol{\eta} \in \{0,1\}^{\mathbb{B}_0(\Lambda)}$. Here $|\boldsymbol{\eta}|$ is the number of bonds in the set $\{b \in \mathbb{B}_0(\Lambda)\colon \eta_b = 1\}$, the product runs over all connected components $C(\boldsymbol{\eta})$ of this set, and

$$\Theta_{\mathrm{free}}(C) = \sum_{m=1}^{q+2} e^{\beta h_m |\mathbb{V}(C)|}, \tag{4.8}$$

for any connected set of bonds C, with $\mathbb{V}\bigl(C(\boldsymbol{\eta})\bigr)$ denoting the set of sites incidental with $C(\boldsymbol{\eta})$.

Similarly, the measure $\mu^{\mathrm{RC}}_{\Lambda,m}$ is obtained by normalizing the weights $W^{\mathrm{RC}}_{\Lambda,m}$ defined for any $\boldsymbol{\eta} \in \{0,1\}^{\mathbb{B}(\Lambda)}$ by the formula

$$W^{\mathrm{RC}}_{\Lambda,m}(\boldsymbol{\eta}) = (e^{\beta J} - 1)^{|\boldsymbol{\eta}|} \prod_{C(\boldsymbol{\eta})} \Theta_{\Lambda,m}\bigl(C(\boldsymbol{\eta})\bigr). \tag{4.9}$$

Here $\Theta_{\Lambda,m}$ is defined as

$$\Theta_{\Lambda,m}(C) = \begin{cases} \Theta_{\mathrm{free}}(C) & \mathbb{V}(C) \cap \Lambda^{\mathrm{c}} = \emptyset \\ e^{\beta h_m |\mathbb{V}(C) \setminus \Lambda^{\mathrm{c}}|} & \text{otherwise,} \end{cases} \tag{4.10}$$

for any connected set of bonds C.

Proof of Theorem III.2(i). We consider Λ to be fixed and omit it temporarily from the notation. In order to prove the strong FKG property of $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}$ and $\mu^{\mathrm{RC}}_{\Lambda,m}$, let us recall a necessary and sufficient condition [FK], the so-called lattice condition

$$W^{\mathrm{RC}}_{\mathrm{free}}\bigl(\boldsymbol{\eta}^{(1)} \vee \boldsymbol{\eta}^{(2)}\bigr) W^{\mathrm{RC}}_{\mathrm{free}}\bigl(\boldsymbol{\eta}^{(1)} \wedge \boldsymbol{\eta}^{(2)}\bigr) \geq W^{\mathrm{RC}}_{\mathrm{free}}\bigl(\boldsymbol{\eta}^{(1)}\bigr) W^{\mathrm{RC}}_{\mathrm{free}}\bigl(\boldsymbol{\eta}^{(2)}\bigr) \tag{4.11}$$

for any pair of configurations $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$, and similarly for W^{RC}_m. Here $\boldsymbol{\eta}^{(1)} \vee \boldsymbol{\eta}^{(2)}$ denotes the maximum and $\boldsymbol{\eta}^{(1)} \wedge \boldsymbol{\eta}^{(2)}$ the minimum of $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$.

It turns out that to verify (4.11), it suffices to consider $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$ that differ just at two bonds. Indeed (see e.g. [CPS]), let

$$\mathcal{R}(\boldsymbol{\zeta},\boldsymbol{\eta}) = \frac{W^{\mathrm{RC}}_{\mathrm{free}}(\boldsymbol{\zeta}\vee\boldsymbol{\eta})}{W^{\mathrm{RC}}_{\mathrm{free}}(\boldsymbol{\zeta})} \tag{4.12}$$

and note that (4.11) can be rewritten as $\mathcal{R}\big(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)}\big) \ge \mathcal{R}\big(\boldsymbol{\eta}^{(1)}\wedge\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(2)}\big)$. Hence, the lattice condition (4.11) is true once we verify that $\mathcal{R}(\boldsymbol{\zeta},\boldsymbol{\eta})$ is increasing in $\boldsymbol{\zeta}$, for any fixed $\boldsymbol{\eta}$. Let us introduce, for any bond b, the configuration $\boldsymbol{\eta}^{(b)}$ by setting $\eta^{(b)}_b = 1$ and $\eta^{(b)}_{b'} = 0$ for any $b' \ne b$. Ordering the set $\mathbb{B}_1(\boldsymbol{\eta})$ into a sequence $\{b_1,\dots,b_{|\mathbb{B}_1(\boldsymbol{\eta})|}\}$, we have

$$\mathcal{R}(\boldsymbol{\zeta},\boldsymbol{\eta}) = \prod_{k=1}^{|\mathbb{B}_1(\boldsymbol{\eta})|} \mathcal{R}\big(\boldsymbol{\zeta}\vee\boldsymbol{\eta}^{(b_1)}\vee\dots\vee\boldsymbol{\eta}^{(b_{k-1})},\boldsymbol{\eta}^{(b_k)}\big). \tag{4.13}$$

Hence, it suffices to prove monotonicity of $\mathcal{R}(\boldsymbol{\zeta},\boldsymbol{\eta})$ for any $\boldsymbol{\eta}$ that is zero except possibly at one bond. Moreover, it suffices to prove the growth when flipping $\boldsymbol{\zeta}$ at a single bond from 0 to 1, i.e., $\boldsymbol{\zeta}$ with $\zeta_b = 0$ to $\boldsymbol{\zeta}^b = \boldsymbol{\zeta}\vee\boldsymbol{\eta}^{(b)}$. The verification of the needed bound, $\mathcal{R}(\boldsymbol{\zeta}^b,\boldsymbol{\eta}^{(b')}) \ge \mathcal{R}(\boldsymbol{\zeta},\boldsymbol{\eta}^{(b')})$, for any pair of bonds b and b', now boils down to the special case of (4.11) with $\boldsymbol{\eta}^{(1)} = \boldsymbol{\zeta}^b$ and $\boldsymbol{\eta}^{(2)} = \boldsymbol{\zeta}\vee\boldsymbol{\eta}^{(b')}$ that differ only at bonds b and b'.

Let thus $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$ be such that

$$\begin{aligned} &\eta^{(1)}_b = \eta^{(2)}_b \qquad && b \ne b_1, b_2 \\ &\eta^{(1)}_{b_1} = \eta^{(2)}_{b_2} = 0 \qquad && \eta^{(1)}_{b_2} = \eta^{(2)}_{b_1} = 1. \end{aligned} \tag{4.14}$$

Since the number of 1-bonds is equal on both sides of (4.11), the nontrivial issue is therefore to check (4.11) for the product over the connected components. Let us suppose, without loss of generality, that there exist *disjoint* connected components A_1 and A_2 of $\boldsymbol{\eta}^{(1)}\wedge\boldsymbol{\eta}^{(2)}$ (possibly isolated sites) that become connected when b_1 is flipped from 0 to 1, and, similarly, B_1, B_2 for the components connected by flipping b_2. (The only other possibility is that both endpoints of b_1, or alternatively b_2, lie in a *single* component of $\boldsymbol{\eta}^{(1)}\wedge\boldsymbol{\eta}^{(2)}$, in which case both sides of (4.11) are equal.) With this proviso, there are only three generic situations:

(a) $\mathbb{V}(A_1)\cup\mathbb{V}(A_2)$ is disjoint from $\mathbb{V}(B_1)\cup\mathbb{V}(B_2)$,
(b) $\mathbb{V}(A_1) = \mathbb{V}(B_1)$ but $\mathbb{V}(A_2)\cap\mathbb{V}(B_2) = \emptyset$,
(c) $\mathbb{V}(A_1) = \mathbb{V}(B_1)$ and $\mathbb{V}(A_2) = \mathbb{V}(B_2)$.

We will prove (4.11) separately for (a), (b), and (c). For notational brevity, we use $\Theta(C)$ for both $\Theta_{\mathrm{free}}(C)$ and $\Theta_m(C)$.

In the case (a) both sides of (4.11) reduce to the same term

$$\Theta(A_1\cup A_2)\Theta(B_1\cup B_2)\Theta(A_1)\Theta(A_2)\Theta(B_1)\Theta(B_2). \tag{4.15}$$

Hence, (4.11) is fulfilled with the equality sign.

Next, consider (b). We denote by C the common component (i.e., $C = A_1 = B_1$) and use A and B to denote the other components. Then (4.11) boils down to the inequality

$$\Theta(C)\Theta(C \cup A \cup B) \geq \Theta(C \cup A)\Theta(C \cup B). \tag{4.16}$$

The unified notation (4.8) and (4.10) allows us to prove this at the same time for *free* and *m-wired* boundary conditions. Set $\bar{m}$ equal to 0 in the former case and to m in the latter. Let for any connected set of bonds A and $m \in \{1, \ldots, q+2\}$

$$\chi_{\Lambda,\bar{m}}(A, m) = \begin{cases} 1 & \mathbb{V}(A) \cap \Lambda^{\mathrm{c}} = \emptyset \quad \text{or} \quad m = \bar{m} \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Let

$$\begin{aligned} \mathfrak{a}_m &= e^{\beta h_m |\mathbb{V}(A) \smallsetminus \Lambda^{\mathrm{c}}|} \chi_{\Lambda,\bar{m}}(A, m) \\ \mathfrak{b}_m &= e^{\beta h_m |\mathbb{V}(B) \smallsetminus \Lambda^{\mathrm{c}}|} \chi_{\Lambda,\bar{m}}(B, m) \qquad \forall m \in \{1, \ldots, q+2\}. \\ \mathfrak{c}_m &= e^{\beta h_m |\mathbb{V}(C) \smallsetminus \Lambda^{\mathrm{c}}|} \chi_{\Lambda,\bar{m}}(C, m) \end{aligned} \tag{4.18}$$

The condition (4.16) is now, for all $\bar{m} \in \{0, 1, \ldots, q+2\}$, equivalent to

$$\left(\sum_{m=1}^{q+2} \mathfrak{c}_m\right)\left(\sum_{m'=1}^{q+2} \mathfrak{a}_{m'}\mathfrak{b}_{m'}\mathfrak{c}_{m'}\right) \geq \left(\sum_{m=1}^{q+2} \mathfrak{a}_m\mathfrak{c}_m\right)\left(\sum_{m'=1}^{q+2} \mathfrak{b}_{m'}\mathfrak{c}_{m'}\right). \tag{4.19}$$

Let us assume that the fields are ordered in an increasing order, $h_1 \leq h_2 \leq \cdots \leq h_{q+2}$. Taking into account that $h_{\bar{m}} = h_{\max}$ once $\bar{m} \neq 0$, we can assume, without loss of generality, that either $\bar{m} = 0$ or $\bar{m} = q+2$. As a consequence, $\mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \cdots \leq \mathfrak{a}_{q+2}$ and $\mathfrak{b}_1 \leq \mathfrak{b}_2 \leq \cdots \leq \mathfrak{b}_{q+2}$ for both free and m-wired boundary conditions. By writing the expression (4.19) as an inequality for a bilinear form in $\mathfrak{c}_m\mathfrak{c}_{m'}$, the sufficient requirement that all the independent coefficients of this form be non-negative reduces to

$$(\mathfrak{a}_m - \mathfrak{a}_{m'})(\mathfrak{b}_m - \mathfrak{b}_{m'}) \geq 0 \qquad \forall m, m', \tag{4.20}$$

which is immediate by our preceding assumptions.

It remains to establish (4.11) under (c). In this case, there are only two components in the game: A and B. Inequality (4.11) is then implied by $\Theta(A \cup B) \leq \Theta(A)\Theta(B)$. By using $\mathfrak{a}_m$ and $\mathfrak{b}_m$ literally as defined in (4.18), this boils down to the inequality

$$\sum_{m=1}^{q+2} \mathfrak{a}_m\mathfrak{b}_m \leq \left(\sum_{m=1}^{q+2} \mathfrak{a}_m\right)\left(\sum_{m'=1}^{q+2} \mathfrak{b}_{m'}\right), \tag{4.21}$$

which is obviously satisfied. $\square$

Remark. The necessity of $h_m = h_{\max}$ for the strong FKG property of $\mu^{\mathrm{RC}}_{\Lambda,m}$ arises from (4.19). Namely, suppose A connects to the boundary (i.e., $\mathbb{V}(A) \cap \Lambda^{\mathrm{c}} \neq \emptyset$), whereas B and C do not. Then $\mathfrak{a}_m = 0$ unless $m = \bar{m} \neq 0$, in which case (4.19) reduces to

$$\mathfrak{b}_{\bar{m}} \sum_{m=1}^{q+2} \mathfrak{c}_m \geq \sum_{m=1}^{q+2} \mathfrak{b}_m \mathfrak{c}_m. \tag{4.22}$$

It is not difficult to convince oneself that one can choose C in such a way that this is satisfied for all B only when $\mathfrak{b}_{\bar{m}} = \max_m \mathfrak{b}_m$. Consequently, $h_{\bar{m}}$ must be equal to $h_{\max}$ for the lattice condition (4.11) or, equivalently, the strong FKG to hold.

Proof of Theorem III.2(ii). Let $\Lambda \subset \Delta \subset \mathbb{Z}^d$ be two finite sets. We claim that

$$\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\,\cdot\,) \underset{\mathrm{FKG}}{\leq} \mu^{\mathrm{RC}}_{\Delta,\mathrm{free}}(\,\cdot\,) \tag{4.23}$$

and

$$\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(\,\cdot\,) \underset{\mathrm{FKG}}{\geq} \mu^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}(\,\cdot\,). \tag{4.24}$$

For free boundary conditions, the inequality (4.23) follows, with the help of (i), immediately from the fact that

$$\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\,\cdot\,) = \mu^{\mathrm{RC}}_{\Delta,\mathrm{free}}(\,\cdot\,|\,\mathcal{D}_\Lambda), \tag{4.25}$$

where $\mathcal{D}_\Lambda$ is the FKG-decreasing event

$$\mathcal{D}_\Lambda = \bigl\{\boldsymbol{\eta}\colon \eta_b = 0\,\forall b \in \mathbb{B}_0(\Lambda)^{\mathrm{c}}\bigr\}. \tag{4.26}$$

For m-wired boundary conditions, the proof is more complicated, since conditioning on the FKG-increasing event

$$\mathcal{O}_\Lambda = \bigl\{\boldsymbol{\eta}\colon \eta_b = 1\,\forall b \in \mathbb{B}(\Lambda)^{\mathrm{c}}\bigr\} \tag{4.27}$$

leads to the state $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$ only if Λ is a volume without 'holes', i.e. if Λ^{c} has only one (infinite) component. If Λ^{c} has finite components $H_1, \dots, H_k$, we use the following trick: for each 'hole' H_i, we introduce an additional bond b_i with one endpoint in H_i and the other in Δ^{c}. Introducing the set

$$\mathbb{B}^*(\Delta) = \mathbb{B}(\Delta) \cup \{b_1, \dots, b_k\}, \tag{4.28}$$

we then define $\bar{\mu}^{\mathrm{RC}}_{\Lambda,m}$ as the random-cluster marginal of $\mu^{\mathrm{ES}}_{\Delta,\mathbb{B}^*(\Delta)}(\,\cdot\,|\,\boldsymbol{\sigma}^m_{\Lambda^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}^*(\Delta)^{\mathrm{c}}})$, where $\boldsymbol{\eta}$ is a configuration in $\mathbb{B}(\mathbb{Z}^d) \cup \{b_1, \dots, b_k\}$ such that $\eta_b = 1$ for all b.

With this definition we get

$$\begin{aligned}
\mu^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}(\,\cdot\,) &= \bar{\mu}^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}\bigl(\,\cdot\,|\,\eta_b = 0\ \forall b \in \mathbb{B}^*(\Delta)\setminus\mathbb{B}(\Delta)\bigr) \\
&\underset{\mathrm{FKG}}{\leq} \bar{\mu}^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}\bigl(\,\cdot\,|\,\eta_b = 1\ \forall b \in \mathbb{B}^*(\Delta)\setminus\mathbb{B}(\Delta)\bigr) \\
&\underset{\mathrm{FKG}}{\leq} \bar{\mu}^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}\bigl(\,\cdot\,|\,\eta_b = 1\ \forall b \in \bigl(\mathbb{B}^*(\Delta)\setminus\mathbb{B}(\Delta)\bigr)\cup\mathbb{B}(\Lambda)^{\mathrm{c}}\bigr) \\
&= \mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(\,\cdot\,),
\end{aligned} \tag{4.29}$$

proving the desired inequality (4.24). Here the first step uses that the strong-FKG measures conditioned on taking a fixed configuration $\boldsymbol{\eta}_A$ in a set A are FKG increasing in $\boldsymbol{\eta}_A$. The second inequality is by the FKG of $\mu^{\mathrm{RC}}_{\Delta,\mathrm{maxwir}}$.

As a consequence of (4.23) and (4.24), the net $\{\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}\}$ (resp. $\{\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}\}$) increases (resp. decreases) as Λ increases (in the order defined by the set inclusion), yielding the existence of the desired limits as well as their translation invariance for all monotone quasilocal functions. Since the latter generate all quasilocal functions, the claim is established. $\square$

In order to prove the claim (iii) of Theorem III.2, we first need some more definitions. Let Λ be a finite set, $\partial\Lambda$ its external boundary, $\partial\Lambda = \{x \in \mathbb{Z}^d \mid \mathrm{dist}(x,\Lambda) = 1\}$, and $\boldsymbol{\sigma}$ be an arbitrary spin configuration. Recalling that $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(\,\cdot\mid\boldsymbol{\sigma}_{\Lambda^c},\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c})$ does not depend on $\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^c}$, we abbreviate notation by calling this measure simply $\mu^{\mathrm{ES}}_{\Lambda,\boldsymbol{\sigma}}$ and using $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}$ to denote its random-cluster marginal.

Let $B \subset \partial\Lambda$ be an arbitrary set of boundary sites. Then we define a *partially wired* RC measure $\mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}$ by taking the $\boldsymbol{\eta}$ marginal of the ES measure

$$\mu^{\mathrm{ES}}_{\Lambda,m}\bigl(\,\cdot\,|\,\eta_b = 0\ \forall b \in \mathbb{B}(\Lambda)\setminus\mathbb{B}_0(\Lambda\cup B)\bigr) \tag{4.30}$$

for any m with $h_m = h_{\max}$. Note that $\mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}$ is a measure on configurations in $\mathbb{B}_0(\Lambda\cup B)\setminus\mathbb{B}_0(B)$ and that, being defined as the conditional measure of a strong FKG measure, it is strong FKG.

