Transitions in active rotator systems

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Coupling noisy excitable systems

Example of noisy excitable system: neurons.

- Finite dimensional dynamical systems.
- They have a stable rest position, which could be a limit cycle, but noise may drive the system sufficiently far away (threshold phenomenon) and then the system follows a complex trajectory ("it fires") before getting back to to rest.

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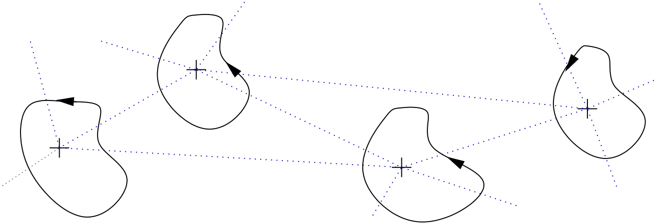
One can of course recognize here the fundamental paradigm of statistical mechanics...

Example: synchronization of interacting dynamical systems



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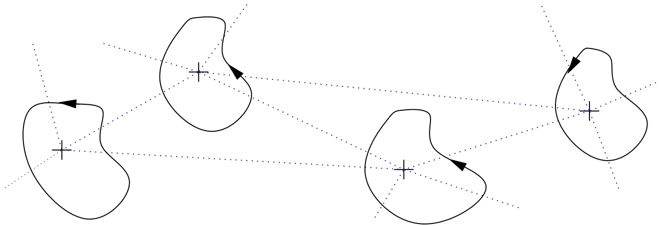

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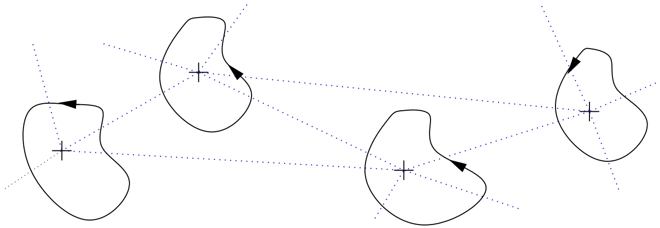

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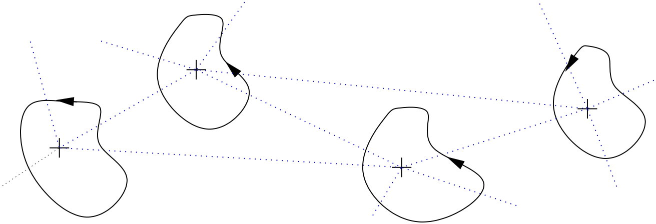

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Can we remain in a neighborhood of equilibrium statmech?

Active rotator model

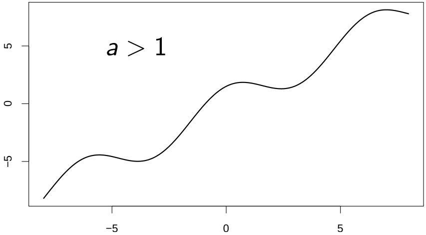
[Kuramoto, Sakaguchi, Shinomoto] 80s and 90s
Consider the N-dimensional diffusion:

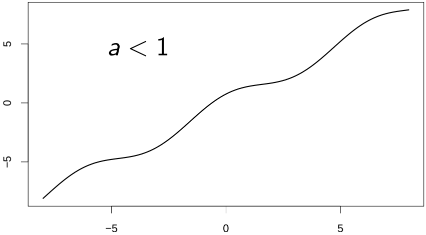
$$\mathrm{d}\varphi_j(t) = -\delta V'(\varphi_j(t))\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,,$$

for $j = 1, 2, \ldots, N$, where

1. $V(\theta) = \theta - a\cos(\theta)$, $\delta, \sigma, K \geq 0$.
2. $\{w_j(\cdot)\}_{j=1,2,\ldots}$ are IID standard Brownian motions.

