Localization in Classical Statistical Mechanics

Nick Crawford; The Technion

June 1, 2011

BIRS

Talk Outline

1. Model of Interest and Basic Question
2. Some Background
3. Kac Interactions and Our Results
4. Proof Outline
5. Research In Progress

The Classical XY model

- Let $\mathcal{G} = (\mathbb{Z}^d, \mathcal{E})$ with distance $\|x-y\|_2$ ($d \geq 2$).
- For $x \in \mathbb{Z}^d$, $\sigma_x \in \mathbb{S}^1$, $\sigma = (\sigma_x)_{x \in \mathbb{Z}^d} \in \Omega$
  Equipped with i.i.d. uniform measure $\prod_{x \in \mathbb{Z}^d} \mathrm{d}\nu(\sigma_x)$.
  Define $-\mathcal{H}_L : \Omega \to \mathbb{R}$ by

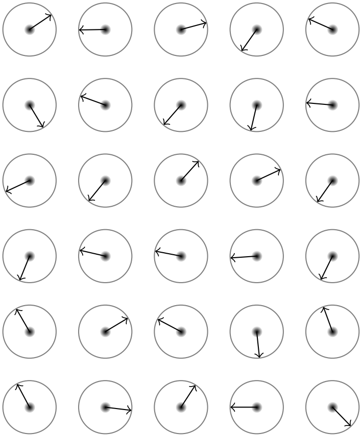
$$-\mathcal{H}_L(\sigma) = \sum_{\|x-y\|_2 = 1,\, x,y \in \Lambda_L} \sigma_x \cdot \sigma_y$$

- Gibbs Weight

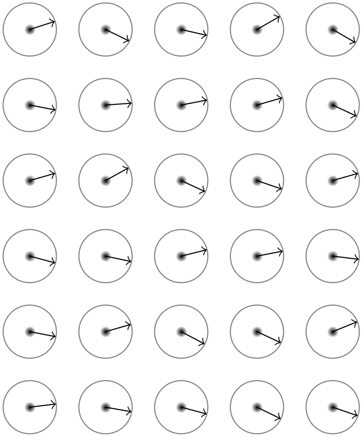
$$\mathrm{d}\mu_L(\sigma) = Z^{-1} e^{-\beta \mathcal{H}_L(\sigma)} \prod_{x \in \Lambda_L} \mathrm{d}\nu(\sigma_x)$$

- Averages denoted by

$$\langle \cdot \rangle.$$



High Temperature, β small



Low Temperature, β large

Ordering in Gibbs States

- Given $A \subset \mathbb{Z}^d$ finite, how does

$$m_L(A) = \langle \frac{1}{|A|} \sum_{x \in A} \sigma_x \rangle_L$$

behave as A varies. Are there preferred directions?

- Technically, need **Boundary Conditions**:

$$-\mathcal{H}(\sigma|\eta) = \sum_{\|x-y\|_2=1, x,y \in \Lambda_L} \sigma_x \cdot \sigma_y + \sum_{\|x-y\|_2=1, x \in \Lambda_L, y \notin \Lambda_L} \sigma_x \cdot \eta_y \quad (1)$$

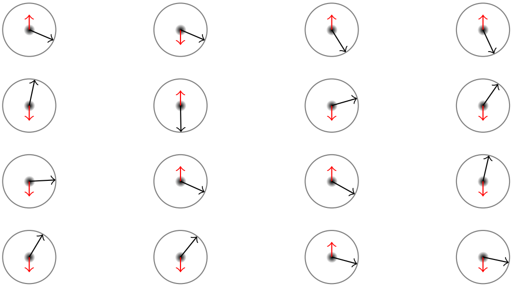
Natural choices: $\eta_x \equiv u, \quad u \in \mathbb{S}^1$.

Local Fields

For $x \in \mathbb{Z}^d$, $\alpha_x(\omega)$ i.i.d. $\{\pm 1\}$ Bernoulli.

$$-\mathcal{H}^\omega(\sigma|\eta) = \sum_{\|x-y\|_2=1, x,y\in\Lambda_L} \sigma_x \cdot \sigma_y + \epsilon \sum_{x\in\Lambda_L} \alpha_x(\omega) e_2 \cdot \sigma_x + \sum_{\|x-y\|_2=1, x\in\Lambda_L, y\notin\Lambda_L} \sigma_x \cdot \eta_y \quad (2)$$

Gibbs State as before.