The measures $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}$ and $\mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}$ satisfy the following FKG bounds:

Lemma IV.1. *Let Λ be a finite set. Then for any $\boldsymbol{\sigma}$ on its complement*

$$\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}(\,\cdot\,) \underset{\mathrm{FKG}}{\leq} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(\,\cdot\,). \tag{4.31}$$

Moreover, let $B \subset \partial\Lambda$. Then

$$\mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}(\,\cdot\,) \underset{\mathrm{FKG}}{\geq} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\,\cdot\,). \tag{4.32}$$

Proof. Given $\boldsymbol{\sigma}$, let us define a partition of $\partial\Lambda$ into sets $\partial_i\Lambda = \{x \in \partial\Lambda\colon \sigma_x = i\}$. Similarly to $\mu^{\mathrm{RC}}_{\Lambda,m}$, the measure $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}$ can be obtained by normalizing a suitable weight $W^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}(\boldsymbol{\eta})$ defined for any $\boldsymbol{\eta} \in \{0,1\}^{\mathbb{B}(\Lambda)}$ by the formula

$$W^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}(\boldsymbol{\eta}) = (e^{\beta J}-1)^{|\boldsymbol{\eta}|} \prod_{i<j} \mathbb{1}_{\{\partial_i\Lambda \not\leftrightarrow \partial_j\Lambda\}}(\boldsymbol{\eta}) \prod_{C(\boldsymbol{\eta})} \Theta_{\Lambda,\boldsymbol{\sigma}}\big(C(\boldsymbol{\eta})\big). \tag{4.33}$$

Here

$$\Theta_{\Lambda,\boldsymbol{\sigma}}(C) = \begin{cases} \Theta_{\mathrm{free}}(C) & \mathbb{V}(C) \cap \Lambda^{\mathrm{c}} = \emptyset \\ e^{\beta h_m |\mathbb{V}(C)\setminus\Lambda^{\mathrm{c}}|} & \mathbb{V}(C) \cap \partial_m\Lambda \neq \emptyset \end{cases} \tag{4.34}$$

and $\partial_i\Lambda \not\leftrightarrow \partial_j\Lambda$ denotes the event that the sets $\partial_i\Lambda$ and $\partial_j\Lambda$ are not connected by a path of occupied bonds.

Using the representation (4.33), it is now easy to see that the measure $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}$ can also be recast as

$$\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\sigma}}(\,\cdot\,) = \frac{\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(\,\cdot\;g)}{\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(\,g)}, \tag{4.35}$$

where

$$g(\boldsymbol{\eta}) = \prod_{i<j} \mathbb{1}_{\{\partial_i\Lambda \not\leftrightarrow \partial_j\Lambda\}}(\boldsymbol{\eta}) \prod_m \prod_{\substack{C:\\ \mathbb{V}(C)\cap\partial_m\Lambda\neq\emptyset}} e^{-(h_{\max}-h_m)|\mathbb{V}(C)\setminus\Lambda^{\mathrm{c}}|} \tag{4.36}$$

for any $\boldsymbol{\eta} \in \{0,1\}^{\mathbb{B}(\Lambda)}$. It turns out that the function g is FKG-decreasing. Indeed, each indicator $\mathbb{1}_{\{\partial_i\Lambda \not\leftrightarrow \partial_j\Lambda\}}(\boldsymbol{\eta})$ is clearly decreasing. The same is true for the remaining factor as is seen by noting that

$$\sum_{\substack{C:\\ \mathbb{V}(C)\cap\partial_m\Lambda\neq\emptyset}} |\mathbb{V}(C)\setminus\Lambda^{\mathrm{c}}|, \tag{4.37}$$

being equal to the number of sites connected to $\partial\Lambda_m$, is an increasing function of $\boldsymbol{\eta}$. Since $h_{\max} \geq h_m$ and since the product of non-negative decreasing functions is decreasing, the monotonicity of g is established. Since $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$ is FKG, (4.31) is proved.

To prove (4.32), it is enough to observe that

$$\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\,\cdot\,) = \mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}\big(\,\cdot \,|\, \eta_b = 0\,\forall b \in \mathbb{B}_0(\Lambda)^c\big) \tag{4.38}$$

by (4.30). Again, since $\mu^{\mathrm{RC}}_{\Lambda,B,\mathrm{maxwir}}$ is FKG, the proof is finished by observing that the event on the r.h.s. of (4.38) is decreasing. $\square$

Apart from the previous Lemma, the second part of Theorem III.2(iii) requires also some control over the possible values of the spins that can be attained on the infinite clusters. Observe first that since $\nu\big(\{(\boldsymbol{\sigma},\boldsymbol{\eta})\colon \sigma_x \neq \sigma_y,\, \eta_{\langle x,y\rangle} = 1\}\big) = 0$ for each $\nu \in \mathcal{G}^{\mathrm{ES}}$, each connected component has a constant spin-value almost surely. Let us use $\mathfrak{S}(\boldsymbol{\sigma},\boldsymbol{\eta})$ to denote the set of possible spin-values attained at the infinite clusters in a configuration $(\boldsymbol{\sigma},\boldsymbol{\eta})$.

Proposition IV.2. *Let $\nu \in \mathcal{G}^{\mathrm{ES}}$. Then $\mathfrak{S} \subseteq \{m: h_m = h_{\max}\}$ ν-almost surely.*

Before we prove this claim, let us formulate a technical lemma.

Lemma IV.3. *Let $(a_k)_{k\geq 1}$ be a sequence of numbers such that $1 \leq a_k \leq Ck^n$ for some constant $C < \infty$ and an integer $n \geq 0$. Then for each $\epsilon > 0$ and any $\bar{k} \geq C(n+1)^n \epsilon^{-(n+1)}$*

$$a_k \leq \epsilon \sum_{k' \leq k} a_{k'} \tag{4.39}$$

holds true for at least one $k \in \{\bar{k}, \dots, (n+1)\bar{k}\}$.

Proof. Suppose $a_k > \epsilon \sum_{k' \leq k} a_{k'}$ for all $k \in \{\bar{k}, \dots, (n+1)\bar{k}\}$. Since $a_{k'} \geq 1$, this implies $a_k > \epsilon \bar{k}$ for all $k \in \{\bar{k}, \dots, 2\bar{k}\}$ and, using induction, $a_k > \epsilon^\ell \bar{k}^\ell$ for all $k \in \{\ell\bar{k}, \dots, (\ell+1)\bar{k}\}$, with $\ell \in \{1, \dots, n\}$. In particular, $a_{(n+1)\bar{k}} > \epsilon^{n+1}\bar{k}^{n+1}$. However, this is in contradiction with the assumption $a_{(n+1)\bar{k}} \leq C(n+1)^n \bar{k}^n$ whenever $\bar{k} \geq C(n+1)^n \epsilon^{-(n+1)}$. $\square$

Proof of Proposition IV.2. Let $m \in \{1, \dots, q+2\}$ with $h_m < h_{\max}$ and suppose that there is $\nu \in \mathcal{G}^{\mathrm{ES}}$ with $\nu(m \in \mathfrak{S}) > 0$. Since $\mathcal{G}^{\mathrm{ES}}$ as well as the described event are invariant w.r.t. spatial shifts, we can suppose without loss of generality that the event

$$\Omega_m^0 = \Big\{(\boldsymbol{\sigma}, \boldsymbol{\eta}) \colon \exists C(\boldsymbol{\eta}),\ \big|C(\boldsymbol{\eta})\big| = \infty,\ \mathbb{V}\big(C(\boldsymbol{\eta})\big) \ni 0,\ \sigma_0 = m\Big\} \tag{4.40}$$

has a positive probability under ν, i.e., $\nu(\Omega_m^0) > 0$. Let Λ_k be the box of site-length $2k+1$ centered at the origin and, for each $(\boldsymbol{\sigma}, \boldsymbol{\eta}) \in \Omega_m^0$ and each $k \geq 1$, let

$$a_k = \Big|\mathbb{V}_k(\boldsymbol{\eta}) \cap \partial\Lambda_{k-1}\Big|, \tag{4.41}$$

where $\mathbb{V}_k$ is the set of sites in Λ_k that are connected to the origin within $\mathbb{B}_0(\Lambda_k)$. Note that $\big|\mathbb{V}_k(\boldsymbol{\eta})\big| \geq \sum_{k' \leq k} a_{k'}$ and that $1 \leq a_k \leq |\partial\Lambda_{k-1}| \leq 2d(2k+1)^{d-1} \leq 3^d dk^{d-1}$, with the latter bound using that $k \geq 1$. Hence, by Lemma IV.3, we know that for each $\epsilon > 0$ and each $\bar{k} \geq (3d/\epsilon)^d$ there is at least one k, with $\bar{k} \leq k \leq d\bar{k}$, such that

$$\big|\mathbb{V}_k(\boldsymbol{\eta}) \cap \partial\Lambda_{k-1}\big| \leq \epsilon\,\big|\mathbb{V}_k(\boldsymbol{\eta})\big|. \tag{4.42}$$

By (4.42) and the subadditivity of the measure we have for $\bar{k} \geq (3d/\epsilon)^d$ that

$$\nu(\Omega_m^0) \leq \mu(\cup_{\bar{k}\leq k\leq d\bar{k}} \Omega_{m,k}^0) \leq \sum_{\bar{k}\leq k\leq d\bar{k}} \nu(\Omega_{m,k}^0), \tag{4.43}$$

with $\Omega_{m,k}^0$ denoting the event

$$\Omega_{m,k}^0 = \Big\{(\boldsymbol{\sigma}, \boldsymbol{\eta}) \colon \sigma_0 = m,\ 0 \leftrightarrow \partial\Lambda_{k-1},\ \big|\{x \in \partial\Lambda_{k-1} \colon x \underset{\Lambda_k}{\leftrightarrow} 0\}\big| \leq \epsilon\,\big|\{x \in \Lambda_k \colon x \underset{\Lambda_k}{\leftrightarrow} 0\}\big|\Big\}. \tag{4.44}$$

Here $x \underset{\Lambda_k}{\leftrightarrow} 0$ indicates the connection within $\mathbb{B}_0(\Lambda_k)$. As a result, for each $\epsilon > 0$ there is a deterministic set $\mathbb{N}_\epsilon \subset \mathbb{N}$, $|\mathbb{N}_\epsilon| = \infty$, such that for any $k \in \mathbb{N}_\epsilon$ one has

$$\nu(\Omega^0_{m,k}) \geq \frac{1}{dk}\nu(\Omega^0_m), \tag{4.45}$$

by the pigeon-hole principle as applied to (4.43).

On the other hand, since $\Omega^0_{m,k}$ is a $\Lambda_k, \mathbb{B}_0(\Lambda_k)$-cylinder event, we can estimate $\nu(\Omega^0_{m,k})$ using (3.7). Recall that $\mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}$ is the specification (3.5) with a special choice $\Lambda = \Lambda_k$ and $\mathbb{B} = \mathbb{B}(\Lambda_k)$ and the spin boundary condition $\boldsymbol{\sigma}$ (the $\boldsymbol{\eta}$ boundary condition is irrelevant in this case). Then (3.7) reads

$$\nu(\Omega^0_{m,k}) = \int \nu(\mathrm{d}\boldsymbol{\sigma}, \mathrm{d}\boldsymbol{\eta}) \mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}(\Omega^0_{m,k}). \tag{4.46}$$

Fix $\epsilon > 0$ such that $dJ\epsilon + h_m < h_{\max}$ and pick $\tilde{m}$ with $h_{\tilde{m}} = h_{\max}$. Then we have for any $\boldsymbol{\sigma}$

$$\begin{aligned}
\mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}(\Omega^0_{m,k}) &\leq \mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}\Biggl(\mathbb{1}_{\Omega^0_{m,k}} \prod_{\substack{\langle x,y\rangle\colon x\in\Lambda_k^{\mathrm{c}}\\ y\in\mathbb{V}_k}} e^{\beta J}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}}\Biggr)\\
&= \mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}\Biggl(\mathbb{1}_{\Omega^0_{\tilde{m},k}} e^{-\beta(h_{\max}-h_m)|\mathbb{V}_k|} \prod_{\substack{\langle x,y\rangle\colon x\in\Lambda_k^{\mathrm{c}}\\ y\in\mathbb{V}_k}} e^{\beta J}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}}\Biggr)\\
&\leq \mu^{\mathrm{ES}}_{\Lambda_k,\boldsymbol{\sigma}}\Bigl(\mathbb{1}_{\Omega^0_{\tilde{m},k}} e^{-\beta(h_{\max}-h_m-dJ\epsilon)|\mathbb{V}_k|}\Bigr) \leq e^{-\beta(h_{\max}-h_m-dJ\epsilon)k}.
\end{aligned} \tag{4.47}$$

Here, in the first step we inserted the factor $e^{\beta J}$ in order to convert an arbitrary configuration at the boundary bonds of the set $\mathbb{V}_k$ to the vacant bond-state. More explicitly, we carried out the following estimate

$$\begin{aligned}
&\sum_{\eta_{\langle x,y\rangle}=0,1} \bigl(\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}} + (e^{\beta J}-1)\delta_{\sigma_x,\sigma_y}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=1\}}\bigr)\\
&\qquad = (e^{\beta J}-1)\delta_{\sigma_x,\sigma_y} + 1 \leq e^{\beta J} = \sum_{\eta_{\langle x,y\rangle}=0,1} e^{\beta J}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}}\\
&\qquad = \sum_{\eta_{\langle x,y\rangle}=0,1} e^{\beta J}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}}\bigl(\mathbb{1}_{\{\eta_{\langle x,y\rangle}=0\}} + (e^{\beta J}-1)\delta_{\sigma_x,\sigma_y}\mathbb{1}_{\{\eta_{\langle x,y\rangle}=1\}}\bigr).
\end{aligned} \tag{4.48}$$

at every boundary bond (note that the unconstraint summation over the bond configuration is in place because $\Omega^0_{\tilde{m},k}$ does not depend on these bonds). This allows us to flip σ_x at each $x \in \mathbb{V}_k$ from m to $\tilde{m}$, resulting in the exponential factor in the second line. The estimate is finished by noting that, on $\Omega^0_{\tilde{m},k}$, the number of flipped bonds does not exceed $d|\mathbb{V}_k \cap \partial\Lambda_{k-1}| \leq d\epsilon|\mathbb{V}_k|$ and that $|\mathbb{V}_k| \geq k$.

By putting (4.45), (4.46) and (4.47) together we get that

$$\tfrac{1}{dk}\nu(\Omega^0_m) \le \nu(\Omega^0_{m,k}) \le e^{-\beta(h_{\max}-h_m-dJ\epsilon)k} \qquad \forall k\in\mathbb{N}_\epsilon. \tag{4.49}$$

However, since $|\mathbb{N}_\epsilon| = \infty$ and k can thus be arbitrary large, this leads to a contradiction whenever $\nu(\Omega^0_m) > 0$. Hence, no such m with $h_m < h_{\max}$ can exist and $\mathfrak{S} \subseteq \{m\colon h_m = h_{\max}\}$ ν-almost surely. $\square$

Proof of Theorem III.2(iii). Let us consider an ES Gibbs measure ν and use μ to denote its $\boldsymbol{\eta}$ marginal. Applying the DLR-equations (3.7) for ν, we get

$$\begin{aligned}\mu(f) = \nu(f) = \int \nu(\mathrm{d}\boldsymbol{\sigma},\mathrm{d}\boldsymbol{\eta})\,\mu^{\text{ES}}_{\Lambda,\mathbb{B}(\Lambda)}(f\,|\,\boldsymbol{\sigma}_{\Lambda^{\mathrm{c}}},\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^{\mathrm{c}}}) &= \int \nu(\mathrm{d}\boldsymbol{\sigma},\mathrm{d}\boldsymbol{\eta})\,\mu^{\text{RC}}_{\Lambda,\boldsymbol{\sigma}_{\partial\Lambda}}(f)\\ &\le \int \nu(\mathrm{d}\boldsymbol{\sigma},\mathrm{d}\boldsymbol{\eta})\,\mu^{\text{RC}}_{\Lambda,\text{maxwir}}(f) = \mu^{\text{RC}}_{\Lambda,\text{maxwir}}(f)\end{aligned} \tag{4.50}$$

for any increasing cylinder function $f(\boldsymbol{\eta})$ supported on $\tilde{\mathbb{B}} \subset \mathbb{B}(\Lambda)$. Here, the inequality follows by (4.31). Applying now (3.10), we get (3.12).

In order to prove (3.13), we have to work a bit harder. Let $(\Delta_n)_{n\ge1}$ be an increasing sequence of boxes centered at the origin and let

$$\Lambda_n(\boldsymbol{\eta}) = \{x\in\Delta_n\colon x \not\leftrightarrow \Delta_n^{\mathrm{c}}\} \cup \{x\in\Delta_n\colon x\leftrightarrow\infty\}. \tag{4.52}$$

By the assumption on μ, either $q_0 = 1$ or there is at most one infinite cluster. The spin on the component(s) is uniquely defined in both cases under consideration: $\sigma_x = m$ with $h_m = h_{\max}$ for all x in the boundary set

$$B_n(\boldsymbol{\eta}) = \partial\Lambda_n(\boldsymbol{\eta}) \cap \{x\leftrightarrow\infty\}. \tag{4.52}$$

Let $\bar{\boldsymbol{\eta}}$ be a configuration and let ν be an ES measure with at most one infinite cluster whenever $q_0 > 1$. Denote $\bar\Lambda_n = \Lambda_n(\bar{\boldsymbol{\eta}})$ and $\bar B_n = B_n(\bar{\boldsymbol{\eta}})$. Then we claim that

$$\nu\bigl(\,\cdot\,|\{\Lambda_n(\,\cdot\,)=\bar\Lambda_n\}\cap\{\boldsymbol{\eta}_{\mathbb{B}_0(\bar\Lambda_n)^{\mathrm{c}}} = \bar{\boldsymbol{\eta}}_{\mathbb{B}_0(\bar\Lambda_n)^{\mathrm{c}}}\}\bigr) \underset{\text{FKG}}{\ge} \mu^{\text{RC}}_{\bar\Lambda_n,\bar B_n,\text{maxwir}}(\,\cdot\,), \tag{4.53}$$

as random-cluster measures in $\mathbb{B}_0(\bar\Lambda_n)$. Namely, if there are no infinite clusters, then $\{\Lambda_n(\,\cdot\,) = \bar\Lambda_n\}$ is clearly an event independent of the bond-values in $\mathbb{B}_0(\bar\Lambda_n)$ and the induced measure is precisely $\mu^{\text{RC}}_{\bar\Lambda_n,\bar B_n,\text{maxwir}}$. On the other hand, if there is an infinite component, then the conditioning on $\{\Lambda_n(\,\cdot\,) = \bar\Lambda_n\}$ *and* $\{\boldsymbol{\eta}_{\mathbb{B}_0(\bar\Lambda_n)^{\mathrm{c}}} = \bar{\boldsymbol{\eta}}_{\mathbb{B}_0(\bar\Lambda_n)^{\mathrm{c}}}\}$ induces the additional requirement that no 'dangling end' reaching outside Δ_n be disconnected within $\mathbb{B}_0(\bar\Lambda_n)$ from the backbone of the infinite cluster. Since the latter is an increasing event and since $\mu^{\text{RC}}_{\bar\Lambda_n,\bar B_n,\text{maxwir}}$ is strong FKG (being given by conditioning from a strong FKG measure), the bound (4.53) follows.