Isolated deterministic system: $\dot{\psi}_t = -V'(\psi_t)$





Simulation 1: $N = 4000$, $K = 2$, $a = 0.7$, $\delta = 0.5$

Simulation 2: $N = 4000$, $K = 2$, $a = 1.4$, $\delta = 0.5$

Simulation 3: $N = 4000$, $K = 2$, $a = 1.1$, $\delta = 0.5$

Large N limit

Notation: empirical probability measure

$$\nu_{N,t}(\mathrm{d}\theta) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t)}(\mathrm{d}\theta)$$

Still for

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We have

$$\partial_t p_t^{\delta}(\theta) \;=\; \frac{\sigma^2}{2}\partial_\theta^2 p_t^{\delta}(\theta) - \partial_\theta\left[p_t^{\delta}(\theta)(J * p_t^{\delta})(\theta)\right] + \delta\partial_\theta\left[p_t^{\delta}(\theta)V'(\theta)\right]$$

with $J(\theta) = -K\sin(\theta)$.

Phenomenon is captured: $K = 2$, $a = 1.1$, $\delta = .5$

Let us look at the PDE

General form:

$$\partial_t p_t^\delta(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J * p_t^\delta)(\theta)\right] + \delta G[p_t^\delta]$$

where $G \in C^1(L_1^2, H_{-1})$, where $L_1^2 = \{u \in L^2 : \int_{\mathbb{S}} u = 1\}$.

Examples include

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with $h \in L^\infty$, as well as generalizations like $\partial_\theta[p(\theta)\int_{\mathbb{S}} h(\theta,\theta',\theta'')p(\theta')p(\theta'')]$ and so on.

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Particular case: $\delta = 0$, $\sigma = 1$, i.e. "reversible Kuramoto PDE"

$$\partial_t p_t^0(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t^0(\theta) - \partial_\theta\left[p_t^0(\theta)(J * p_t^0)(\theta)\right] \qquad (2)$$

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Facts on the semigroup defined by (2):

- the associated particle system is reversible
- it is a gradient flow (Lyapunov function)
- it is rotation invariant, that is if $p_t(\theta)$ is a solution to (2), so is $p_t(\theta + \theta_0)$

A remark on the nonlocal PDE

This equation can be rewritten in *gradient form*

$$\partial_t p_t(\theta) = \nabla \left[p_t(\theta) \nabla \left(\frac{\delta \mathcal{F}(p_t)}{\delta p_t}(\theta) \right) \right]$$

where

$$\mathcal{F}(p) := \frac{1}{2} \int_{\mathbb{S}} p(\theta) \log p(\theta) \, \mathrm{d}\theta + \frac{1}{2} \int_{\mathbb{S}^2} \widetilde{J}(\theta - \theta') p(\theta) p(\theta') \, \mathrm{d}\theta \, \mathrm{d}\theta'$$

our case $\widetilde{J}(\cdot) = K \cos(\cdot)$, and it appears in different contexts, among them

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In all cases it is the simple case for a more general non reversible dynamics / non gradient system

On the stationary solutions of reversible Kuramoto

All stationary (probability) solutions are (up to rotation invariance)

$$q(\theta) = \frac{\exp(2Kr\cos(\theta))}{2\pi I_0(2Kr)}, \quad \text{with } r = \Psi(2Kr)$$

where $\Psi(x) = I_1(x)/I_0(x)$, where $2\pi I_j(x) = \int_{\mathbb{S}} \cos(j\theta)\exp(x\cos(\theta))\,\mathrm{d}\theta$.

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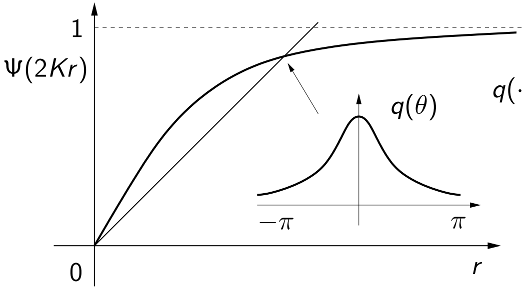
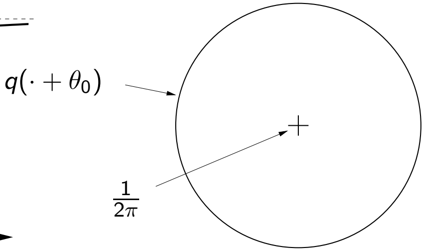
In particular $r = 0$ is always a solution and if $K > 1$:





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The manifold (a circle)

$$M_0 := \{q(\cdot + \theta_0) =: q_{\theta_0}(\cdot) : \theta_0 \in \mathbb{S}\}$$

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Notably, the linearized (around q) evolution operator (on L^2_0)

$$-L_q u := \frac{1}{2} u'' - [uJ * q + qJ * u]'$$

is self-adjoint in $H_{-1,1/q}$ with compact resolvent and spectrum in $[0, \infty)$. In fact, q' is a simple eigenvector for the eigenvalue 0, so there is a spectral gap $\lambda(K) > 0$ (in fact, explicit estimate). [Bertini G P,10]

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Put otherwise, M is a “Stable Normally Hyperbolic Manifold” and SNHMs are rather robust structures.