Basic Question

- Given ϵ, does this model order at β large (T low)?
- Given $A \subset \mathbb{Z}^d$ finite, how does

$$m_L(A) = \langle \frac{1}{|A|} \sum_{x \in A} \sigma_x \rangle_L$$

behave as A varies. Are there preferred directions?

Basic Question

- Given ϵ, does this model order at β large (T low)?
- Given $A \subset \mathbb{Z}^d$ finite, how does

$$m_L(A) = \langle \frac{1}{|A|} \sum_{x \in A} \sigma_x \rangle_L$$

behave as A varies. Are there preferred directions?
- Believed that ordering occurs in $\hat{\mathrm{e}}_1$ direction.

- Robust: Any symmetric distribution for α_x. Even may take bias, e.g. $\mathbb{P}(\alpha_x = 1) = p$.

- Similar Phenomena should exist in any spin model with continuous spin space.

Why is this Interesting? Two Related Models:

- From now on, we restrict attention to $\mathbb{Z}^d, d = 2$.

  Model 1: As above with $\epsilon = 0$:

$$-\mathcal{H}(\sigma|\eta) = \sum_{\|x-y\|_2=1, x,y\in\Lambda_L} \sigma_x \cdot \sigma_y + \sum_{\|x-y\|_2=1, x\in\Lambda_L, y\notin\Lambda_L} \sigma_x \cdot \eta_y \tag{3}$$

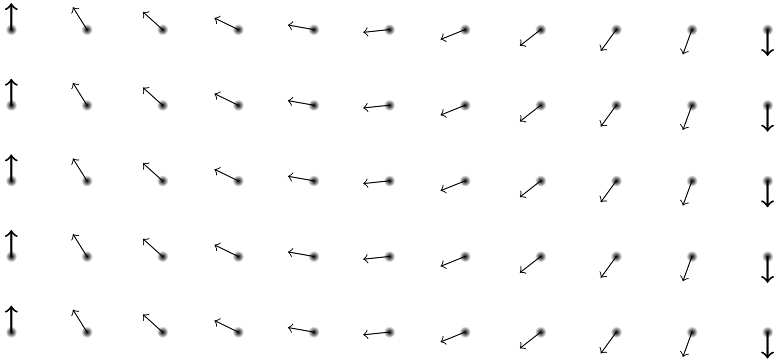
- **Fact:** No matter the η you take,

$$m_L(A) \to 0 \text{ as } L \to \infty$$

ANY $\beta > 0$.

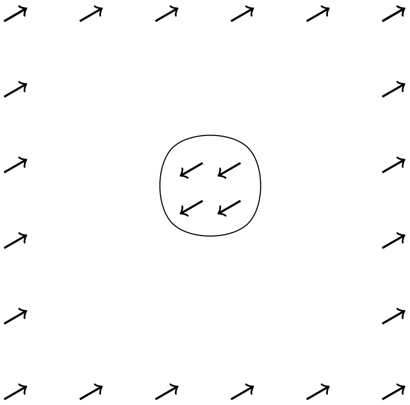
In $L \times L$ slab

$$-\inf_{\sigma} \mathcal{H}(\sigma | \uparrow_L \downarrow_R) + \inf_{\sigma} \mathcal{H}(\sigma | \uparrow_L \uparrow_R) \sim -L^2 \times (1 - \cos(\frac{\pi}{L})) \sim -\frac{\pi^2}{2}.$$



In $L \times L$ slab

$$-\inf_{\sigma} \mathcal{H}(\sigma| \uparrow_L \downarrow_R) + \inf_{\sigma} \mathcal{H}(\sigma| \uparrow_L \uparrow_R) \sim -L^2 \times (1 - \cos(\frac{\pi}{L})) \sim -\frac{\pi^2}{2}.$$



Cost: $\frac{1}{\log L}$ for $d = 2$

L^{d-2} for $d \geq 3$

- Rigorous Statement known as Mermin-Wagner Theorem ['66], Dobrushin-Shlosman ['75] etc.