Let f be a non-negative monotone increasing $\mathbb{B}_0(\Delta)$-cylinder function, where Δ is a finite set. Now we can write

$$\begin{aligned}\mu(f) = \nu(f) \geq \nu\bigl(f \mathbb{1}_{\{\Lambda_n(\,\cdot\,)\supseteq\Delta\}}\bigr) &= \sum_{\substack{\bar\Lambda_n\supseteq\Delta\\ \bar B_n\subseteq\partial\bar\Lambda_n}} \nu\bigl(f \mathbb{1}_{\{\Lambda_n(\,\cdot\,)=\bar\Lambda_n\}}\mathbb{1}_{\{B_n(\,\cdot\,)=\bar B_n\}}\bigr)\\ &\geq \sum_{\substack{\bar\Lambda_n\supseteq\Delta\\ \bar B_n\subseteq\partial\bar\Lambda_n}} \nu\bigl(\mu^{\mathrm{RC}}_{\bar\Lambda_n,\bar B_n,\mathrm{maxwir}}(f)\mathbb{1}_{\{\Lambda_n(\,\cdot\,)=\bar\Lambda_n\}}\mathbb{1}_{\{B_n(\,\cdot\,)=\bar B_n\}}\bigr)\\ &\geq \mu^{\mathrm{RC}}_{\Delta,\mathrm{free}}(f)\,\mu\bigl(\{\Lambda_n(\,\cdot\,)\supseteq\Delta\}\bigr).\end{aligned} \tag{4.54}$$

Here in the first step we used non-negativeness of f, in the third step we used Gibbsianness in $\bar\Lambda_n$ along with the bound (4.53), and then applied (4.32), (4.23) and resummed over the volumes $\bar\Lambda_n$ and the components $\bar B_n$ to get the final expression.

Since $\mu\bigl(\{\Lambda_n(\,\cdot\,)\supseteq\Delta\}\bigr)$ tends to 1 as $n\to\infty$ by the Monotone Convergence Theorem, the proof is finished for $f\geq 0$ by taking that limit followed by $\Delta\nearrow\mathbb{Z}^d$. Arbitrary cylinder f's are handled by noting that $f-\min f\geq 0$. $\square$

Proof of Theorem III.2(iv). Let g be a monotone increasing function, depending only on bonds $\mathbb{B}_0(\Delta)$ for some finite Δ. Let for each Λ define

$$\mathfrak{g}_\Lambda = \sum_{x:\,\tau^x(\Delta)\subset\Lambda} g\circ\tau^x, \tag{4.55}$$

where τ is the shift operator. Let $\mu^{\mathrm{RC},(1),\alpha}_{\Lambda,\mathrm{free}}$ be the state generated by the free boundary condition, however, with the weights in (3.6) multiplied by the function $e^{\alpha\mathfrak{g}_\Lambda}$. By standard subadditivity arguments, it follows that the normalizing factor of such modified weights gives rise to the 'free energy'

$$\mathfrak{F}(\alpha) = \lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\log\Bigl\{\sum_{\bar{\boldsymbol\sigma}_\Lambda,\bar{\boldsymbol\eta}_{\mathbb{B}(\Lambda)}} e^{\alpha\mathfrak{g}_\Lambda(\bar{\boldsymbol\eta}_\Lambda)}\,W(\bar{\boldsymbol\sigma}_\Lambda,\bar{\boldsymbol\eta}_{\mathbb{B}(\Lambda)}\,|\,\boldsymbol\sigma_{\Lambda^{\mathrm{c}}},\boldsymbol\eta_{\mathbb{B}(\Lambda)^{\mathrm{c}}})\Bigr\}, \tag{4.56}$$

where $\mathfrak{F}(\alpha)$ is independent of the boundary condition $\boldsymbol\sigma,\boldsymbol\eta$ and is convex in α. Moreover, by differentiating we find out that

$$\limsup_{\Lambda\nearrow\mathbb{Z}^d}\,\mu^{\mathrm{RC},(1)}_{\Lambda,\mathrm{maxwir}}\Bigl(\frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr)\leq\frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}\alpha^+}(\alpha_1)\leq\frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}\alpha^-}(\alpha_2)\leq\liminf_{\Lambda\nearrow\mathbb{Z}^d}\,\mu^{\mathrm{RC},(1),\alpha}_{\Lambda,\mathrm{free}}\Bigl(\frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr), \tag{4.57}$$

where $0<\alpha_1<\alpha_2<\alpha$ are arbitrary.

Since $\mathfrak{g}_\Lambda$ is increasing, we have from (4.24) and the translation invariance of $\mu^{\mathrm{RC},(1)}_{\mathrm{maxwir}}$ that the left hand side of (4.57) equals $\mu^{\mathrm{RC},(1)}_{\mathrm{maxwir}}(g)$. Thus we just need to show that if α is small enough then $\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC},(1),\alpha}$ is FKG-dominated by $\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC},(2)}$. For that recall that the second measure can be directly generated by the weights $W_{\Lambda,\mathrm{free}}^{(2)}$ defined in (4.7), and the first one by the weights $e^{\alpha\mathfrak{g}_\Lambda}W_{\Lambda,\mathrm{free}}^{(1)}$. Hence, as in the proof of Lemma IV.1, it suffices to ensure that the function

$$\boldsymbol{\eta}\mapsto G_\Lambda(\boldsymbol{\eta})=e^{\alpha\mathfrak{g}_\Lambda(\boldsymbol{\eta})}\frac{W_{\Lambda,\mathrm{free}}^{(1)}(\boldsymbol{\eta})}{W_{\Lambda,\mathrm{free}}^{(2)}(\boldsymbol{\eta})} \tag{4.58}$$

is monotone decreasing in $\boldsymbol{\eta}$. Let us define the variance of g by the formula

$$\operatorname{var}(g)=\sup_{\bar b}\sup_{\substack{\boldsymbol{\eta},\boldsymbol{\eta}':\,\eta_b=\eta_b'\\ \forall b\neq\bar b}}\bigl|\,g(\boldsymbol{\eta})-g(\boldsymbol{\eta}')\bigr|. \tag{4.59}$$

Note that $\operatorname{var}(g)$ is the maximal value that g can jump over by flipping a single bond. Since

$$\frac{W_{\Lambda,\mathrm{free}}^{(1)}}{W_{\Lambda,\mathrm{free}}^{(2)}}(\boldsymbol{\eta})=\left[\frac{e^{\beta J_1}-1}{e^{\beta J_2}-1}\right]^{|\boldsymbol{\eta}|}, \tag{4.60}$$

the decreasingness of G_Λ is guaranteed for instance by $e^{\alpha\operatorname{var}(g)|\mathbb{B}_0(\Delta)|}(e^{\beta J_1}-1)\le(e^{\beta J_2}-1)$, with this achieved, for $J_1<J_2$, by taking α small enough. Thus, for α small positive we have

$$\mu_{\mathrm{maxwir}}^{\mathrm{RC},(1)}(g)\le\liminf_{\Lambda\nearrow\mathbb{Z}^d}\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC},(1),\alpha}\Bigl(\frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr)\le\liminf_{\Lambda\nearrow\mathbb{Z}^d}\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC},(2)}\Bigl(\frac{\mathfrak{g}_\Lambda}{|\Lambda|}\Bigr)\le\mu_{\mathrm{free}}^{\mathrm{RC},(2)}(g), \tag{4.61}$$

where the last inequality follows from $\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC},(2)}\underset{\mathrm{FKG}}{\le}\mu_{\mathrm{free}}^{\mathrm{RC},(2)}$ and the translation invariance of $\mu_{\mathrm{free}}^{\mathrm{RC},(2)}$. Since g was arbitrary, (3.14) is established. $\square$

4.3. Weak limits of the ES Gibbs measures.

Since by Theorem III.2 the limits (3.10) and (3.11) exist for every quasilocal f depending only on the bond configurations $\boldsymbol{\eta}$, to prove Theorem III.3 we just need to extend this to functions of both $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$. It turns out that the possibility of swapping around $\boldsymbol{\sigma}$-dependence and $\boldsymbol{\eta}$-dependence under the expectation w.r.t. the ES Gibbs measures will be convenient for that. Before we formulate this precisely, let us give some definitions.

For any collection $\{\mathcal{F}_i\}_{i=1}^{q+2}$ of pairwise disjoint finite sets $\mathcal{F}_i\subset\mathbb{Z}^d$, let us define

$$F_{\{\mathcal{F}_i\}}^{\mathrm{free}}(\boldsymbol{\eta})=\prod_{i<j}\mathbb{1}_{\{\mathcal{F}_i\nleftrightarrow\mathcal{F}_j\}}(\boldsymbol{\eta})\prod_{m=1}^{q+2}\prod_{\substack{C:\\ \mathbb{V}(C)\cap\mathcal{F}_m\neq\emptyset}}\frac{e^{\beta h_m|\mathbb{V}(C)|}}{\Theta_{\mathrm{free}}(C)}. \tag{4.62}$$

Here, $\mathbb{1}_{\{\mathcal{F}_i\nleftrightarrow\mathcal{F}_j\}}(\boldsymbol{\eta})$ indicates the event that, under $\boldsymbol{\eta}$, none of the points in $\mathcal{F}_i$ is connected to any point in $\mathcal{F}_j$ by a path of occupied bonds, the product over C runs over all components of the set $\mathbb{B}_1(\boldsymbol{\eta})$ with $\mathbb{V}(C)\cap\mathcal{F}_m\neq\emptyset$, and $\Theta_{\mathrm{free}}(C)$ is as in (4.8).

Similarly, given also a finite set Λ with $\mathcal{F} = \cup_{i=1}^{q+2}\mathcal{F}_i \subset \Lambda$, let us define

$$F^{\bar{m}}_{\Lambda,\{\mathcal{F}_i\}}(\boldsymbol{\eta}) = \prod_{i<j} \mathbb{1}_{\{\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j\}}(\boldsymbol{\eta}) \prod_{m=1}^{q+2} \prod_{\substack{C:\\ \mathbb{V}(C)\cap\mathcal{F}_m\neq\emptyset}} \frac{e^{\beta h_m |\mathbb{V}(C)\smallsetminus\Lambda^c|}}{\Theta_{\Lambda,m}(C)}\, \chi_{\Lambda,\bar{m}}(C,m) \tag{4.63}$$

for each $\bar{m} \in \{1,\dots,q+2\}$, where we recall the definitions (4.10) and (4.17).

Remark. In the following, it will be important to remember explicitly which boundary spin the measure $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$ originated from. Therefore we shall henceforth write $\mu^{\mathrm{RC}}_{\Lambda,m}$ instead of $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}$.

Lemma IV.4. *Let $A \subset \mathbb{Z}^d$ be a finite set and let f be a cylinder function in $(A, \mathbb{B}(A))$. Then there are numbers $(a_{\{\mathcal{F}_i\}})$ such that*

$$\mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}(f) = \sum_{\{\mathcal{F}_i\}} a_{\{\mathcal{F}_i\}}\, \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}\big(F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\big) \tag{4.64}$$

$$\mu^{\mathrm{ES}}_{\Lambda,\bar{m}}(f) = \sum_{\{\mathcal{F}_i\}} a_{\{\mathcal{F}_i\}}\, \mu^{\mathrm{RC}}_{\Lambda,\bar{m}}\big(F^{\bar{m}}_{\Lambda,\{\mathcal{F}_i\}}\big) \tag{4.65}$$

for each $\bar{m} \in \{1,\dots,q+2\}$ and all $\Lambda \supset A$ with $\mathbb{B}_0(\Lambda) \supset \mathbb{B}(A)$. Moreover, $a_{\{\mathcal{F}_i\}} = 0$ whenever there is an $x \in \mathcal{F} = \cup_{i=1}^{q+2}\mathcal{F}_i$ with $\operatorname{dist}(x, A) > 1$. In particular, both sums above are finite.

Proof. Let Λ be such that $\Lambda \supset A$ and $\mathbb{B}_0(\Lambda) \supset \mathbb{B}(A)$. Then by using that $\mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}$ and $\mu^{\mathrm{ES}}_{\Lambda,\bar{m}}$ are Gibbs measures we have

$$\mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}(f) = \mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}\big(\mu^{\mathrm{ES}}_{A,\mathbb{B}(A)}(f \,|\, \boldsymbol{\sigma}_{A^c}, \boldsymbol{\eta}_{\mathbb{B}(A)^c})\big), \tag{4.66}$$

and similarly for $\mu^{\mathrm{ES}}_{\Lambda,\bar{m}}(f)$. The finite volume specification $\mu^{\mathrm{ES}}_{A,\mathbb{B}(A)}(f \,|\, \boldsymbol{\sigma}_{A^c}, \boldsymbol{\eta}_{\mathbb{B}(A)^c})$ depends only on spin variables at the exterior boundary ∂A of A. It therefore suffices to prove the claim for functions of the spin variables that are cylinder in $\bar{A} = A \cup \partial A$.

Each such function f can be uniquely recast as $\sum_{\{\mathcal{F}_i\}} a_{\{\mathcal{F}_i\}} f_{\{\mathcal{F}_i\}}$, where $a_{\{\mathcal{F}_i\}}$ are real numbers such that $a_{\{\mathcal{F}_i\}} = 0$ whenever $\mathcal{F} \not\subset \bar{A}$, and

$$f_{\{\mathcal{F}_i\}}(\boldsymbol{\sigma}) = \prod_{m=1}^{q+2} \prod_{x\in\mathcal{F}_m} \delta_{\sigma_x,m}. \tag{4.67}$$

It is now a matter of a direct computation to show that, for all $\bar{m} \in \{1,\dots,q+2\}$,

$$\begin{aligned} \mu^{\mathrm{ES}}_{\Lambda,\mathrm{free}}(f_{\{\mathcal{F}_i\}} \,|\, \boldsymbol{\eta}) &= F^{\mathrm{free}}_{\{\mathcal{F}_i\}}(\boldsymbol{\eta}) \\ \mu^{\mathrm{ES}}_{\Lambda,\bar{m}}(f_{\{\mathcal{F}_i\}} \,|\, \boldsymbol{\eta}) &= F^{\bar{m}}_{\Lambda,\{\mathcal{F}_i\}}(\boldsymbol{\eta}). \end{aligned} \tag{4.68}$$

Namely, the components C of $\mathbb{B}_1(\boldsymbol{\eta})$ such that $\mathbb{V}(C) \cap \mathcal{F}_m \neq \emptyset$ necessary satisfy that $\mathbb{V}(C) \cap \mathcal{F}_i = \emptyset$ for all $i \neq m$. This gives rise to the indicators $\mathbb{1}_{\{\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j\}}$. For $\boldsymbol{\eta}$ such that $\prod_{i<j} \mathbb{1}_{\{\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j\}}(\boldsymbol{\eta}) = 1$, the spin configuration can be integrated out, yielding the ratios $e^{\beta h_m |\mathbb{V}(C)|}/\Theta_{\text{free}}(C)$ resp. $e^{\beta h_m |\mathbb{V}(C) \setminus \Lambda^c|}/\Theta_{\Lambda,\bar{m}}(C)$. However, one gets the latter only when $\mathbb{V}(C) \cap \Lambda^c = \emptyset$ or $m = \bar{m}$. The claim is finished by taking the expectation w.r.t. $\boldsymbol{\eta}$. $\square$

Proof of Theorem III.3. It was shown in Lemma IV.4 that $\boldsymbol{\sigma}$-dependent cylinder functions can be interchanged under the expectation for $\boldsymbol{\eta}$-dependent functions $F^{\text{free}}_{\{\mathcal{F}_i\}}$ and $F^m_{\Lambda,\{\mathcal{F}_i\}}$. Unfortunately, the weak limits (3.10) and (3.11) cannot yet be invoked to conclude the existence of (3.15) and (3.16), the reason being that the $F_{\{\mathcal{F}_i\}}$'s are, in general, not quasilocal (moreover, $F^m_{\Lambda,\{\mathcal{F}_i\}}$ even explicitly depends on the expanding volume). However, both functions $F^{\text{free}}_{\{\mathcal{F}_i\}}$ and $F^m_{\Lambda,\{\mathcal{F}_i\}}$ turn out to be 'almost-surely' quasilocal, in the terminology of [PVdV] and [Gr], which is still sufficient for the limits (3.10) and (3.11) to exist.

For finite sets $\mathcal{F}, \Delta$ with $\mathcal{F} \subset \Delta$, let $\mathfrak{M}_{\Delta,\mathcal{F}}$ be the event

$$\mathfrak{M}_{\Delta,\mathcal{F}} = \bigl\{\boldsymbol{\eta}\colon \forall x, y \in \mathcal{F} \ \ x \leftrightarrow \Delta^c \ \text{ and } \ y \leftrightarrow \Delta^c \ \text{ implies } \ x \underset{\Delta}{\leftrightarrow} y\bigr\}, \tag{4.69}$$

where $x \underset{\Delta}{\leftrightarrow} y$ is the event that there is a path of occupied bonds in $\mathbb{B}(\Delta)$ connecting x and y. Let further

$$\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}} = \bigl\{\boldsymbol{\eta} \in \mathfrak{M}_{\Delta,\mathcal{F}}\colon x \in \mathcal{F} \ \text{ with } \ x \leftrightarrow \Delta^c \ \text{ implies } \ x \in \mathcal{F}_m\bigr\}, \tag{4.70}$$

and recall $q_0 = \#\{m\colon h_m = h_{\max}\}$. For each Δ, $m \in \{1, \dots, q+2\}$, and $\{\mathcal{F}_i\}$ define also a random variable $Q^m_{\Delta,\{\mathcal{F}_i\}}$ by putting

$$Q^m_{\Delta,\{\mathcal{F}_i\}} = \begin{cases} q_0 & \mathcal{F}_m \leftrightarrow \Delta^c \\ 1 & \text{otherwise.} \end{cases} \tag{4.71}$$

The remainder of the proof is based on an approximation of $F_{\{\mathcal{F}_i\}}$'s by quasilocal functions and showing that the error incurred thereby upon the expectations of $F_{\{\mathcal{F}_i\}}$'s is negligible. These claims are formulated in Lemma IV.5 and Lemma IV.6.