Stable Normally Hyperbolic Manifold in L^2

This notion goes together with an evolution in L^2_1: for us (1) and its linearized evolution semigroup $\{\Phi(p^\delta, t)\}_{t\geq 0}$ in L^2_0.

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A SNHM $M_\delta \subset L^2_1$ of characteristics λ_1, λ_2 $(0 \leq \lambda_1 < \lambda_2)$ and $C > 0$ is a C^1 compact connected invariant manifold s.t. for every $v \in M_\delta$ there exists a projection $P^o(v)$ on the tangent space of M_δ at v such that if p^δ_t is on M_δ for $t \geq 0$ and $u \in L^2_0$

1. $$\Phi(p^\delta, t)P^o(p^\delta_0)u \;=\; P^o(p^\delta_t)\Phi(p^\delta, t)u$$

2. $$\|\Phi(p^\delta, t)P^o(p^\delta_0)u\|_2 \;\leq\; C\exp(\lambda_1 t)\|u\|_2$$
   and, for $P^s := 1 - P^o$, we have
   $$\|\Phi(p^\delta, t)P^s(p^\delta_0)u\|_2 \;\leq\; C\exp(-\lambda_2 t)\|u\|_2 \qquad (3)$$

3. there exists a negative continuation of the dynamics $\{p^\delta_t\}_{t\leq 0}$ and of the linearized semigroup and for every such continuation
   $$\|\Phi(p^\delta, -t)P^o(p^\delta_0)u\|_2 \;\leq\; C\exp(-\lambda_1 t)\|u\|_2$$

Robustness of (S)NHMs

Recall

$$\partial_t p_t^\delta(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta \left[p_t^\delta(\theta)(J * p_t^\delta)(\theta) \right] + \delta G[p_t^\delta]$$

where $G \in C^1(L^2_1, H_{-1})$and DG unif. bounded in a L^2-neighborhood of M_0.

Fact: M_0 (explicit) is a SNHM for the $\delta = 0$ case with characteristics 0, $\lambda(K)$ and C (explicit) and explicit projection(s)

$$P_q^o u = \frac{(u, q')_{-1,1/q} q'}{(q', q')_{-1,1/q}}$$

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NHMs are robust structures

- Hirsch, Pugh, Shub "Invariant manifolds" 1977
- Bates, Lu, Zeng "Existence and persistence of inv. manif." 1998
- Sell, You "Dynamics of evolutionary equations" 2002

We use [Sell and You], but need more explicit estimates in order to set-up

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where $G \in C^1(L_1^2, H_{-1})$and DG unif. bounded in a L^2-neighborhood of M_0.

Theorem

There exists $\delta_0 > 0$ such that if $\delta \in [0, \delta_0]$ there exists a SNHM M_δ in L_1^2 with suitable characteristics. Moreover

$$M_\delta \;=\; \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{S}\}\ ,$$

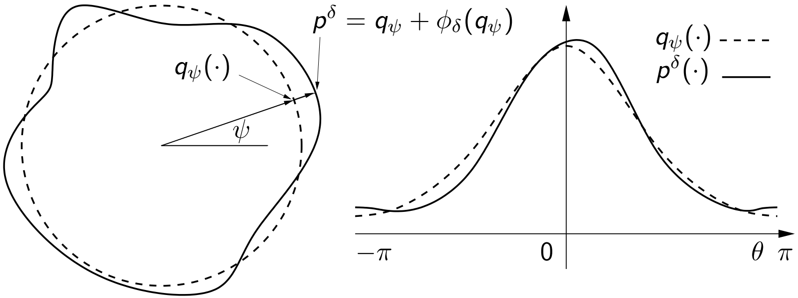
for a suitable function $\phi_\delta \in C^1(M, L_0^2)$ with the properties that