Theorem

In $d = 2$ for any short range spin system with continuous symmetry ALL Gibbs states are invariant.

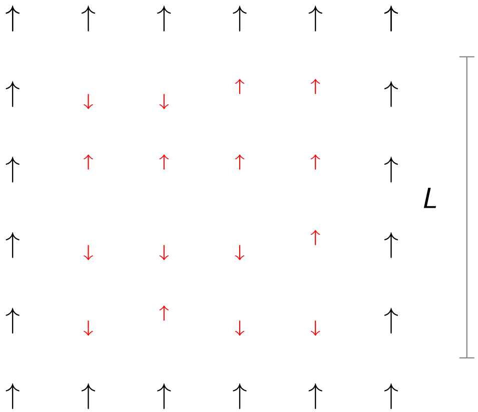
- **Model 2:** $\sigma_x \in \{\pm \hat{e}_2\}$ and $\nu(\sigma_x) = \frac{1}{2}\delta_{e_2}(\sigma_x) + \frac{1}{2}\delta_{-\hat{e}_2}(\sigma_x)$, ϵ field strength.

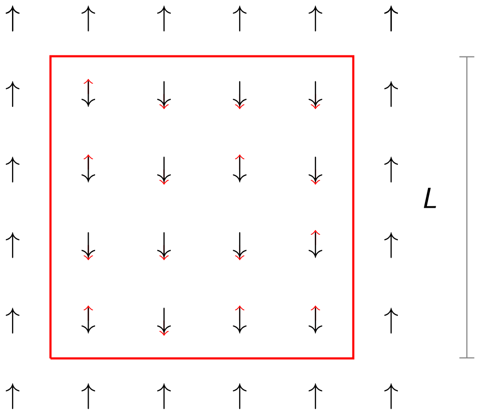
$$-\mathcal{H}^{\omega}(\sigma|\eta) = \sum_{\|x-y\|_2=1, x,y\in\Lambda_L} \sigma_x \cdot \sigma_y + \epsilon \sum_{x\in\Lambda_L} \alpha_x(\omega) e_2 \cdot \sigma_x + \sum_{\|x-y\|_2=1, x\in\Lambda_L, y\notin\Lambda_L} \sigma_x \cdot \eta_y \quad (4)$$

Theorem (**Imry-Ma ’75 Aizenman-Wehr ’89**)

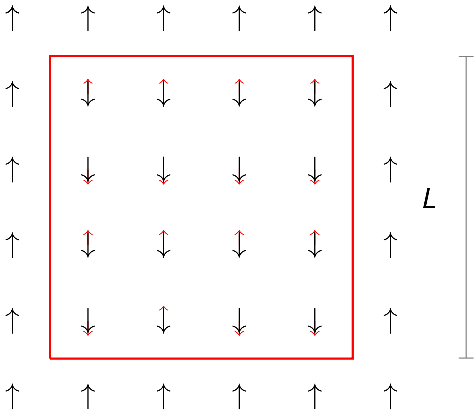
ANY $\beta > 0$, *for all* η *and* $\epsilon > 0$,

$$\mathbb{E}_\alpha \left[\langle \sigma_x \rangle_L \right] \to 0$$


L



Contour Energy: $\sim L^{d-1}$



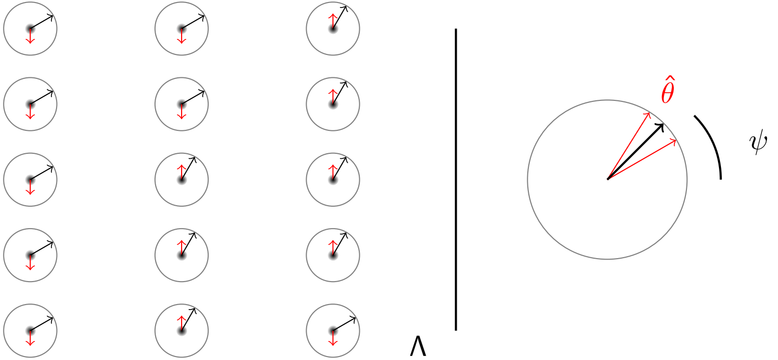
Contour Energy: $\sim L^{d-1}$

Local Field Energy Difference:

$2\epsilon \sum_{x\in\Lambda_L} \alpha_x \sim \epsilon \mathcal{N}_L L^{\frac{d}{2}}$

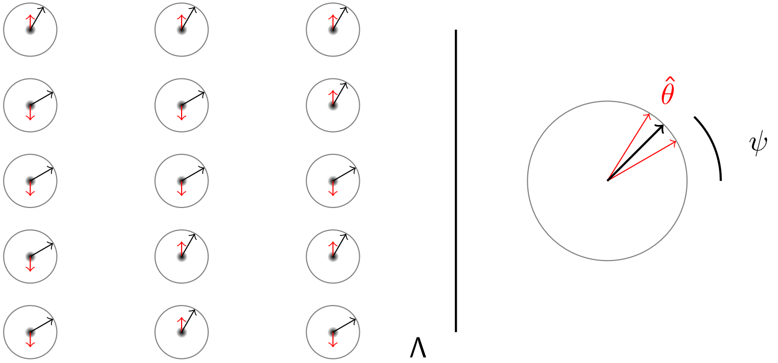
Back to our Question

- Intuitive picture at β large:



Back to our Question

- Intuitive picture at β large:



Fix ψ; write $\theta_x = \psi + \hat{\theta}_x$:

$$-\mathcal{H}_\Lambda(\theta)$$
$$\asymp -\frac{1}{2}\sum_{\langle x,y\rangle\in\Lambda}[\hat{\theta}_x-\hat{\theta}_y]^2+\epsilon\sum_{x\in\Lambda}\frac{1}{\deg_\Lambda(x)}[\alpha_x-\alpha_\Lambda][\cos(\psi)\hat{\theta}_x]+\epsilon|\Lambda|\alpha_\Lambda\sin(\psi) \quad (5$$

$$-\mathcal{H}(\theta) \asymp -\frac{1}{2} \sum_{\langle x,y \rangle \in \Lambda} [\hat{\theta}_x - \hat{\theta}_y]^2 + \epsilon \sum_{x \in \Lambda} [\alpha_x - \alpha_\Lambda][\cos(\psi)\hat{\theta}_x] + |\Lambda| \alpha_\Lambda \sin(\psi)$$

- Optimizing over $\hat{\theta}_x$ gives

$$\epsilon^2 \cos^2(\psi) \sum_{\langle x,y \rangle} [g_x - g_y]^2 |\Lambda| + \alpha_\Lambda \sin(\psi)$$

where

$$-\Delta_N \cdot g_x = \alpha_x - \alpha_\Lambda$$

Our Contribution: Kac Interactions

Let $J : \mathbb{R}^d \to [0, \infty)$, $\int J = 1$ be smooth with compact support and $J_K(u) = K^{-d} J(\frac{u}{K})$.

$$-\mathcal{H}^\omega(\sigma|\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_L} J_K(x-y)\sigma_x \cdot \sigma_y + \epsilon \sum_{x \in \Lambda_L} \alpha_x(\omega) e_2 \cdot \sigma_x + \frac{1}{2} \sum_{x \in \Lambda_L, y \notin \Lambda_L} J_K(x-y)\sigma_x \cdot \eta_y \quad (6)$$

Our Contribution: Kac Interactions

Let $J : \mathbb{R}^d \to [0, \infty)$, $\int J = 1$ be smooth with compact support and $J_K(u) = K^{-d} J(\frac{u}{K})$.

$$-\mathcal{H}^\omega(\sigma|\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_L} J_K(x-y)\sigma_x \cdot \sigma_y + \epsilon \sum_{x \in \Lambda_L} \alpha_x(\omega) e_2 \cdot \sigma_x + \frac{1}{2} \sum_{x \in \Lambda_L, y \notin \Lambda_L} J_K(x-y)\sigma_x \cdot \eta_y \quad (6)$$

Boundary Condition Choice: $\eta_x \equiv e_1$.