Lemma IV.5. *For all finite $\Delta \subset \mathbb{Z}^d$ and any $\{\mathcal{F}_i\}$ with $\mathcal{F} = \cup_i \mathcal{F}_i$*

(i) $F^{\text{free}}_{\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}$ is quasilocal for all $m \in \{1, \dots, q+2\}$.

(ii) $F^{\text{free}}_{\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}}$ is quasilocal.

Proof. (i) Let m be fixed and let $\Lambda \supset \Delta$. Observe that $\mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}} \prod_{i<j} \mathbb{1}_{\{\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j\}}$ is a cylinder function in $\mathbb{B}(\Lambda)$. Hence, only the contributions from the product over the connected components in (4.62) can be altered by flipping a bond $b \notin \mathbb{B}(\Lambda)$. Let us estimate precisely the incurred change.

Let $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^b$ be two configurations differing at a single bond $b \in \mathbb{B}(\Lambda)^{\mathrm{c}}$, $\eta_b = 0, \eta_b^b = 1$. Suppose that $\boldsymbol{\eta} \in \mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}$ is such that $\prod_{i<j} \mathbb{1}_{\{\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j\}}(\boldsymbol{\eta}) = 1$ and that there is a C connecting $\mathcal{F}_m$ with $\mathbb{B}(\Lambda)^{\mathrm{c}}$. By the definition of $\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}$, the configuration $\boldsymbol{\eta}^b$ also satisfies these three conditions. Moreover, the value of $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ is clearly not affected unless $b \in C$. Suppose the latter occurs and denote by C^b the corresponding component under $\boldsymbol{\eta}^b$. Then

$$\left|F^{\mathrm{free}}_{\{\mathcal{F}_i\}}(\boldsymbol{\eta}^b) - F^{\mathrm{free}}_{\{\mathcal{F}_i\}}(\boldsymbol{\eta})\right| \leq \left|\frac{e^{\beta h_m |\mathbb{V}(C^b)|}}{\Theta_{\mathrm{free}}(C^b)} - \frac{e^{\beta h_m |\mathbb{V}(C)|}}{\Theta_{\mathrm{free}}(C)}\right|, \tag{4.72}$$

where we have estimated all ratios by 1, except for the one affected by flipping b.

It turns out that the r.h.s. of (4.72) is exponentially small in $\mathrm{dist}(b,\mathcal{F})$. Indeed, if $h_m < h_{\max}$ this is immediate by estimating $\Theta_{\mathrm{free}}(C) \geq e^{\beta h_{\max}|\mathbb{V}(C)|}$, while for $h_m = h_{\max}$ both ratios tend exponentially fast to $1/q_0$ as $\mathrm{dist}(b,\mathcal{F}) \to \infty$. Thus, the r.h.s. of (4.72) is summable over the positions of b. By the standard telescoping trick, this proves quasilocality (i.e., continuity in product topology) of $F^{\mathrm{free}}_{\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}$, as required by (i).

To prove (ii), it clearly suffices to note that

$$F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\Big[\mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}} - \sum_{m=1}^{q+2} \mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}\Big] \tag{4.73}$$

is a cylinder event in $\mathbb{B}(\Delta)$. Namely, the function in the brackets is zero unless there is no component incident with $\mathcal{F}$ that reaches up to Δ^{c}. In that case, $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ depends only on bonds from $\mathbb{B}(\Delta)$, i.e, it is effectively a local function. $\square$

Lemma IV.6. *For any $\{\mathcal{F}_i\}$ with $\mathcal{F} = \cup_{i=1}^{q+2}\mathcal{F}_i$ and any m such that $h_m = h_{\max}$*

(i)

$$\lim_{\Delta \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\mathfrak{M}_{\Delta,\mathcal{F}}) = 1 \tag{4.74}$$

$$\lim_{\Delta \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,m}(\mathfrak{M}_{\Delta,\mathcal{F}}) = 1. \tag{4.75}$$

(ii) Let $G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m} = F^m_{\Lambda,\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}} - Q^m_{\Delta,\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}} F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$. Then

$$\lim_{\Delta \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,m}\big(G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m}\big) = 0. \tag{4.76}$$

Proof. (i) The inner limits on the l.h.s. exist because $\mathfrak{M}_{\Delta,\mathcal{F}}$ is a cylinder event, and the random-cluster measures have a weak limit by Theorem III.2. The outer limit is then a consequence of the fact that $\mathfrak{M}_{\Delta,\mathcal{F}} \uparrow \mathfrak{M}_{\mathcal{F}}$, where $\mathfrak{M}_{\mathcal{F}}$ is the set of configurations featuring at most one infinite component incidental with $\mathcal{F}$. The limits are thus equal to $\mu^{\mathrm{RC}}_{\mathrm{free}}(\mathfrak{M}_{\mathcal{F}})$ and $\mu^{\mathrm{RC}}_m(\mathfrak{M}_{\mathcal{F}})$, respectively. Now, $\mu^{\mathrm{RC}}_{\mathrm{free}}$ and μ^{RC}_{m} are ergodic (see, e.g., [BoC]) and they satisfy the so-called 'finite-energy' condition. Therefore, it follows from a general argument [BK] that there is almost surely at most one infinite cluster, which means $\mu^{\mathrm{RC}}_{\mathrm{free}}(\mathfrak{M}_{\mathcal{F}}) = 1 = \mu^{\mathrm{RC}}_{m}(\mathfrak{M}_{\mathcal{F}})$. By plugging these observations together the claim (i) is proved.

To prove (ii), take $\{\mathcal{F}_i\}$ and $\Delta \subset \Lambda$ with $\Delta \supset \mathcal{F}$. Then the following three possibilities can occur for configurations under the measure $\mu^{\mathrm{RC}}_{\Lambda,m}$:

(A) $\mathcal{F} \not\leftrightarrow \Delta^{\mathrm{c}}$

(B) $\mathcal{F} \leftrightarrow \Delta^{\mathrm{c}}$ but $\mathcal{F} \not\leftrightarrow \Lambda^{\mathrm{c}}$

(C) $\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}}$.

Clearly, under (A), the absence of components connecting $\mathcal{F}$ with the outside of Δ implies

$$\mathbb{1}_{\mathfrak{M}^{m}_{\Delta,\{\mathcal{F}_i\}}} = \mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}}, \quad Q^{m}_{\Delta,\{\mathcal{F}_i\}} = 1, \quad \text{and} \quad F^{m}_{\Lambda,\{\mathcal{F}_i\}} = F^{\mathrm{free}}_{\{\mathcal{F}_i\}}, \tag{4.77}$$

by the inspection of (4.62) and (4.63). Consequently, all terms in the definition of $G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m}$ cancel and $G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m} = 0$ in this case. If (C) occurs, on the other hand, we have a component $C_{m,\Lambda}$ connecting $\mathcal{F}_m$ to $\partial\Lambda$, and thus

$$\mathbb{1}_{\mathfrak{M}^{m}_{\Delta,\{\mathcal{F}_i\}}} = \mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}}, \quad Q^{m}_{\Delta,\{\mathcal{F}_i\}} = q_0, \quad \text{and} \quad F^{m}_{\Lambda,\{\mathcal{F}_i\}} = q_0 F^{\mathrm{free}}_{\{\mathcal{F}_i\}}, \tag{4.78}$$

where we used that $\frac{1}{q_0} = \frac{e^{\beta h_m |\mathbb{V}(C_{m,\Lambda})|}}{\Theta_{\mathrm{free}}(C_{m,\Lambda})}$ $\mu^{\mathrm{RC}}_{\Lambda,m}$-almost surely. This implies that, under (C), $G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m} = 0$ as well. Hence, the estimates of the expectation $\mu^{\mathrm{RC}}_{\Lambda,m}\big(G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m}\big)$ boil down to the analysis of (B).

Let $\mathfrak{P}^{\mathcal{F}}_{\Lambda,\Delta}$ denote the event (B), i.e., $\mathfrak{P}^{\mathcal{F}}_{\Lambda,\Delta} = \{\boldsymbol{\eta}\colon \mathcal{F} \leftrightarrow \Delta^{\mathrm{c}} \text{ but } \mathcal{F} \not\leftrightarrow \Lambda^{\mathrm{c}}\}$. Then, by the preceding reasoning, $|G^{\{\mathcal{F}_i\}}_{\Lambda,\Delta,m}| \le q_0 \mathbb{1}_{\mathfrak{P}^{\mathcal{F}}_{\Lambda,\Delta}}$ $\mu^{\mathrm{RC}}_{\Lambda,m}$-a.s. Thus, it suffices to prove that

$$\lim_{\Delta\nearrow\mathbb{Z}^d} \lim_{\Lambda\nearrow\mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,m}(\mathfrak{P}^{\mathcal{F}}_{\Lambda,\Delta}) = 0. \tag{4.79}$$

We will establish this by proving that the events (A) or (C) get the full mass under these limits. First we recall the well known characterization

$$\mu^{\mathrm{RC}}_{m}(\mathcal{F} \leftrightarrow \infty) = \lim_{\Lambda\nearrow\mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,m}(\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}}). \tag{4.80}$$

This follows by the inequalities $\mu^{\mathrm{RC}}_{\tilde\Lambda,m}(\mathcal{F} \leftrightarrow \tilde\Lambda^{\mathrm{c}}) \le \mu^{\mathrm{RC}}_{\tilde\Lambda,m}(\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}}) \le \mu^{\mathrm{RC}}_{\Lambda,m}(\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}})$, where the first one is due to monotonicity of $\{\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}}\}$ in Λ and the other one is due to (4.24).

Since $\{\mathcal{F} \not\leftrightarrow \Delta^{\mathrm{c}}\} \uparrow \{\mathcal{F} \not\leftrightarrow \infty\}$ as $\Delta \nearrow \mathbb{Z}^d$, we easily get that

$$\lim_{\Delta\nearrow\mathbb{Z}^d} \lim_{\Lambda\nearrow\mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,m}\big(\{\mathcal{F} \not\leftrightarrow \Delta^{\mathrm{c}}\} \cup \{\mathcal{F} \leftrightarrow \Lambda^{\mathrm{c}}\}\big) = 1, \tag{4.81}$$

proving the desired claim. $\square$

With Lemma IV.5 and IV.6 in the hand, the proof of Theorem III.3 can be concluded. Namely, for any $\epsilon > 0$, finite sets $\bar\Lambda$, $\bar\Delta_1$, $\bar\Delta_2 \subset \mathbb{Z}^d$ can be chosen, such that

$$\begin{gathered}\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\mathfrak{M}_{\Delta,\mathcal{F}}) \geq 1-\epsilon/2\\ \mu^{\mathrm{RC}}_{\Lambda,m}(\mathfrak{M}_{\Delta,\mathcal{F}}) \geq 1-\epsilon/4\\ -\epsilon/4 \leq \mu^{\mathrm{RC}}_{\Lambda,m}\bigl(F^m_{\Lambda,\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}} - Q^m_{\Delta,\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\bigr) \leq \epsilon/4\end{gathered} \tag{4.82}$$

for any $\Lambda \supset \bar\Lambda$ and $\bar\Delta_1 \supset \Delta \supset \bar\Delta_2$, and any m such that $h_m = h_{\max}$. Since both $F^m_{\{\mathcal{F}_i\},\Lambda}$ and $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ are bounded by one, this yields

$$\begin{gathered}\Bigl|\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(F^{\mathrm{free}}_{\{\mathcal{F}_i\}}) - \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}\bigl(F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}}\bigr)\Bigr| \leq \epsilon/2\\ \Bigl|\mu^{\mathrm{RC}}_{\Lambda,m}(F^m_{\Lambda,\{\mathcal{F}_i\}}) - \mu^{\mathrm{RC}}_{\Lambda,m}\bigl(Q^m_{\Delta,\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\bigr)\Bigr| \leq \epsilon/2.\end{gathered} \tag{4.83}$$

Now the functions $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}_{\Delta,\mathcal{F}}}$ and $Q^m_{\Delta,\{\mathcal{F}_i\}}\mathbb{1}_{\mathfrak{M}^m_{\Delta,\{\mathcal{F}_i\}}}F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ are quasilocal by Lemma IV.3 and because $Q^m_{\Delta,\{\mathcal{F}_i\}}$ is cylinder, hence the limit $\Lambda \nearrow \mathbb{Z}^d$ can be performed on their expectations by Theorem III.2. Consequently

$$\begin{gathered}\bigl|\limsup_{\Lambda\nearrow\mathbb{Z}^d}\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(F^{\mathrm{free}}_{\{\mathcal{F}_i\}}) - \liminf_{\Lambda\nearrow\mathbb{Z}^d}\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(F^{\mathrm{free}}_{\{\mathcal{F}_i\}})\bigr| \leq \epsilon\\ \bigl|\limsup_{\Lambda\nearrow\mathbb{Z}^d}\mu^{\mathrm{RC}}_{\Lambda,m}(F^m_{\Lambda,\{\mathcal{F}_i\}}) - \liminf_{\Lambda\nearrow\mathbb{Z}^d}\mu^{\mathrm{RC}}_{\Lambda,m}(F^m_{\Lambda,\{\mathcal{F}_i\}})\bigr| \leq \epsilon.\end{gathered} \tag{4.84}$$

The arbitrariness of ϵ finishes the claim. $\square$

4.4. Gibbs uniqueness and absence of percolation.

Proof of Theorem III.4(i). We shall prove that any $\nu \in \mathcal{G}^{\mathrm{ES}}$ not exhibiting percolation is equal to the limiting measure $\mu^{\mathrm{ES}}_{\mathrm{free}}$ whose existence was established previously. The proof of this item goes along the lines of the argument in (4.51–54).

Let the sequences (Δ_n) and $(\Lambda_n(\boldsymbol{\eta}))$ be defined as in (4.51). Since there are no infinite components ν-a.s., the set $B_n(\boldsymbol{\eta}) = \emptyset$ for all $n \geq 1$ and ν-almost all $\boldsymbol{\eta}$. Assume a cylinder function f and given $\epsilon > 0$ take Δ large enough such that f is cylinder in Δ and

$$\bigl|\mu^{\mathrm{ES}}_{\Delta',\mathrm{free}}(f) - \mu^{\mathrm{ES}}_{\mathrm{free}}(f)\bigr| \leq \epsilon \tag{4.85}$$

for all $\Delta' \supset \Delta$. By the precisely same argument as in (4.54) we get the estimate

$$\begin{aligned}\nu(f\mathbb{1}_{\{\Lambda_n(\,\cdot\,)\not\supset\Delta\}}) &+ \bigl[\mu^{\mathrm{ES}}_{\mathrm{free}}(f)-\epsilon\bigr]\,\nu(\mathbb{1}_{\{\Lambda_n(\,\cdot\,)\supset\Delta\}})\\ &\leq \nu(f) = \nu(f\mathbb{1}_{\{\Lambda_n(\,\cdot\,)\not\supset\Delta\}}) + \sum_{\bar\Lambda_n\supset\Delta}\nu\bigl(\mu^{\mathrm{ES}}_{\bar\Lambda_n,\mathrm{free}}(f)\mathbb{1}_{\{\Lambda_n(\,\cdot\,)=\bar\Lambda_n\}}\bigr)\\ &\leq \nu(f\mathbb{1}_{\{\Lambda_n(\,\cdot\,)\not\supset\Delta\}}) + \bigl[\mu^{\mathrm{ES}}_{\mathrm{free}}(f)+\epsilon\bigr]\,\nu(\mathbb{1}_{\{\Lambda_n(\,\cdot\,)\supset\Delta\}}).\end{aligned} \tag{4.86}$$

Since f is bounded and $\Lambda_n \nearrow \mathbb{Z}^d$ ν-a.s., the Bounded Convergence Theorem yields

$$\left|\nu(f) - \mu^{\mathrm{ES}}_{\mathrm{free}}(f)\right| \le \epsilon. \tag{4.87}$$

The arbitrariness of ϵ finishes the claim. □

Proof of Theorem III.4(ii). This is an easy corollary of the previous item and the bound (3.12). Namely, if there is no percolation under $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}$ then by (3.12) all ES Gibbs measures lie in the set $\mathcal{G}^{\mathrm{ES}}_{\emptyset}$. Since by (i) there is at most one measure in $\mathcal{G}^{\mathrm{ES}}_{\emptyset}$, there is only one ES Gibbs state. □

Proof of Theorem III.4(iii). Since the if part is item (ii), we need only to concentrate on the only-if part. Suppose $q_0 \ge 2$ and $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(N_\infty > 0) > 0$. Then we claim that the set $\{\mu^{\mathrm{ES}}_m\colon h_m = h_{\max}\}$ has q_0 elements. Namely, by Proposition IV.2 each infinite cluster has a unique color sampled from $\{m\colon h_m = h_{\max}\}$. By the limiting characterization

$$\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(0 \leftrightarrow \infty) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(0 \leftrightarrow \Lambda^{\mathrm{c}}), \tag{4.88}$$

as already used in the proof of Lemma IV.6, we have for any $m \ne \tilde{m}$ such that $h_m, h_{\tilde{m}} = h_{\max}$

$$\begin{aligned}\mu^{\mathrm{ES}}_m(\sigma_0 = m) &= \mu^{\mathrm{ES}}_m(\sigma_0 = m,\, 0 \not\leftrightarrow \infty) + \mu^{\mathrm{ES}}_m(\sigma_0 = m,\, 0 \leftrightarrow \infty)\\ &= \mu^{\mathrm{ES}}_m(\sigma_0 = \tilde{m}) + \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(0 \leftrightarrow \infty),\end{aligned} \tag{4.89}$$

where in the term with $0 \not\leftrightarrow \infty$ we used that there is a complete symmetry between m and $\tilde{m}$ on finite clusters, whereas on $0 \leftrightarrow \infty$ the spin at 0 is fixed to m by (4.88). Whenever $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(0 \leftrightarrow \infty) > 0$, which is by translation invariance equivalent to $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(N_\infty > 0) > 0$, this implies that the symmetry between the m's with $h_m = h_{\max}$ is broken and thus q_0 distinct phases coexist. □

Before we embark on proving item (iv) of Theorem III.4, we shall establish the following useful claim.