- $\phi_\delta(q) \in \mathcal{R}(L_q)$;
- *there exists $C > 0$ such that both* $\sup_\psi \|\phi_\delta(q_\psi)\|_2$ *and* $\sup_\psi \|\partial_\psi \phi_\delta(q_\psi)\|_2$ *are bounded by $C\delta$.*

Robustness of (S)NHMs

Theorem

There exists $\delta_0 > 0$ such that if $\delta \in [0, \delta_0]$ there exists a SNHM M_δ in L^2_1. Moreover $M_\delta = \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{S}\}$ for a $\phi_\delta \in C^1(M, L^2_0)$ with the properties that $\phi_\delta(q) \in \mathcal{R}(L_q)$ and $\sup_\psi \|\phi_\delta(q_\psi)\|_2 \le C\delta$ and $\sup_\psi \|\partial_\psi \phi_\delta(q_\psi)\|_2 \le C\delta$.



Dynamics on M_δ

Theorem

For $\delta \in [0, \delta_0]$ we have that $t \mapsto \psi_t^\delta$ (the "phase") is C^1 and

$$\dot{\psi}_t^\delta + \delta \frac{\left(G[q_{\psi_t^\delta}], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q', q')_{-1,1/q}} = O(\delta^2)$$

with $O(\delta^2)$ uniform in t. Moreover if we call n_ψ the unique solution of

$$L_{q_\psi} n_\psi = G[q_\psi] - \frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}} q'_\psi \quad \text{and} \quad \left(n_\psi, q'_\psi\right)_{-1,1/q_\psi} = 0$$

we have

$$\sup_\psi \|\phi_\delta(q_\psi) - \delta n_\psi\|_{H_1} = O(\delta^2)$$

Dynamics on M_δ

Actually a fully satisfactory result demands more than

$$R_\delta(\psi_t^\delta) := \dot{\psi}_t^\delta + \delta \frac{\left(G[q_{\psi_t^\delta}], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q', q')_{-1,1/q}} = O(\delta^2)\,,$$

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$$R_\delta(\psi_t^\delta) := \dot{\psi}_t^\delta + \delta \frac{\left(G[q_{\psi_t^\delta}], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q', q')_{-1,1/q}} = O(\delta^2)\,,$$

In fact, we have the explicit expression

$$R_\delta(\psi) \; := \; \frac{\left([\phi_\delta(q_\psi) J * \phi_\delta(q_\psi)]' + \delta\left(G\left[q_\psi + \phi_\delta(q_\psi)\right] - G\left[q_\psi\right]\right)\,,\; q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}}$$

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Theorem

Assuming in addition that DG (recall that $G \in C^1(L^2_1; H_{-1})$) is uniformly continuous in a L^2-neighborhood of M_0, we have that there exists $\ell(\cdot)$, with $\ell(\delta) = o(1)$ as $\delta \searrow 0$, such that

$$\sup_{\psi \in \mathbb{S}} |R'_\delta(\psi)| \;\leq\; \delta\, \ell(\delta)$$

A generic dynamics on the circle

Theorem

For a generic dynamics on the circle $\dot{\psi}_t = -f(\psi_t)$ *with* f *a trigonometric polynomial s. t. if* $f(0) = 0$ *then* $f'(0) \neq 0$ *and for any value of* $K > 1$ *there exists* V *(explicit!) such that for* δ *small enough, the phase dynamic on* M_δ *is equivalent to* $\dot{\psi}_t = -f(\psi_f)$ *with speed factor* δ*.*

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Proof. If

$$V'(\theta) \;=\; a_0 + \sum_{k=1}^{n} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right)$$

then a straightforward calculation gives

$$\frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}} \;=\; a_0 + \frac{I_0}{I_0^2 - 1} \sum_{k=1}^{n} \left(I_k a_k \cos(k\psi) + I_k b_k \sin(k\psi)\right)$$

A generic dynamics on the circle

So the phase dynamics on M_δ is equivalent to $\dot{\psi}_t = -f(\psi_t)$ with

$$f(\theta) = A_0 + \sum_{k=1}^{n} (A_k \cos(k\theta) + B_k \sin(k\theta))$$

where $A_0 = a_0$ and

$$\frac{A_k}{a_k} = \frac{B_k}{b_k} = \frac{I_0 I_k}{I_0^2 - 1}$$

□

The link can be made more explicit:

$$f = a_0 + D(K)\, q_0 * (V' - a_0)$$

where $D(K) = I_0^2/(I_0^2 - 1)$.