Our Contribution: Kac Interactions

Let $J : \mathbb{R}^d \to [0, \infty)$, $\int J = 1$ be smooth with compact support and $J_K(u) = K^{-d} J(\frac{u}{K})$.

$$-\mathcal{H}^\omega(\sigma|\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_L} J_K(x-y)\sigma_x \cdot \sigma_y + \epsilon \sum_{x \in \Lambda_L} \alpha_x(\omega) e_2 \cdot \sigma_x + \frac{1}{2} \sum_{x \in \Lambda_L, y \notin \Lambda_L} J_K(x-y)\sigma_x \cdot \eta_y \quad (6)$$

Boundary Condition Choice: $\eta_x \equiv e_1$.

Block Averages:

$$M_z = K^{-d(1+\delta)} \sum_{x \in B_{K^{1+\delta}}} \sigma_x.$$

Theorem (Main Theorem)

Let $d \geq 2$ be fixed. For ϵ, ξ small and $\beta > \beta(\epsilon, \xi)$, $K \geq K(\epsilon)$,

$$\| \langle M_z \rangle_L^{\omega, \rightarrow} - \rho(\beta, \epsilon) e_1 \|_2 \leq \xi$$

as $L \rightarrow \infty$ except for $z \in \mathcal{D}^\omega \subset \mathbb{Z}^d$ – ω a.s.

Theorem (Main Theorem)

Let $d \geq 2$ be fixed. For ϵ, ξ small and $\beta > \beta(\epsilon, \xi)$, $K \geq K(\epsilon)$,

$$\|\langle M_z \rangle_L^{\omega,\rightarrow} - \rho(\beta,\epsilon) e_1\|_2 \leq \xi$$

as $L \to \infty$ except for $z \in \mathcal{D}^\omega \subset \mathbb{Z}^d$ – ω a.s.

There is $\gamma > 0$ so that

$$|\mathcal{D}^\omega \cap \Lambda_L| \leq e^{-\epsilon^{-\gamma}} L^d$$

for all $L \geq L(\omega)$.

Theorem (Main Theorem)

Let $d \geq 2$ be fixed. For ϵ, ξ small and $\beta > \beta(\epsilon, \xi)$, $K \geq K(\epsilon)$,

$$\|\langle M_z \rangle_L^{\omega, \rightarrow} - \rho(\beta, \epsilon) e_1 \|_2 \leq \xi$$

as $L \to \infty$ except for $z \in \mathcal{D}^\omega \subset \mathbb{Z}^d$ – ω a.s.

There is $\gamma > 0$ so that

$$|\mathcal{D}^\omega \cap \Lambda_L| \leq e^{-\epsilon^{-\gamma}} L^d$$

for all $L \geq L(\omega)$.

$$\rho(\beta, \epsilon) \to \sqrt{1 - \epsilon^2} \text{ as } \beta \to \infty.$$

Proof Outline

- Mean Field Analysis $\rightsquigarrow -\phi$.
- Coarse Graining and Large Deviation Estimates for Magnetization Profiles.
- Randomness and Contour Identification.
- Peierls Contour Counting Estimates.

Mean Field Theory

- For $\delta > 0$, let $k_< = K^{1-\delta}$. All $x \in B_{k_<}$ feel similar interaction.

Mean Field Theory

- For $\delta > 0$, let $k_< = K^{1-\delta}$. All $x \in B_{k_<}$ feel similar interaction.
- Motivates study of Complete Graph type model:

$$\frac{1}{2} K^{-d(1-\delta)} \sum_{x,y \in B_{k_<}} \sigma_x \cdot \sigma_y + \epsilon \sum_{x \in B_{k_<}} \alpha_x e_2 \cdot \sigma_x$$

Mean Field Theory

- For $\delta > 0$, let $k_< = K^{1-\delta}$. All $x \in B_{k_<}$ feel similar interaction.
- Motivates study of Complete Graph type model:

$$\frac{1}{2} K^{-d(1-\delta)} \sum_{x,y \in B_{k_<}} \sigma_x \cdot \sigma_y + \epsilon \sum_{x \in B_{k_<}} \alpha_x e_2 \cdot \sigma_x$$

$$= \frac{1}{2} \sum_{x,y \in B_{k_<}} \sigma_x \cdot \sigma_y + \epsilon \sum_{x \in B^+_{k_<}} \hat{e}_2 \cdot \sigma_x - \epsilon \sum_{x \in B^-_{k_<}} e_2 \cdot \sigma_x$$