Lemma IV.7. *The measure $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}$ is strongly mixing and, in particular, ergodic w.r.t. translations in any of the lattice principal directions.*

Proof. Let τ denote the translation in one of the lattice principal directions. We shall show that $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(f\, g \circ \tau^n) \to \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(f)\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(g)$ for all L^2-functions f and g. As is well known, it actually suffices to verify this for cylinder functions (which are dense in L^2), and since we have a space with a natural ordering, we can even restrict ourselves to f, g monotone.

Let f, g be non-negative monotone increasing cylinder functions and let $g_n = g \circ \tau^n$. Then $f g_n$ is also monotone increasing and hence for any box Δ centered at the origin, an integer n and another box Λ large enough (taken in this order)

$$\begin{aligned}\mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}(f g_n) \le \mu^{\mathrm{RC}}_{\Lambda,\mathrm{maxwir}}\bigl(f g_n | \{\boldsymbol{\eta}_{\mathbb{B}(\partial\Delta)} = 1\}\bigr) = \mu^{\mathrm{ES}}_{\Lambda,m}\bigl(f g_n | \{\boldsymbol{\eta}_{\mathbb{B}(\partial\Delta)} = 1\}\bigr)\\ = \mu^{\mathrm{ES}}_{\Delta,\mathrm{maxwir}}(f)\, \mu^{\mathrm{ES}}_{\tau^{-n}\Lambda,\mathrm{maxwir}}\bigl(g | \{\boldsymbol{\eta}_{\mathbb{B}(\tau^{-n}\partial\Delta)} = 1\}\bigr),\end{aligned} \tag{4.90}$$

where we used Gibbsiannes of $\mu^{\mathrm{ES}}_{\Lambda,m}$ (here m is such that $h_m = h_{\max}$) to 'split' f from g. The latter expectation tends in the limit $\Lambda \nearrow \mathbb{Z}^d$ and $n \to \infty$ to $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}(g)$ as follows from (4.24), so we get

$$\limsup_{n\to\infty} \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(f\, g \circ \tau^n) \le \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(f)\, \mu^{\mathrm{RC}}_{\mathrm{maxwir}}(g), \tag{4.91}$$

where we have also taken the limit $\Delta \nearrow \mathbb{Z}^d$ on the r.h.s. of (4.90). Since the complementary inequality follows from FKG, the strong-mixing property of $\mu^{\mathrm{RC}}_{\mathrm{maxwir}}$ is established. □

Proof of Theorem III.4(iv). Suppose that $J > J_c$ and $d = 2$. Then the first condition implies that there is percolation under $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$. Moreover, since $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$ satisfies the following items

(1) $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$ is separately ergodic in all lattice directions
(2) $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$ is invariant under lattice reflections and rotations
(3) $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$ is FKG,

(as has been proved previously) the powerful result of [GKR] asserts that the infinite cluster is unique under $\mu^{\mathrm{RC},J}_{\mathrm{maxwir}}$. Moreover, by a corollary to this result, the cluster contains an infinite series of nested circuits that (eventually) encircle any point of the lattice.

Suppose moreover $q_0 = 1$. Then (3.13) yields that the random-cluster marginal $\nu^{\mathrm{RC},J}$ of any $\nu \in \mathcal{G}^{\mathrm{ES}}$ at the coupling constant J is bounded below by $\mu^{\mathrm{RC},J}_{\mathrm{free}}$. Let $J > J_1 > J_c$. Then

$$\nu^{\mathrm{RC},J}(\,\cdot\,) \underset{\mathrm{FKG}}{\ge} \mu^{\mathrm{RC},J}_{\mathrm{free}}(\,\cdot\,) \underset{\mathrm{FKG}}{\ge} \mu^{\mathrm{RC},J_1}_{\mathrm{maxwir}}(\,\cdot\,), \tag{4.92}$$

where the second inequality is Theorem III.2(iv). In particular, *all* measures at J exhibit an infinite cluster as well as the above circuits running around the origin, because the latter is an FKG-increasing event.

The proof is concluded along very much the same lines as was the argument (4.85-87), but now with a different definition of volumes Λ_n. Let (Δ_n) be an increasing sequence of boxes centered at the origin and let $\Lambda_n(\boldsymbol{\eta})$ be the *interior* sites of the maximal closed circuit of occupied bonds surrounding the origin and contained wholly in $\mathbb{B}_0(\Delta_n)$. As before, the event $\{\Lambda_n(\,\cdot\,) = \bar{\Lambda}_n\}$ does not depend on $\boldsymbol{\eta}_{\mathbb{B}(\bar{\Lambda}_n)}$, hence the conditioning argument of (4.86) along with the existence of the limiting state (3.15) proves that $\nu = \mu^{\mathrm{ES}}_m$, where m is the unique spin such that $h_m = h_{\max}$. □

4.5 Relations to the Gibbs random-cluster measures.

Proof of Theorem III.5(i). Since the free measure $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}$ can be directly defined by the weights (4.7), it is not hard to verify that $\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}$ satisfies the DLR condition

$$\mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(f) = \int \mu^{\mathrm{RC}}_{\Lambda,\mathrm{free}}(\mathrm{d}\boldsymbol{\eta})\mu^{\mathrm{RC}}_{\mathbb{B}}(f\,|\,\boldsymbol{\eta}_{\mathbb{B}^c}), \tag{4.93}$$

for any $\mathbb{B}$-cylinder function f and any $\mathbb{B} \subset \mathbb{B}(\Lambda)$.

Recalling (4.69), we now claim that $\boldsymbol{\eta} \mapsto \mathbb{1}_{\mathfrak{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\boldsymbol{\eta})\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\boldsymbol{\eta})$ is a quasilocal function for any $\Delta \supset \mathbb{V}(\mathbb{B})$. Namely, under $\mathfrak{M}_{\Delta,\mathbb{V}(\mathbb{B})}$, there is at most one infinite cluster connecting $\mathbb{B}$ with Δ^{c} and if there is one, then its weight tends exponentially fast to q_0 as $\Delta \nearrow \mathbb{Z}^d$. As in the proof of Lemma IV.5, this is sufficient for the quasilocality of $\boldsymbol{\eta} \mapsto \mathbb{1}_{\mathfrak{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\boldsymbol{\eta})\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\boldsymbol{\eta})$.

Let f be bounded. Then the proof is finished by the following array of identities

$$\begin{aligned}\mu_{\mathrm{free}}^{\mathrm{RC}}(f) = \lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC}}(f) &= \lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC}}\big(\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\,\cdot\,)\big)\\ &= \lim_{\Delta\nearrow\mathbb{Z}^d}\lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC}}\big(\mathbb{1}_{\mathfrak{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\,\cdot\,)\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\,\cdot\,)\big)\\ &= \lim_{\Delta\nearrow\mathbb{Z}^d}\mu_{\mathrm{free}}^{\mathrm{RC}}\big(\mathbb{1}_{\mathfrak{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\,\cdot\,)\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\,\cdot\,)\big) = \mu_{\mathrm{free}}^{\mathrm{RC}}\big(\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\,\cdot\,)\big),\end{aligned} \tag{4.94}$$

where the five equalities follow by successively applying (3.11), (4.93), (4.74) with the boundedness of $\mu_{\mathbb{B}}^{\mathrm{RC}}(f\,|\,\cdot\,)$, (3.11) and the quasilocality of the involved function, and finally using again (3.11). Hence, $\mu_{\mathrm{free}}^{\mathrm{RC}}$ satisfies (3.17), i.e., $\mu_{\mathrm{free}}^{\mathrm{RC}} \in \mathcal{G}^{\mathrm{RC}}$. The case of $\mu_{\Lambda,\mathrm{maxwir}}^{\mathrm{RC}}$ is completely analogous. $\square$

Proof of Theorem III.5(ii). We claim that $\mu^{\mathrm{RC}}(\boldsymbol{\eta}_{\mathbb{B}}|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}})$ is strong FKG (as a measure in $\mathbb{B}$). In order to prove that we note that since the weight (3.17) is essentially a rewrite of (4.8), we can use that (4.8) gives rise to a strong FKG measure. Namely, pick $\boldsymbol{\eta}$ and define $\boldsymbol{\eta}^{(\Delta)}$ by the formula

$$\eta_b^{(\Delta)} = \begin{cases} \eta_b & b \in \mathbb{B}_0(\Delta)\\ 0 & \text{otherwise.}\end{cases} \tag{4.95}$$

Then we have $W_{\mathbb{B}_0(\Delta)}^{\mathrm{RC}}\big(\boldsymbol{\eta}_{\mathbb{B}_0(\Delta)}^{(\Delta)}|\,\boldsymbol{\eta}_{\mathbb{B}_0(\Delta)^{\mathrm{c}}}^{(\Delta)}\big) = e^{-\beta h_{\max}|\Delta|}W_{\Delta,\mathrm{free}}^{\mathrm{RC}}\big(\boldsymbol{\eta}^{(\Delta)}\big)$. Consequently,

$$\mu_{\mathbb{B}(\Delta)}^{\mathrm{RC}}\big(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}_0(\Delta)^{\mathrm{c}}}^{(\Delta)}\big) = \mu_{\Delta,\mathrm{free}}^{\mathrm{RC}}\big(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}(\Delta)^{\mathrm{c}}}^{(\Delta)}\big). \tag{4.96}$$

Since the latter measure is strong FKG and since $(\mu_{\mathbb{B}}^{\mathrm{RC}})$ form a consistent family of specifications, $\mu_{\mathbb{B}}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}}^{(\Delta)})$ is strong FKG as well for any $\mathbb{B} \subset \mathbb{B}(\Delta)$ (this is because conditioned strong FKG measures are still strong FKG).

It follows that $\mu_{\mathbb{B}}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}}^{(\Delta)}) \to \mu_{\mathbb{B}}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}})$ as $\Delta \nearrow \mathbb{Z}^d$. This is because for each $\boldsymbol{\eta}$ there is Δ such that the number of infinite components reaching from $\mathbb{V}(\mathbb{B})$ outside Δ is equal to the number of infinite components touching $\mathbb{V}(\mathbb{B})$ (here we used that there is only finitely many infinite clusters connected to $\mathbb{B}$). Hence $\mu_{\mathbb{B}}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}})$ is strong FKG for all $\boldsymbol{\eta}$. In particular, $\mu_{\mathbb{B}}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}^{\mathrm{c}}})$ is increasing in the boundary condition (the specifications are consistent) and

$$\mu_{\Lambda,\mathrm{free}}^{\mathrm{RC}}(\,\cdot\,) \underset{\mathrm{FKG}}{\leq} \mu_{\mathbb{B}(\Lambda)}^{\mathrm{RC}}(\,\cdot\,|\,\boldsymbol{\eta}_{\mathbb{B}(\Lambda)^{\mathrm{c}}}) \underset{\mathrm{FKG}}{\leq} \mu_{\Lambda,\mathrm{maxwir}}^{\mathrm{RC}}(\,\cdot\,), \tag{4.97}$$

where we identified the specification with the RC measures for the free and m-wired boundary condition. The claim is finished by applying this to the DLR condition for any monotone cylinder event, followed by the limit $\Lambda \nearrow \mathbb{Z}^d$. $\square$

Proof of Theorem III.5(iii). Let $\nu \in \mathcal{G}^{\mathrm{ES}}$ be extremal (without loss of generality) with the additional assumption that $\nu(N_\infty > 1) = 0$ whenever $q_0 > 1$. By the Backward Martingale Theorem [Ge, Theorem 7.12], $\mu^{\mathrm{ES}}_{\Delta_n,\mathbb{B}(\Delta_n)}(\,\cdot\,|\boldsymbol{\sigma}_{\Delta_n^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Delta_n)^{\mathrm{c}}}) \to \nu$ for ν-almost all $(\boldsymbol{\sigma},\boldsymbol{\eta})$ along any *increasing* sequence $\Delta_n \nearrow \mathbb{Z}^d$. Let $\Lambda \subset \Delta$ be finite volumes and let us recall that $\lim_{\Delta\nearrow\mathbb{Z}^d} \nu(\mathfrak{M}_{\Delta,\Lambda}) = 1$ (see (4.69) for the definition of $\mathfrak{M}_{\Delta,\Lambda}$) whenever $q_0 > 1$. Hence, with an overwhelming probability provided n and Δ are large enough, the boundary of Λ 'sees' at most one of the spins from $\partial\Delta_n$ above. It is easy to verify that, under this condition, the random-cluster measure in $\mathbb{B}(\Lambda)$ induced from $\mu^{\mathrm{ES}}_{\Delta_n,\mathbb{B}(\Delta_n)}(\,\cdot\,|\boldsymbol{\sigma}_{\Delta_n^{\mathrm{c}}}, \boldsymbol{\eta}_{\mathbb{B}(\Delta_n)^{\mathrm{c}}})$ above is Gibbsian for the specification (3.17). The claim is finished by taking $n \to \infty$ followed by $\Delta \nearrow \mathbb{Z}^d$. $\square$

Proof of Theorem III.5(iv). Recall the definition of $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ in (4.62). It turns out that $F^{\mathrm{free}}_{\{\mathcal{F}_i\}}$ satisfies the following identity:

$$\sum_{m=1}^{q+2} F^{\mathrm{free}}_{\{\mathcal{F}_1,\dots,\mathcal{F}_{m-1},\mathcal{F}_m\cup\{x\},\mathcal{F}_{m+1},\dots,\mathcal{F}_{q+2}\}}(\boldsymbol{\eta}) = F^{\mathrm{free}}_{\{\mathcal{F}_i\}}(\boldsymbol{\eta}) \tag{4.98}$$

for each $\{\mathcal{F}_i\}$, any $x \notin \mathcal{F} = \cup_i \mathcal{F}_i$ and any $\boldsymbol{\eta}$. Namely, let $\mathcal{F}_i \not\leftrightarrow \mathcal{F}_j$ for $i \neq j$ in $\boldsymbol{\eta}$ and suppose $x \leftrightarrow \mathcal{F}_m$ for some m. Then the sum on the l.h.s. of (4.98) degenerates to the m-th summand, which is easily identified with the r.h.s. On the other hand, if $x \not\leftrightarrow \mathcal{F}_m$ for all m, then the sum in (4.98) can be propagated through the products in (4.62) up to the last term, where the desired identity then follows by taking also (4.8) into account.

The relation (4.98) enables us to define a joint measure on $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$. Let $\mu \in \mathcal{G}^{\mathrm{RC}}$ and let $\mathfrak{A}_{\{\mathcal{F}_i\}}$ denote the event

$$\mathfrak{A}_{\{\mathcal{F}_i\}} = \big\{\boldsymbol{\sigma} \colon \sigma_x = m \ \forall x \in \mathcal{F}_m\big\}. \tag{4.99}$$

Note that $\mathfrak{A}_{\{\mathcal{F}_i\}}$ is a cylinder event in $\mathcal{F}$. Consider the set function ν, for the sets on the product space of configurations $(\boldsymbol{\sigma},\boldsymbol{\eta})$, defined as

$$\nu\big(\mathfrak{A}_{\{\mathcal{F}_i\}} \times \mathfrak{B}\big) = \mu\big(\,F^{\mathrm{free}}_{\{\mathcal{F}_i\}} \mathbb{1}_{\mathfrak{B}}\big), \tag{4.100}$$

where $\mathfrak{B}$ stands for any cylinder event on configurations $\boldsymbol{\eta}$. Due to the fact that μ is a measure on $\boldsymbol{\eta}$ and due to (4.98), the set function defined in (4.100) satisfies the consistency condition for all finite-volume projections and, by the Kolmogorov theorem, it thus gives rise to a measure on $(\boldsymbol{\sigma},\boldsymbol{\eta})$.

Clearly, the $\boldsymbol{\eta}$-marginal of ν is μ, by the very construction, so it remains to show that $\nu \in \mathcal{G}^{\mathrm{ES}}$. Let Λ be a finite set of sites and let $(\Lambda_i)_{i=1}^{q+2}$ be a partition thereof. We would like to use Gibbsianness of μ under the expectation in (4.100). However, in order to keep things mathematically just, we first need to mollify $F^{\mathrm{free}}_{\{\Lambda_i\}}$ for its non-quasilocality by inserting the event $\mathfrak{M}_{\Delta,\Lambda}$, where $\Delta \supset \Lambda$. Then we claim that for any $\mathbb{B} \supset \mathbb{B}(\Delta)$

$$\begin{aligned}\nu\big(\mathfrak{A}_{\{\Lambda_i\}} \times (\mathfrak{B}\cap\mathfrak{M}_{\Delta,\Lambda})\big) &= \mu\big(\mu^{\mathrm{RC}}_{\mathbb{B}}(\,F^{\mathrm{free}}_{\{\Lambda_i\}} \mathbb{1}_{\mathfrak{B}\cap\mathfrak{M}_{\Delta,\Lambda}})\big)\\ &= \mu\Big(\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}\big(\mathfrak{A}_{\{\Lambda_i\}} \times (\mathfrak{B}\cap\mathfrak{M}_{\Delta,\Lambda})\big)\Big) = \nu\Big(\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}\big(\mathfrak{A}_{\{\Lambda_i\}} \times (\mathfrak{B}\cap\mathfrak{M}_{\Delta,\Lambda})\big)\Big).\end{aligned} \tag{4.101}$$

Here in the first equality we used (4.100) and the fact that $\mu \in \mathcal{G}^{\mathrm{RC}}$. In the second equality we used that

$$F^{\mathrm{free}}_{\{\Lambda_i\}}(\boldsymbol{\eta})\mu^{\mathrm{RC}}_{\mathbb{B}}(\boldsymbol{\eta}_{\mathbb{B}}|\,\boldsymbol{\eta}_{\mathbb{B}^c}) = \mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}\big(\mathfrak{A}_{\{\Lambda_i\}} \times \{\boldsymbol{\eta}_{\mathbb{B}}\}|\boldsymbol{\sigma}_{\Lambda^c}, \boldsymbol{\eta}_{\mathbb{B}^c}\big). \tag{4.102}$$

As follows by inspection of (3.6), the last term in (4.102) either vanishes, if $\boldsymbol{\sigma}$ is inconsistent with $\boldsymbol{\eta}$, or it is independent of $\boldsymbol{\sigma}_{\Lambda^c}$. Since the former event has probability zero under ν, the expectation over the last term can equally be taken w.r.t. μ or ν, which justifies the third equality in (4.101).