A first particular case

Let us look at

$$\partial_t p_t^\delta(\theta) = \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J * p_t^\delta)(\theta)\right] + \delta\partial_\theta\left[p_t^\delta(\theta)V'(\theta)\right]$$

with $V(\theta) = \theta - a\cos(\theta)$.

$a = 0$: last term is just a drift and $q_{\delta t}(\cdot)$ is solution.

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$a = 0$: last term is just a drift and $q_{\delta t}(\cdot)$ is solution.

$a > 0$: look at phase dynamics

$$\dot{\psi}_t^\delta = -\delta \left(1 + \frac{a}{a_c(K)} \sin(\psi_t^\delta) \right) + O(\delta^2).$$

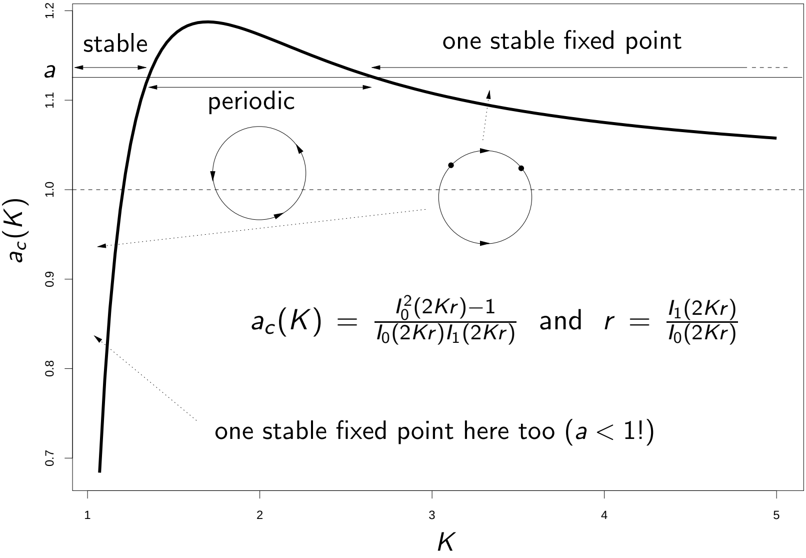
where

$$a_c(K) := \frac{I_0^2(2Kr) - 1}{I_0(2Kr) I_1(2Kr)}$$

and recall that $r = r(K)$.

So if $a_c(K) > a$ then for δ small the solution is periodic.

Dynamics on the manifold



The period

Always for

$$\partial_t p_t^\delta(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J * p_t^\delta)(\theta)\right] + \delta\partial_\theta\left[p_t^\delta(\theta)V'(\theta)\right]$$

with $V(\theta) = \theta + a\cos(\theta)$.

For $a_c(K) > a$ the solution is periodic of period T_δ

$$T_\delta \;=\; \frac{\tau}{\delta} + O(1)$$

(actually one can show $O(\delta)$!) with

$$\tau := \frac{2\pi}{\sqrt{1-(a/a_c(K))^2}}$$

It is a pulsating wave...

Numerics on T_δ

For $K = 2$ and $a = 1.1$, so $\tau = 18.0779\dots$

δ	T_δ	τ/δ
0.005	3615.5928	3615.6229
0.010	1807.7964	1807.8566
0.020	903.8982	904.0186
0.040	451.9491	452.1901
0.080	225.9745	226.4577
0.160	112.9873	113.9624
0.320	56.4936	58.5178
0.640	28.2468	33.0278

This is in agreement with

$$\frac{\delta T_\delta - \frac{\tau}{\delta}}{\frac{\tau}{\delta}} = \frac{\delta \tau_\delta}{\tau} - 1 \sim c\delta^2$$

with $c = 0.333\dots$

Active rotators with $V(\theta) = \theta - a\cos(j\theta)/j$, $j = 2, 3, \ldots$

Large scale phase dynamics lead by

$$f(\psi) = -\left(1 + \frac{a}{a_{c,j}(K)}\sin(j\psi)\right)$$

with $a_{c,j}(K) = (I_0^2 - 1)/I_0 I_j$.