The formula (4.102) is proved by noting that if $\boldsymbol{\eta}$ is such that $\Lambda_i \not\leftrightarrow \Lambda_j$ for any $i \neq j$, then

$$\begin{aligned} F^{\mathrm{free}}_{\{\Lambda_i\}}(\boldsymbol{\eta})\mu_{\mathbb{B}}(\boldsymbol{\eta}_{\mathbb{B}}|\,\boldsymbol{\eta}_{\mathbb{B}^c}) = &\\ \frac{1}{\mathcal{Z}^{\boldsymbol{\eta}}_{\Lambda,\mathbb{B}}}\,(e^{\beta J}-1)^{|\mathbb{B}_1(\boldsymbol{\eta})\cap\mathbb{B}|} &\prod_{\substack{\mathbb{V}(C(\boldsymbol{\eta}))\cap\mathbb{V}(\mathbb{B})\neq\emptyset\\ \mathbb{V}(C(\boldsymbol{\eta}))\cap\Lambda=\emptyset}} e^{-\beta h_{\max}|\mathbb{V}(C(\boldsymbol{\eta}))|}\,\Theta_{\mathrm{free}}\big(C(\boldsymbol{\eta})\big)\\ &\times\prod_{m=1}^{q+2}\prod_{\mathbb{V}(C(\boldsymbol{\eta}))\cap\Lambda_m\neq\emptyset} e^{-\beta(h_{\max}-h_m)|\mathbb{V}(C(\boldsymbol{\eta}))|}, \end{aligned} \tag{4.103}$$

where $\mathcal{Z}^{\boldsymbol{\eta}}_{\Lambda,\mathbb{B}}$ is some suitable normalization. Since the exponenents in the terms $e^{-\beta h_{\max}|\mathbb{V}(C(\boldsymbol{\eta}))|}$ combine to $\sum_{C(\boldsymbol{\eta})\cap\mathbb{B}\neq\emptyset}|\mathbb{V}(C(\boldsymbol{\eta}))|$, which depends neither on $\boldsymbol{\eta}_{\mathbb{B}}$ nor on the partition (Λ_i), the r.h.s.'s of (4.102) and (4.103) are easily identified.

The proof of (iv) is now completed by using that (4.101) is satisfied for any events of the type $\mathfrak{C} \cap \mathfrak{M}_{\Delta,\Lambda}$, where $\mathfrak{C}$ is any cylinder event in Λ of both $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$. By neglecting the event $\mathfrak{M}_{\Delta,\Lambda}$ on the r.h.s. of (4.101) and by recalling that $\lim_{\Delta\nearrow\mathbb{Z}^d}\nu(\mathfrak{M}_{\Delta,\Lambda}) = 1$ from the assumption $\mu(N_\infty > 1) = 0$, we get the inequality

$$\nu(\mathfrak{C}) \le \nu\big(\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}(\mathfrak{C}|\,\cdot\,)\big), \tag{4.104}$$

for any $\Lambda,\mathbb{B}$-cylinder event $\mathfrak{C}$ on configurations $(\boldsymbol{\sigma},\boldsymbol{\eta})$. Here we used the consistency of the specifications $\mu^{\mathrm{ES}}_{\Lambda,\mathbb{B}}$ to keep $\mathbb{B}$ fixed while Δ is expanding (note that (4.101) was derived under the assumption that $\mathbb{B} \supset \mathbb{B}(\Delta)$). By going to the complementary event, the DLR equation for ν is proved. $\square$

5. COLORED RANDOM-CLUSTER REPRESENTATIONS AND THE CONTOUR MODEL

5.1. Colored random-cluster model.

In the preceding section, we studied the random-cluster model and the connection between its percolation properties and the uniqueness of the spin Gibbs states. The random-cluster measures, being just the $\boldsymbol{\eta}$-marginal of the ES Gibbs measures, were 'colorless' because no information about the incidental spin values was retained. In this section to introduce an alternative random-cluster representation, where some information about the spins remains attached to the bond configurations. Even though such a 'colored' representation can be introduced for general values of external fields h_m, $m = 1, \ldots, q+2$, it will be convenient to restrict oneself to the case with only two nonvanishing fields.

Let us consider the spin system as described by the Hamiltonian (2.1), where we assume that $h_3 = \cdots = h_{q+2} = 0$. We shall consider the following marginal of the ES Gibbs measures defined in Section 3. With each $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ we associate the bond configuration $\boldsymbol{\omega} \in \{I, J, Q, D\}^{\mathbb{B}(\mathbb{Z}^d)}$ by putting for each bond $b \equiv \langle x, y \rangle$

$$\omega_b = \begin{cases} I & \eta_b = 1,\ \sigma_x = \sigma_y = 1 \\ J & \eta_b = 1,\ \sigma_x = \sigma_y = 2 \\ Q & \eta_b = 1,\ \sigma_x = \sigma_y \in \mathcal{Q} \\ D & \text{otherwise,} \end{cases} \tag{5.1}$$

where we denote $\mathcal{Q} = \{3, \ldots, q+2\}$, i.e., the spin values unaffected by the external fields. The mapping $(\boldsymbol{\sigma}, \boldsymbol{\eta}) \mapsto \boldsymbol{\omega}$ is measurable w.r.t. the natural σ-algebra on configurations $(\boldsymbol{\sigma}, \boldsymbol{\eta})$, hence it induces a well-defined marginal, which we call the *colored* random cluster measure.

The colored random-cluster marginal will play an important role in the computation of (some part of) the phase diagram performed in the next Section. For that it will be useful to have explicit formulas for the weigths of the graphical configurations. Therefore, similarly as in (3.4), we employ the standard Fortuin-Kasteleyn trick in the 'colored' version,

$$e^{\beta J \delta_{\sigma_x, \sigma_y}} = 1 + (e^{\beta J} - 1)\delta^{\mathcal{Q}}_{\sigma_x, \sigma_y} + (e^{\beta J} - 1)\delta_{\sigma_x, 1}\delta_{\sigma_y, 1} + (e^{\beta J} - 1)\delta_{\sigma_x, 2}\delta_{\sigma_y, 2}. \tag{5.2}$$

Here, $\delta^{\mathcal{Q}}_{\sigma_x, \sigma_y}$ indicates that the spins at x and y are equal *and* belong to $\mathcal{Q}$. The four terms correspond exactly to the four distinct types of bonds in (5.1), here in the order D, Q, I, and J.

By using (5.2) we get a natural graphical representation of the partition function $Z = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}$ as a sum $\sum_{\boldsymbol{\omega}} W(\boldsymbol{\omega})$ with suitable weights $W(\boldsymbol{\omega})$, and similarly for the Gibbs measures of the model. To make this precise, we will consider a finite volume Λ, and specify boundary conditions on $\partial\mathbb{B}(\Lambda) = \mathbb{B}(\Lambda) \setminus \mathbb{B}_0(\Lambda)$. Actually, we will restrict ourselves to constant boundary coditions L (corresponding to constant and free boundary conditions of the spin model)—with L standing for I, J, Q, or D—defined by fixing all bonds on the boundary to be L-bonds. If $L = Q$, I, or J we call such a boundary condition *wired*, while for $L = D$ we call it *free*. Let $\Omega^L_{\mathbb{B}(\Lambda)}$ be the space of all possible configurations with boundary condition L. Note that $\Omega^L_{\mathbb{B}(\Lambda)} \neq \{I, J, Q, D\}^{\mathbb{B}(\Lambda)}$ because no two ordered bonds of different color can meet—they are always interlaced by at least one D bond.

To determine the weights $W_{\Lambda,L}(\boldsymbol{\omega})$ for graphical configurations $\boldsymbol{\omega} \in \Omega^L_{\mathbb{B}(\Lambda)}$ in finite Λ with fixed boundary condition L, we sum the Boltzman weights over all $\boldsymbol{\sigma}$ consistent with $\boldsymbol{\omega}$. Up to a boundary factor independent of $\boldsymbol{\omega} \in \Omega^L_{\mathbb{B}(\Lambda)}$, we get

$$W_{\Lambda,L}(\boldsymbol{\omega}) = (e^{\beta J} - 1)^{|\mathbb{B}_Q(\boldsymbol{\omega})| + |\mathbb{B}_I(\boldsymbol{\omega})| + |\mathbb{B}_J(\boldsymbol{\omega})|} \\ \times e^{\beta h_1 |\mathbb{V}_I(\boldsymbol{\omega})|} \, e^{\beta h_2 |\mathbb{V}_J(\boldsymbol{\omega})|} \, (q + e^{\beta h_1} + e^{\beta h_2})^{|\mathbb{V}_D(\boldsymbol{\omega})|} \, q^{C_Q(\boldsymbol{\omega})}. \tag{5.3}$$

Here, for any $\bar{L} \in \{Q, I, J\}$, we use $\mathbb{B}_{\bar{L}}(\boldsymbol{\omega}) \subset \mathbb{B}(\Lambda)$, to denote the set of $\bar{L}$-bonds in $\boldsymbol{\omega}$. The set $\mathbb{V}_I(\boldsymbol{\omega}) \cap \Lambda$ consists of all sites incident with the bonds in $\mathbb{B}_I(\boldsymbol{\omega})$ (and similarly for $\mathbb{V}_J(\boldsymbol{\omega})$), while $\mathbb{V}_D(\boldsymbol{\omega}) \cap \Lambda$ is the set of all lattice sites in Λ such that all adjacent bonds are of D-type. Furthermore, $\mathbb{V}_{\bar{L}}(\boldsymbol{\omega}) \cap \partial\Lambda = \partial\Lambda$ if $\bar{L}$ equals the boundary condition L and $\mathbb{V}_{\bar{L}}(\boldsymbol{\omega}) \cap \partial\Lambda = \emptyset$, otherwise. Finally, $C_Q(\boldsymbol{\omega})$ stands for the number of connected components of Q-bonds (excluding, however, the boundary component).

5.2. Contour model.

In this subsection we reformulate the graphical representation into a labeled-contour model. We essentially adopt the definition presented in [BKM] (see also [K] for a pedagogical treatment on these issues in the context of the standard Potts model).

Given a configuration $\boldsymbol{\omega}$ on $\mathbb{B}(\mathbb{Z}^d)$, a closed k-dimensional unit hypercube $c \subset \mathbb{R}^d$ with vertices in $\mathbb{Z}^d$ is called *occupied* if all bonds $b \subset c$ are ordered (i.e., if they are not of type D). In particular, occupied one-dimensional hypercubes are just the ordered bonds themselves. Let a boundary condition L be given and let $\boldsymbol{\omega} \in \Omega^L_{\mathbb{B}(\Lambda)}$. To introduce contours, let $U(\boldsymbol{\omega}) \subset \mathbb{R}^d$ be the 1/3-neighborhood of the union of all occupied k-dimensional hypercubes, $k = 1, \dots, d$, i.e.,

$$U(\boldsymbol{\omega}) = \{x \in \mathbb{R}^d : \exists c \text{ occupied, with } \operatorname{dist}(x, c) < 1/3\}. \tag{5.4}$$

Since no two ordered bonds of different type can share a common endpoint, all bonds in a component of $U(\boldsymbol{\omega})$ are of the same type, inducing thus a label Q, I, or J to each connected component of $U(\boldsymbol{\omega})$. The complement $\mathbb{R}^d \setminus U(\boldsymbol{\omega})$ is labeled by D. *Contours* (denoted by Γ) of $\boldsymbol{\omega}$ are then the components of the boundary of $U(\boldsymbol{\omega})$.

Note that, due to $\boldsymbol{\omega} \in \Omega^L_{\mathbb{B}(\Lambda)}$, each contour Γ is an orientable surface that splits $\mathbb{R}^d \setminus \Gamma$ into a bounded component, $\operatorname{Int}\Gamma$, and an unbounded one, $\operatorname{Ext}\Gamma$, i.e., $\operatorname{Ext}\Gamma \cup \Gamma \cup \operatorname{Int}\Gamma = \mathbb{R}^d$. Moreover, Γ is touching one component of $U(\boldsymbol{\omega})$ and one component of $\mathbb{R}^d \setminus U(\boldsymbol{\omega})$ and is thus endowed with two *labels*—interior and exterior ones, denoted by i and e. The whole triple $\gamma = (\Gamma, i, e)$ is called a *labeled contour*. Notably, while one of the labels is of the ordered type (Q, I, or J), the other is always disordered. Given a labeled contour $\gamma = (\Gamma, i, e) = (\Gamma_\gamma, i_\gamma, e_\gamma)$, we use $\mathbb{B}(\gamma) = \mathbb{B}(\Gamma_\gamma)$ to denote the set of all bonds intersecting Γ_γ. Notice that, neccesarily, $\omega(b) = D$ for each $b \in \mathbb{B}(\gamma)$. Finally, we use the notation $\operatorname{Int}\gamma = \operatorname{Int}\Gamma_\gamma$ and $\operatorname{Ext}\gamma = \operatorname{Ext}\Gamma_\gamma$.

Clearly, there is a one-to-one correspondence between graphical configurations $\boldsymbol{\omega} \in \Omega^L_{\mathbb{B}(\Lambda)}$ and sets $\partial(\boldsymbol{\omega})$ of labeled contours, for which

(1) no pair of contours $\gamma_1, \gamma_2 \in \partial(\boldsymbol{\omega})$ spatially overlap
(2) the contour labels match (i.e., the labels of contours attached to a component of $\mathbb{R}^d \setminus \cup_{\gamma\in\partial}\gamma$ coincide)
(3) the exterior labels of exterior contours equal to L.

Correspondingly, we need to recast the weights $W_{\Lambda,L}(\boldsymbol{\omega})$ as the products of some contour weights and the 'ground state' contributions associated with the regions of constant configurations in the complement of the contours. There are two strategies for separating these volume terms depending on whether one regards the 'volume' as the number of bonds contained in it, or the number of its sites. These approaches lead to different contour weights, which will allow us to study the model in two different parameter domains. Namely, it turns out that the former is suitable for all temperatures but limited external fields, while the latter is suitable for arbitrary fields but only a restricted range of temperatures.

Bond model. To introduce the first model, we redistribute the site weights in (5.2) uniformly over all adjacent bonds. Namely, by using that every point in a set $\mathbb{V}$ of lattice sites is shared by $2d$ bonds in $\mathbb{B}(\mathbb{V})$, we can write

$$2d|\mathbb{V}_I| = 2|\mathbb{B}_I| + \sum_{\gamma: i_\gamma \text{ or } e_\gamma = I} |\gamma| + R_{I,L} \tag{5.5}$$

and similarly for J. Here $|\gamma| = |\Gamma_\gamma|$ denotes the 'area' of γ defined as the number of intersection points of Γ_γ with bonds from $\mathbb{B}(\gamma)$. (Note that any bond $b \in \mathbb{B}_D$, such that both its endpoints are in $\mathbb{V}_I$, has two intersections with Γ_γ and therefore contributes twice to $|\gamma|$.) The boundary term $R_{\bar{L},L}$, defined for any $L, \bar{L} \in \{I, J, Q, D\}$ as

$$R_{\bar{L},L} = \begin{cases} 2d(|\Lambda| + |\partial\Lambda|) - |\mathbb{B}(\Lambda)| & \bar{L} = L \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

arises because the sites in $\partial\Lambda$ are contained in less then $2d$ bonds in $\mathbb{B}(\Lambda)$. Similarly, we get

$$2d|\mathbb{V}_D| = 2|\mathbb{B}_D| - \sum_{\gamma: i_\gamma \text{ or } e_\gamma = D} |\gamma| + R_{D,L}, \tag{5.7}$$

where $\mathbb{B}_D$ is the set of all disordered bonds, $\mathbb{B}_D = \mathbb{B}(\Lambda) \setminus (\mathbb{B}_I \cup \mathbb{B}_J \cup \mathbb{B}_Q)$. The difference in sign in front of the sum originates from the fact that both endpoints of bonds from $\mathbb{B}_I$ lie in $\mathbb{V}_I$, whereas $\mathbb{B}_D$ contains also bonds with one endpoint or even no endpoints in $\mathbb{V}_D$. Substituting these formulas into (5.3), we get, up to a fixed boundary factor stemming from the $R_{\cdot,L}$'s, the expression

$$W_{\mathbb{B}(\Lambda)}(\boldsymbol{\omega}) = (e^{\beta J}-1)^{|\mathbb{B}_Q(\boldsymbol{\omega})|}\,\big[e^{\beta h_1/d}(e^{\beta J}-1)\big]^{|\mathbb{B}_I(\boldsymbol{\omega})|}\,\big[e^{\beta h_2/d}(e^{\beta J}-1)\big]^{|\mathbb{B}_J(\boldsymbol{\omega})|} \\ \times\,\big[(q+e^{\beta h_1}+e^{\beta h_2})^{1/d}\big]^{|\mathbb{B}_D(\boldsymbol{\omega})|}\prod_{\gamma\in\partial(\boldsymbol{\omega})}\varrho(\gamma). \quad (5.8)$$

Here the contour weights $\varrho(\gamma)$, for contours with the labels $e_\gamma \to i_\gamma$, are given by

$$\varrho(\gamma) = \begin{cases} (q+e^{\beta h_1}+e^{\beta h_2})^{-\frac{|\gamma|}{2d}} & Q \to D \\ (q+e^{\beta h_1}+e^{\beta h_2})^{-\frac{|\gamma|}{2d}}\,q & D \to Q \\ \big[e^{-\beta h_1}(q+e^{\beta h_1}+e^{\beta h_2})\big]^{-\frac{|\gamma|}{2d}} & D \to I,\ I \to D \\ \big[e^{-\beta h_2}(q+e^{\beta h_1}+e^{\beta h_2})\big]^{-\frac{|\gamma|}{2d}} & D \to J,\ J \to D. \end{cases} \quad (5.9)$$

Note that the additional q for $D \to Q$ contours accounts for the number $C_Q(\boldsymbol{\omega})$ of Q-components, as it appears in (5.2).

Site model. The alternative definition is again based on (5.5), but this time expressing all bulk terms by means of site-numbers $|\mathbb{V}_L(\boldsymbol{\omega})|$ rather than bond-numbers $|\mathbb{B}_L(\boldsymbol{\omega})|$. We get, again up to a fixed boundary factor, the expression

$$\widetilde{W}_{\mathbb{B}(\Lambda)}(\boldsymbol{\omega}) = (e^{\beta J}-1)^{d|\mathbb{V}_Q(\boldsymbol{\omega})|}\,\big[e^{\beta h_1}(e^{\beta J}-1)^d\big]^{|\mathbb{V}_I(\boldsymbol{\omega})|}\,\big[e^{\beta h_2}(e^{\beta J}-1)^d\big]^{|\mathbb{V}_J(\boldsymbol{\omega})|} \\ \times\,\big[q+e^{\beta h_1}+e^{\beta h_2}\big]^{|\mathbb{V}_D(\boldsymbol{\omega})|}\prod_{\gamma\in\partial(\boldsymbol{\omega})}\tilde{\varrho}(\gamma) \quad (5.10)$$

with new contour weights

$$\tilde{\varrho}(\gamma) = \begin{cases} q\big(e^{\beta J}-1\big)^{-\frac{|\gamma|}{2}} & D \to Q \\ \big(e^{\beta J}-1\big)^{-\frac{|\gamma|}{2}} & \text{otherwise.} \end{cases} \quad (5.11)$$

In order to avoid the reader's confusion, let us emphasize once more that both expressions (5.8) for $W_{\mathbb{B}(\Lambda)}(\boldsymbol{\omega})$ and (5.10) for $\widetilde{W}_{\mathbb{B}(\Lambda)}(\boldsymbol{\omega})$ agree with (5.3) for $W_{\Lambda,L}(\boldsymbol{\omega})$ only modulo a fixed (and boundary-condition dependent) term.

An important feature of the decompositions (5.8) and (5.10) is that the bulk terms associated with each of the phases exhaust the entire volume ($\mathbb{B}(\Lambda)$ or Λ)—the weights (5.9) and (5.11) then represent the true *energy gain* rendered by introducing a contour. With this observation, (5.8) and (5.9), or alternatively (5.10) and (5.11), establish a standard Pirogov-Sinai setup; the only thing now required for all machinery to set into motion is an appropriate upper bound for the contour weights:

Lemma V.1. *Let $d \geq 2$.*

(A) *Let $h_1, h_2 \leq \frac{2}{3\beta} \log q$. Then $\varrho(\gamma) \leq q^{-\frac{|\gamma|}{6d}}$.*

(B) *Let $e^{\beta J} - 1 \geq q^{\frac{1}{3d}}$. Then $\tilde{\varrho}(\gamma) \leq q^{-\frac{|\gamma|}{6d}}$.*

Proof. (A) While the bound on $\varrho(\gamma)$ is for the case $Q \to D$ valid for any value of external fields h_1, h_2, the validity for the cases $D \leftrightarrow I$ and $D \leftrightarrow J$ is clearly ensured by the assumed condition on h_1 and h_2. In the remaining case of contours of the type $D \to Q$, the independence of the prefactor q entails that the bound is the most restrictive for the smallest possible *closed* contour. Observing that the area of such a contour is $|\gamma_{\min}| = 4d - 2$ (the contour surrounding just one bond), we get

$$\varrho(\gamma) = q(q + e^{\beta h_1} + e^{\beta h_2})^{-\frac{|\gamma|}{2d}} \leq q^{|\gamma|(\frac{1}{4d-2} - \frac{1}{2d})} = q^{-|\gamma| \frac{1}{2d} \frac{d-1}{2d-1}}, \tag{5.12}$$

where the middle inequality follows by neglecting $e^{\beta h_1}$ and $e^{\beta h_2}$ and using that $1 \leq \frac{|\gamma|}{|\gamma_{\min}|} = \frac{|\gamma|}{4d-2}$. Now, $\frac{d-1}{2d-1} \geq \frac{1}{3}$ for $d \geq 2$, which yields the claim.

(B) Again, taking into account the area of the smallest contour of type $D \to Q$, we easily derive that

$$\tilde{\varrho}(\gamma) \leq \left((e^{\beta J} - 1)^{-\frac{1}{2}} q^{\frac{1}{4d-2}}\right)^{|\gamma|}. \tag{5.13}$$

The claim now follows by employing the conditions $e^{\beta J} - 1 \geq q^{\frac{1}{3d}}$ and $d \geq 2$. $\square$

6. EXPLICIT PHASE DIAGRAM COMPUTATIONS

This section is more or less a standard exercise in Pirogov-Sinai theory. First we recast the partition functions of the model (5.8) in terms of the so-called polymer models with weights $z_L(\gamma)$, $L = I, J, Q, D$ (resp. $\tilde{z}_L(\gamma)$ for the model (5.10)) to be introduced below. By banning the contours with disfavorably large weights, one eventually arrives at the 'metastable free energies' f_O, f_I, f_J, and f_D, in whose terms the phase diagram is rigorously given and, moreover, can recursively be determined to an arbitrary degree of accuracy with the aid of cluster expansion. In particular, in Subsections 6.3 and 6.4, the explicit computations corroborating the phase diagrams (Fig.$\tilde{1}$–3) as well as the Gibbs uniqueness in the one-field case and $h_1 \to \infty$ and/or high temperature are carried out.

6.1. Labeled-contour models.

Let us return to the expressions (5.8) and (5.10) and introduce the partition functions of labeled contour models in the form that is the initial point of the Pirogov-Sinai theory. For any set $V \subset \mathbb{R}^d$, we define $\mathbb{V}(V) = V \cap \mathbb{Z}^d$ and $\mathbb{B}(V)$ as the set of all bonds whose middle point belongs to V. Notice that if γ is a disordered contour ($e_\gamma = D$), the set $\mathbb{B}(\operatorname{Int}\gamma)$ contains only bonds whose both endpoints belong to $\operatorname{Int}\gamma$, while for an ordered contour ($e_\gamma \in \{Q, I, J\}$), the set $\mathbb{B}(\operatorname{Int}\gamma)$ contains also bonds from $\mathbb{B}(\gamma)$ with one endpoint in $\operatorname{Int}\gamma$ and the other in $\operatorname{Ext}\gamma$.

Further, we use $\mathcal{E}_L$ to denote the 'ground state' contribution to the metastable free energy arising from the bulk terms in (5.8) (or (5.10)). Namely,

$$\begin{aligned}
\mathcal{E}_Q &= -\tfrac{1}{\beta}\log(e^{\beta J}-1)\\
\mathcal{E}_I &= \mathcal{E}_Q - \tfrac{h_1}{d}\\
\mathcal{E}_J &= \mathcal{E}_Q - \tfrac{h_2}{d}\\
\mathcal{E}_D &= -\tfrac{1}{\beta d}\log(q+e^{\beta h_1}+e^{\beta h_2}).
\end{aligned} \tag{6.1}$$

These definitions enable us to express the partition function in a unified fashion, embodying a variety of choices of volumes and boundary conditions. For any bounded $V\subset\mathbb{R}^d$ and any $L\in\{Q,I,J,D\}$, we define

$$Z_L(V) = \sum_{\partial\in\mathcal{M}_L(V)}\prod_{\gamma\in\partial}\rho(\gamma)\prod_{\bar L} e^{-\beta\mathcal{E}_{\bar L}|\mathbb{B}_{\bar L}(\partial,V)|}. \tag{6.2}$$

Here the sum is over the set $\mathcal{M}_L(V)$ of all families of matching labeled contours contained in V, such that their exteriormost label is L and $\mathbb{B}(\gamma)\cap\mathbb{B}(\partial V)=\emptyset$ for each its contour (with $\mathbb{B}(\partial V)$ denoting the set of all bonds intersecting ∂V)[12]. The set $\mathbb{B}_{\bar L}(\partial,V)\subset\mathbb{B}(V)$, for any $\bar L\in\{Q,I,J,D\}$, is the set of bonds in the state $\bar L$ for the given labeled contour configuration ∂. Finally, in a similar fashion we introduce

$$\tilde Z_L(V) = \sum_{\partial\in\mathcal{M}_L(V)}\prod_{\gamma\in\partial}\tilde\rho(\gamma)\prod_{\bar L} e^{-\beta d\mathcal{E}_{\bar L}|\mathbb{V}_{\bar L}(\partial,V)|}. \tag{6.3}$$

Now, for any (geometric) contour Γ such that there exists a labeled contour γ of the type $L\to D$, $L\in\{I,J,Q\}$, with $\Gamma_\gamma=\Gamma$, we define the weight

$$z_L(\Gamma) = \rho(\gamma)\frac{Z_D(\operatorname{Int}\Gamma)}{Z_L(\operatorname{Int}\Gamma)}. \tag{6.4}$$

If, on the other hand, Γ corresponds to a disordered contour, there exist labeled contours γ_I, γ_J and γ_Q such that $\Gamma=\Gamma_{\gamma_L}$ and $i_{\gamma_L}=L$, with $L=I,J,Q$. Then we define

$$z_D(\Gamma) = \frac{\rho(\gamma_I)Z_I(\operatorname{Int}\Gamma)+\rho(\gamma_J)Z_J(\operatorname{Int}\Gamma)+\rho(\gamma_Q)Z_Q(\operatorname{Int}\Gamma)}{Z_D(\operatorname{Int}\Gamma)}. \tag{6.5}$$

The weights $\tilde z_L(\Gamma)$ are defined analogously in terms of $\tilde\rho(\gamma)$ and $\tilde Z_L(\operatorname{Int}\Gamma)$. Notice that the contours in z_L and $\tilde z_L$ are regarded just as geometric objects without any additional labels.

[12] Notice that the last condition is trivially satisfied when defining $Z_{i_\gamma}(\operatorname{Int}\gamma)$ for any given labeled contour γ. On the other hand, it disallows contours in $\operatorname{Int}\gamma$ to get 'too close' to Γ_γ in the definition of $Z_{e_\gamma}(\operatorname{Int}\gamma)$ (i.e. when the boundary condition on $\operatorname{Int}\gamma$ is as if the contour γ were not present). In the standard Pirogov-Sinai approach, this amounts to a version of the *diluted* and *crystalic* partition functions (see [Z]).

It is now easy (and quite standard) to prove by induction that

$$Z_L(V) = e^{-\beta\mathcal{E}_L|\mathbb{B}(V)|} \sum_{\Delta\in\mathcal{M}(V)} \prod_{\Gamma\in\Delta} z_L(\Gamma) \tag{6.6}$$

and

$$\tilde{Z}_L(V) = e^{-\beta d\mathcal{E}_L|\mathbb{V}(V)|} \sum_{\Delta\in\mathcal{M}(V)} \prod_{\Gamma\in\Delta} \tilde{z}_L(\Gamma) \tag{6.7}$$

for any bounded $V \subset \mathbb{R}^d$ and any $L \in \{I, J, Q, D\}$. Here, the sums are running over the set $\mathcal{M}(V)$ of all collections Δ of L-contours in V (i.e., those with $e_\gamma = L$) such that

$$\begin{aligned} \mathbb{B}(\Gamma)\cap\mathbb{B}(\partial V) = \emptyset \qquad & \forall\,\Gamma \in \Delta \\ \mathbb{B}(\Gamma_1)\cap\mathbb{B}(\Gamma_2) = \emptyset \qquad & \forall\,\Gamma_1,\Gamma_2 \in \Delta,\ \Gamma_1 \neq \Gamma_2, \end{aligned} \tag{6.8}$$

where the second line is the so called compatibility condition. To emphasize that the model is now based on purely geometric objects without any labels, we refer to it as a *polymer model* (with weights z_L resp. $\tilde{z}_L$).

Lemma V.1. *$z_L(\Gamma) = \tilde{z}_L(\Gamma)$ for all L and all L-contours Γ.*

Proof. Fix $L \in \{Q, I, J\}$ and define a partial order $\prec$ on L-contours, by putting $\Gamma \prec \Gamma'$ whenever $\Gamma' \subset \operatorname{Int}\Gamma'$. Pick an L-contour Γ and suppose that $z_L(\tilde{\Gamma}) = \tilde{z}_L(\tilde{\Gamma})$ for any $\tilde{\Gamma} \prec \Gamma$. We shall show that this implies that the weights must be equal even for Γ. Namely, using (6.6) resp. (6.7) with $V = \operatorname{Int}\Gamma$ in the definitions (6.4) we see that the condition $z_L(\Gamma) = \tilde{z}_L(\Gamma)$ boils down to

$$\rho(\Gamma)\frac{e^{-\beta\mathcal{E}_D|\mathbb{B}(\operatorname{Int}\Gamma)|}}{e^{-\beta\mathcal{E}_L|\mathbb{B}(\operatorname{Int}\Gamma)|}} = \tilde{\rho}(\Gamma)\frac{e^{-\beta d\mathcal{E}_D|\mathbb{V}(\operatorname{Int}\Gamma)|}}{e^{-\beta d\mathcal{E}_L|\mathbb{V}(\operatorname{Int}\Gamma)|}}, \tag{6.9}$$

because the sums in (6.6) and (6.7) over the contours inside $\operatorname{Int}\Gamma$ cancel out by the induction assumption.

The obtained condition is easily checked from the definition of the contour weights (5.8–11). By proceding to larger and larger contours, the claim is established. The case when $L = D$ is completely analogous. $\square$

6.2. Cluster expansion.

By neglecting the difference between the models (5.8) and (5.10), as follows from Lemma V.1, we see that Lemma V.1(A) guarantees that contour weights decay fast enough with the size of the contour, provided q is large and either the fields are not too large positive or the temperature is not too large. As then follows, e.g., from [Z,BI], the phase digram can be computed by equating the metastable free energies f_O, f_I, f_J, f_D corresponding to the phases Q, I, J, and D.

The latter quantities are evaluated using a truncated polymer model with weights $\bar{z}_L$ defined by putting

$$\bar{z}(\Gamma) = \begin{cases} z(\Gamma) & z(\Gamma) \le e^{-\tau|\Gamma|} \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

Here τ is a large number growing linearly with dimension. The evaluation is performed by means of the cluster expansion. It will be convenient to invoke here the version [KP] based on the Möbius formula (see also [DKS] for a related setup), because, as will be appreciated later, in this way a partial resummation of the otherwise clumsy expansion can be accomplished.

We begin by extending the definition of the polymer model partition function to any finite set $\mathbb{B}$ of bonds,

$$\mathcal{Z}_L(\mathbb{B}) = \sum_{\Delta \in \mathcal{M}(\mathbb{B})} \prod_{\Gamma \in \Delta} \bar{z}_L(\Gamma). \tag{6.11}$$

With a slight abuse of notation, $\mathcal{M}(\mathbb{B})$ denotes here the set of compatible collections Δ of L-contours such that $\mathbb{B}(\Gamma) \subset \mathbb{B}$ for each $\Gamma \in \Delta$. Then we get the cluster terms

$$\mathfrak{z}_L^{\mathrm{T}}(\mathbb{C}) = \sum_{\mathbb{B} \subset \mathbb{C}} (-1)^{|\mathbb{C}|-|\mathbb{B}|} \log\bigl(\mathcal{Z}_L(\mathbb{B})\bigr) \tag{6.12}$$

defined for each finite set of bonds $\mathbb{C}$. Due to (6.6–7) and (6.12), the cluster terms $\mathfrak{z}_L^{\mathrm{T}}(\mathbb{C})$ vanish whenever the set $\mathbb{C}$ is not connected and decay with the size of the set $\mathbb{C}$,

$$|\mathfrak{z}_L^{\mathrm{T}}(\mathbb{C})| \le \epsilon^{|\mathbb{C}|}, \tag{6.13}$$

where for ϵ we get an estimate of the order $q^{-\frac{1}{6d}}$ both for the Bond and the Site model, in the respective regions as articulated in Lemma V.1. In terms of these contributions we get

$$f_L = d\mathcal{E}_L - \frac{d}{\beta} \sum_{\mathbb{C} \ni b} \frac{\mathfrak{z}_L^{\mathrm{T}}(\mathbb{C})}{|\mathbb{C}|} \tag{6.14}$$

for any $L = Q, I, J, D$, where b is a fixed bond.

6.3. Pirogov-Sinai expansion.

We begin by evaluating first couple of terms in the expansion of the ordered metastable free energies, i.e., the ones for $L = Q, I, J$. This is feasible to do all at once, because the three phases are of a very similar nature. Throughout the computation we neglect the difference between the functionals z (resp. $\tilde{z}$) and $\bar{z}$. This has no influence upon the resulting phase diagram, because, as follows e.g. from Theorem 1.7(i) in [Z], $\min\{\mathrm{diam}(\gamma)\colon e_\gamma = L,\, z_L(\gamma) \ne \bar{z}_L(\gamma)\}$ is inversely proportional to the difference $f_L - \min_{\bar{L}} f_{\bar{L}}$, i.e., it diverges at the points where L gives rise to a ‘stable’ phase.

The simplest possible contour inside an ordered phase corresponds to a single bond being flipped from L to D. Then $\mathbb{C}$ above contains only one bond, in which case we denote it pictorially as $\rightsquigarrow$. Since there are no non-trivial sub-terms to $\rightsquigarrow$, we get $\mathcal{Z}_L(\rightsquigarrow) = 1 + z_L(\rightsquigarrow)$, which yields, for all $L = Q, I, J$,

$$\mathfrak{z}_L^{\mathrm{T}}(\rightsquigarrow) = \log\left(1 + \frac{1}{e^{\beta J} - 1}\right). \tag{6.15}$$

The next simplest contours correspond to flipping two, three, four or more neighboring sites, but without having 'liberated' any single site from the L-phase. It is not difficult to see from (6.8) that all these contours have a vanishing $\mathfrak{z}_L^{\mathrm{T}}$ functional. Namely, $\mathcal{Z}_L(\mathbb{C}) = [1 + z_L(\rightsquigarrow)]^{|\mathbb{C}|}$ in that case, and the same applies also to all subsets $\mathbb{B} \subset \mathbb{C}$, resulting in enormous cancelations due to the term $(-1)^{|\mathbb{C}|-|\mathbb{B}|}$. This is acturally the reason why it is more convenient to work with $\mathbb{C}$'s rather than the contours themselves, because otherwise this effect would be concealed under complicated geometry considerations and would appear, if at all, just as a sheer miracle.

The next non-trivial contribution therefore comes from 'liberating' a single site. Then the corresponding set $\mathbb{C}$ has a shape of a d-dimensional coordinate 'star', i.e., the set of all bonds adjacent to a single site, denoted pictorially as $\divideontimes$. In that case we have $\mathcal{Z}_L(\divideontimes) = [1 + z_L(\rightsquigarrow)]^{2d} + z_L(\divideontimes) - z_L(\rightsquigarrow)^{2d}$. By observing that $z_L(\divideontimes) = \mathcal{D}_L\,[z_L(\rightsquigarrow)]^{2d}$, with

$$\mathcal{D}_L = (q + e^{\beta h_1} + e^{\beta h_2}) \times \begin{cases} 1 & L = Q \\ e^{-\beta h_1} & L = I \\ e^{-\beta h_2} & L = J, \end{cases} \tag{6.16}$$

we get

$$\mathfrak{z}_L^{\mathrm{T}}(\divideontimes) = \log\left(1 + (\mathcal{D}_L - 1)\Big[\frac{z_L(\rightsquigarrow)}{1 + z_L(\rightsquigarrow)}\Big]^{2d}\right). \tag{6.17}$$

This is where we stop evaluating the cluster terms. However, in order to estimate correctly the error incurred thereby, let us find out what is the next contributing cluster. It turns out that (as before, but now a little less transparently) larger $\mathbb{C}$'s that do not 'liberate' an additional site get again zero under (6.8). This is because their partition sums $\mathcal{Z}_L(\mathbb{C})$ factorize into products $\mathcal{Z}_L(\divideontimes)\mathcal{Z}_L(\rightsquigarrow)^{|\mathbb{C}\setminus\divideontimes|}$. The next contributing $\mathbb{C}$ then consists of two coordinate 'stars' positioned next to each other. Since the induced contour has $|\gamma| \geq 4d - 2$, we get an error term of size $q^{-\frac{4d-2}{6d}}$, i.e.,

$$f_L = d\mathcal{E}_L - \frac{d}{\beta}\,\mathfrak{z}_L^{\mathrm{T}}(\rightsquigarrow) - \frac{1}{\beta}\,\mathfrak{z}_L^{\mathrm{T}}(\divideontimes) + \mathcal{O}(q^{-\frac{1}{2}}) \tag{6.18}$$

for arbitrary dimension $d \geq 2$ and $L = Q, I, J$.

The computation of f_D is fairly analogous in spirit, so let us stay sketchy. It turns out that, in order to get below the error exhibited in (6.18), it suffices to analyze only the simplest possible cluster $\mathbb{C}$, with $|\mathbb{C}| = 1$. In the graphical language, this corresponds to connecting two disordered sites by an ordered bond—we denote the corresponding $\mathbb{C}$ pictorially as $\Leftrightarrow$. Then we have

$$\mathfrak{z}_D^{\mathrm{T}}(\Leftrightarrow) = \log\left(1 + (e^{\beta J} - 1)\frac{q + e^{2\beta h_1} + e^{2\beta h_2}}{(q + e^{\beta h_1} + e^{\beta h_2})^2}\right). \tag{6.19}$$

The higher-order clusters correspond to contours with the surface of at least $6d-4$, whence we obtain

$$f_D = d\mathcal{E}_D - \frac{d}{\beta}\mathfrak{z}_D^{\mathrm{T}}(\Leftrightarrow) + \mathcal{O}(q^{-\frac{3}{4}}), \tag{6.20}$$

for all $d \geq 2$.

6.4. High field/temperature expansion.

So far we have dealt with fields or temperatures taking not too large positive values. The opposite extremes are handled by the high-fugacity and the high-temperature expansions [Ru]. In the former case, with the most interesting application to the one-field problem (see Fig. 1), this has to be done back in the original spin language, because the graphical representation keeps on exhibiting a phase transition—a percolation transition inherent to the Bernoulli measure used to generate the graphical representation.

Let us sketch how such an expansion is devised in the 'almost' one-field case (i.e., when $h_1 \gg \max\{h_2, 0\}$). We shall do this by a reduction to a contour model, whose means of control have been elucidated already in Subsections 6.1 and 6.2. Let us assume the original spin model with Hamiltonian (2.1), and, given a spin configuration $\boldsymbol{\sigma}$, let us declare x to be a *bad site* whenever $\sigma_x \neq 1$. *Contours* are then the connected components of the set of all bad sites. Each geometrical contour Γ then carries the weight

$$\hat{\varrho}(\Gamma) = \sum_{\substack{\boldsymbol{\sigma}_\Gamma \\ \sigma_x \neq 1\, \forall x \in \Gamma}} \exp\bigl(-\beta H_\Gamma^1(\boldsymbol{\sigma}_\Gamma)\bigr), \tag{6.21}$$

with H_Γ^1 denoting the restriction of (2.1) to $\Gamma \cup \partial\Gamma^{\mathrm{c}}$ with the boundary condition at $\partial\Gamma^{\mathrm{c}}$ set equal to 1. An analogous expression to (6.2) then holds for the partition function of the model, however, with only *one* (equal to 1) 'ground state' configuration entering the product over $\bar{L}$. Moreover, all internal/external colors of the contours are automatically matching, being equal to 1, so the only consistency condition required upon the contours is that they do not touch each other.

The associated weights z_1 are then given by the expression

$$z_1(\Gamma) = e^{-\beta h_1 |\Gamma|}\hat{\varrho}(\Gamma), \tag{6.22}$$

where $|\Gamma|$ is the number of sites in Γ, with z_1 fulfilling the bound

$$z_1(\Gamma) \le \left[e^{-\beta h_1}(q + e^{\beta h_2})e^{2\beta J d}\right]^{|\Gamma|}, \tag{6.23}$$

derived by estimating the interactions within Γ by their absolute minimum. Thence, for h_1 large enough, the contour model with weights z_1 can be cluster-expanded as indicated in Subection 6.2, showing thus that each Gibbs state features a decisive abundance of 1-sites—a typical configuration under any measure looks as a 'sea' of 1's with small 'islands' of other values of the spins. Consequently, there is a uniqueness of the infinite volume Gibbs states, provided $h_1 \gg \max\{h_2, 0, 2Jd, \frac{1}{\beta}\log q\}$.

The expansion associated with *high temperatures* (i.e., small β) is developed analogously, however, one can stay with the graphical representation in this case. Namely, as is seen from (5.3), each ordered bonds is assigned a weight proportional to βJ, for β small enough, so by declaring all ordered *bonds* to be bad we are in the same situation as in the high-field case. Contours are then the connected components of the set of all ordered bonds. In this way, uniqueness of the Gibbs state for the full *Edwards-Sokal* measure can be proved, yielding as a corollary the uniqueness of the associated Gibbs state.

6.5. Proof of Theorem II.1.

The claim is, in fact, for the most part already proved. Namely, by inspection of the Lemma V.1, there are $\bar{h}, \bar{\beta}$ positive, such that the expansion methods converge in the parameter set

$$\bar{\mathfrak{R}} = \{(\beta, h)\colon h \le \bar{h}\} \cup \{(\beta, h)\colon \beta \ge \bar{\beta}\}. \tag{6.24}$$

Notably, here $\bar{\beta}$ is about a third of the critical temperature of the q-state Potts model. By the expansions from the previous subsections, the corresponding phases as well as the coexistence lines are established in $\bar{\mathfrak{R}}$.

As to the percolation statements, first note that the percolation lines coincide with the first-order transition lines in $\bar{\mathfrak{R}}$, because all contours are finite almost surely. The statement about the Gibbs uniqueness then follows from Theorem III.4(i). The result for the one-field case and $d = 2$ is then an adaptation of Theorem III.4(iv).

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SUMMARY

Understanding phase transitions, both qualitatively and quantitatively, is one of the most frequented topics of investigation on the borderline of statistical physics and probability theory. This thesis offers four studies on the subject: in four papers written in collaboration with my two promotors F. den Hollander and R. Kotecký and with L. Chayes from UCLA and C. Borgs and J. Chayes from Microsoft Research, various aspects of phase transitions are studied by rigorous mathematical methods.

The unifying framework of these papers is the theory of infinite-volume Gibbs measures, with the latter describing the large-volume asymptotics of the physical systems of interest. The occurrence of a phase transition means that this asymptotics is not unique, but depends crucially on the choice of the parameters and on the boundary condition. In order to control such phenomena, different tools have to be applied in different contexts. In particular, we demonstrate three techniques 'in action', applied to phase transitions in a model of a heteropolymer at an interface (Chapter 2) and in lattice models related to the Potts model (Chapters 3–5). Below we proceed by summarizing the problems that are addressed, with no intention of being exhaustive—the reader is urged to consult the Introduction and the respective chapters for a more detailed commentary.

The first paper 'A Heteropolymer near a linear interface' deals with the thermodynamics of a disordered chain of two species of monomers, with an interaction tending to place each monomer on one side of a one-dimensional interface. The 'disorder' is quenched, in the sense that the monomer arrangement is fixed and is sampled from an i.i.d. random sequence of ± 1. In the parameter region where previous work shows a localization property in terms of the free energy, the existence of a unique Gibbs measure and some of its properties are established, for instance, the exponential tail for the distribution of the typical distance of a monomer away from the interface. The heteropolymer is described by a stationary process which makes coupling theory and ergodic theory available. The latter are indeed our main technical tools throughout this chapter.

The second paper 'Coexistence of partially disordered/ordered phases in an extended Potts model' studies the phase diagram of a 'crossbreed' between the Ising model and the Potts model, related to the 'fuzzy' Potts model with two 'fuzzy' spin-values. More precisely, the q Potts spins are split into two families labeled by the 'fuzzy' spin ± 1, with an additional Ising interaction and an external field acting on the latter variables. However, unlike previous studies, we are predominantly concerned with the case when the families are of different sizes. In this case, the phase diagram exhibits two distinct triple points at intermediate temperatures, connected by the first-order phase transition line between the low-temperature ordered phase concentrated

on the spins of one family and the high-temperature disordered phase concentrated on the spins of the other family. The complexity of the spin system is reduced by resorting to a 'graphical representation', which allows for characterizations of the phases in terms of percolation.

The third paper 'Reflection positivity of the random-cluster measure invalidated for noninteger q' is linked with the previous one, in the sense that it disqualifies its main technical tool whenever one tries to cover the 'extension' of the graphical representation to noninteger spin numbers (the so-called random-cluster model). Since reflection positivity is associated with nice convexity properties of the system (this is the way it arises in quantum field theory), this suggest that the random-cluster model is not a 'nice' field theory for q noninteger. The lack of reflection positivity serves thus as another example where an 'obvious' analytic continuation fails, as has been elaborated on in related contexts by Griffiths and Gujrati.

The fourth paper 'Gibbs structure and phase coexistence in the Potts model with external fields' devises a novel approach to the Gibbs theory of coupled Potts-spin and random-cluster measures—the so-called Edwards-Sokal (ES) measures. In the case when each of the Potts spins is coupled to an external field, we prove several structure theorems on (1) the relation between the number of Potts-spin and ES Gibbs measures, (2) the FKG properties of the random-cluster projections of the ES measures, (3) the existence of the thermodynamic limit for 'extremal' boundary conditions, and (4) the characterization of phase transitions in terms of percolation. Using the powerful expansion technique, based on a merging of Pirogov-Sinai theory with 'colored' random-cluster representations, we further establish large parts of the phase diagram in the case of two fields taking on nonzero values.

SAMENVATTING

Het begrijpen van faseovergangen, zowel kwalitatief als kwantitatief, is een van de meest intensief onderzochte onderwerpen op het grensvlak tussen de statistische fysica en de kansrekening. Dit proefschrift presenteert vier studies op dit gebied: in vier artikelen, die geschreven werden in samenwerking met mijn twee promotores F. den Hollander en R. Kotecký en met L. Chayes van UCLA en C. Borgs en J. Chayes van Microsoft Research, worden verschillende aspecten van faseovergangen bestudeerd met wiskundig strenge methoden.

Het verbindende thema van deze artikelen is de theorie van Gibbsmaten op oneindige ruimten, waarbij deze laatste het limietgedrag representeren van de onderzochte systemen wanneer het volume groot is. Het optreden van een faseovergang betekent dat dit limietgedrag niet uniek is, maar afhangt van de keuze van de parameters en van de randvoorwaarden. Om greep te krijgen op dergelijke verschijnselen moeten afhankelijk van de situatie verschillende gereedschappen gebruikt worden. We laten in het bijzonder drie technieken 'in actie' zien, toegepast op faseovergangen in een model van een heteropolymeer in de nabijheid van een grensvlak (hoofdstuk 2) en in roostermodellen verwant aan het Pottsmodel (hoofdstuk 3–5). Hieronder gaan we verder met het samenvatten van de vraagstellingen die we behandelen, zonder daarbij uitputtend te willen zijn–de lezer wordt dringend verzocht om voor gedetailleerder commentaar er de inleiding en de desbetreffende hoofdstukken op na te slaan.

Het eerste artikel—over een heteropolymeer in de nabijheid van een lineair grensvlak—behandelt de thermodynamica van een ongeordende keten van twee soorten monomeren, met een wisselwerking die de neiging heeft om de monomeren elk aan een kant van het eendimensionale grensvlak te plaatsen. De ongeordendheid is 'quenched', in de zin dat de volgorde van de monomeren vaststaat en getrokken is uit een o.i.v. rij van ± 1. In een parametergebied waar eerder werk een localisatie-eigenschap laat zien in termen van de vrije energie, wordt het bestaan van een unieke Gibbsmaat en een paar van zijn eigenschappen aangetoond, bijvoorbeeld de exponentiele staart van de verdeling van de typische afstand van een monomeer tot het grensvlak. Het heteropolymeer wordt beschreven met een stationair proces dat het gebruik van koppelingstheorie en ergodentheorie beschikbaar maakt. Deze laatste twee zijn in feite onze belangrijkste stukken gereedschap in dit hoofdstuk.

Het tweede artikel—over het naast elkaar bestaan van gedeeltelijk geordende/ongeordende fasen in een uitbreiding van het Pottsmodel—onderzoekt hat fasediagram van een kruising tussen het Isingmodel en het Pottsmodel, gerelateerd aan het 'fuzzy' Pottsmodel met twee 'fuzzy' spinwaarden. Om precies te zijn worden de q Potts-spins verdeeld in twee families, aangeduid met de 'fuzzy' spin ± 1, met daaraan toegevoegd een Ising-wisselwerking en een uitwendig

veld dat werkt op deze laatste grootheden. In tegenstelling tot eerder onderzoek houden wij ons echter hoofdzakelijk bezig met het geval dat de families van verschillende grootte zijn. In dit geval laat het fasediagram bij gematigde temperaturen twee verschillende driefasenpunten zien, verbonden door de lijn van de eerste-orde faseovergang tussen de lage-temperatuur geordende fase geconcentreerd op de spins uit één familie, en de hoge-temperatuur ongeordende fase geconcentreerd op de spins uit de andere familie. De complexe natuur van het spinsysteem wordt gereduceerd door middel van een 'grafische representatie', die het mogelijk maakt de fasen in termen van percolatie te karakteriseren.

Het derde artikel—over het uitgesloten zijn van reflectie-positiviteit voor de random-clustermaat voor niet gehele q—houdt verband met het vorige, in de zin dat het voornaamste daar gebruikte stuk gereedschap buiten spel wordt gezet zodra men probeert om ook de 'uitbreiding' te behandelen van de grafische representatie naar niet gehele aantallen spins (het zogenaamde random cluster model). Aangezien reflectie-positiviteit verband houdt met 'prettige' convexiteitseigenschappen van het systeem (dit is hoe het in de quantumveldentheorie opduikt), suggereert dit dat het random cluster model geen 'prettige' veldentheorie is voor niet gehele q. Het ontbreken van reflectiepositiviteit geeft dus een voorbeeld van een geval waarin een 'voor de hand liggende' analytische voortzetting faalt, zoals in verwante context is uitgewerkt door Griffiths en Gujrati.

Het vierde artikel—over Gibbs structuur en het naast elkaar bestaan van fasen in het Pottsmodel met uitwendige velden—ontwerpt een nieuwe aanpak van de Gibbstheorie van gekoppelde Pottsspin- en random-cluster-maten—de zogenaamde Edwards-Sokal (ES) maten. In het geval dat elk van de Pottsspins gekoppeld is aan een uitwendig veld, bewijzen we verschillende structuurstellingen aangaande het aantal Pottsspin- en ES-Gibbs-maten, FKG-eigenschappen van de random-cluster-projecties van de ES-maten, het bestaan van de thermodynamische limiet voor 'extremale' randvoorwaarden, en de karakterisatie van faseovergangen in termen van percolatie. Met gebruikmaking van de krachtige expansietechniek, gebaseerd op een samengaan van Pirogov-Sinai-theorie met 'gekleurde' random-cluster-representaties, bepalen we verder grote delen van het fasediagram in het geval van twee velden die waarden ongelijk aan nul aannemen.

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As should be clear to those who have already experienced it, the preparation of a doctorate thesis involves a larger circuit of people than just the author himself. Here I would like to thank all those who contributed to my having reached the stage of writing the ‘Acknowledgments’.

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To Karel I am grateful for many hours of inspiring discussions. I certainly learned from you most of what I know about the extensions of Pirogov-Sinai theory. When do we write ours?

From Bálint I learned all what I know about quantum Heisenberg models, as well as that the Czechs and the Hungarians are very alike, despite the threatening language barrier.

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Being aware that the last paragraph of the 'Acknowledgments' is often the most frequently glanced at, I reserved it for showing my gratitude to those without whom not only this thesis would not exist but also my life would not be as it is. I would like to thank my parents, my brother and sister and all my family for encouragement and support and, in particular, my sweet wife Iva, who not only is always there to cheer me up in times when my self-assurance is wavering but also gives a broader sense to all what I am trying to accomplish. I dedicate this thesis to her.

CURRICULUM VITÆ

I was born on July 27, 1971 in Brno, Czech Republic. Having finished my high-school education in 1989, I was admitted to the Mathematics and Physics Faculty of Charles University, Prague. In June 1994 I graduated in Theoretical Physics, with a diploma thesis on disordered Potts systems.

The academic year 1994-95 I spent in the Netherlands, participating in the 'Master Class' program on 'Mathematical Physics'. The latter I successfully completed with a test problem generalizing the concept of subshifts of finite type to non-discrete single-spin state-spaces.

Since 1995 I have been a postgraduate student jointly at Charles University and University of Nijmegen. This arrangement instigated an extensive communication with many people. During the years 1995-99 I visited several places and presented contributed talks on several conferences and workshops. In 1998 I spent four months at Microsoft Research (Redmond) and two months at the Fields Institute (Toronto).

Since 1998 I am married to Iva Biskupová.

After July 1, 1999, we both return to Microsoft, where I will be a post-doc for two years in the 'Theory' group.