Mean Field Theory

- For $\delta > 0$, let $k_< = K^{1-\delta}$. All $x \in B_{k_<}$ feel similar interaction.
- Motivates study of Complete Graph type model:

$$\frac{1}{2} K^{-d(1-\delta)} \sum_{x,y \in B_{k_<}} \sigma_x \cdot \sigma_y + \epsilon \sum_{x \in B_{k_<}} \alpha_x e_2 \cdot \sigma_x$$

$$= \frac{1}{2} \sum_{x,y \in B_{k_<}} \sigma_x \cdot \sigma_y + \epsilon \sum_{x \in B^+_{k_<}} \hat{e}_2 \cdot \sigma_x - \epsilon \sum_{x \in B^-_{k_<}} e_2 \cdot \sigma_x$$

- Typically $\frac{|B^\pm_{k_<}|}{|B_{k_<}|} = \frac{1}{2} + O(|B_{k_<}|^{-\frac{1}{2}})$

- Random variables of interest:

$$M^{\pm} = \frac{1}{|B^{\pm}_{k_<}|} \sum_{x \in B^{\pm}_{k_<}} \sigma_x$$

- Large Deviation Principle for (M^+, M^-)

$$\langle \mathbf{1}_{\{M^{\pm} \sim m^{\pm}\}} \rangle \sim e^{-\beta |B_{k_<}| (\phi(m^+, m^-) - \inf \phi)}.$$

$$\phi(m^+, m^-) = -\frac{1}{8} \|m^+ + m^-\|_2^2 - \frac{\epsilon}{2} \hat{\mathrm{e}}_2 \cdot (m^+ - m^-) - \frac{1}{2\beta}(S(m^+) + S(m^-)) \quad (7)$$

and

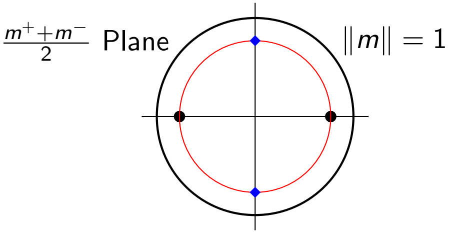
$$S(m) = \inf_{h \in \mathbb{R}^2} (\log \int_{\mathbb{S}^1} \mathrm{d}\nu(\sigma) e^{\sigma \cdot h} - m \cdot h)$$

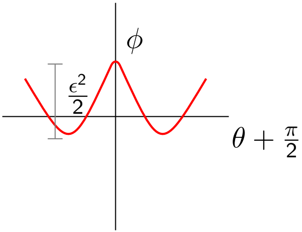
The Limits of this Method

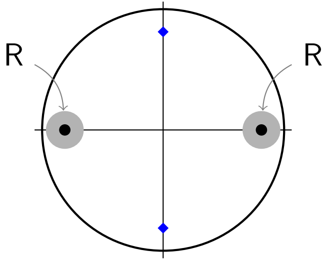
Want to bring LDP

$$\langle \mathbf{1}_{\{M^\pm \sim m^\pm\}} \rangle \sim e^{-\beta |B_{k_<}|(\phi(m^+,m^-) - \inf \phi)}.$$

to $\mathbb{Z}^d$.
Optimizers of $\phi(m^+, m^-)$ characterized by $\frac{m^+ + m^-}{2}$.

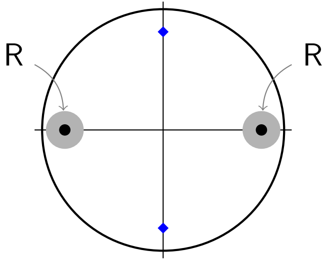






Best we can hope for:

$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2}.$$

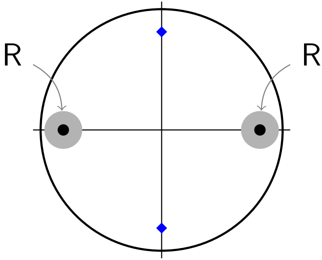


Best we can hope for:

$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2}.$$

Recall $\frac{|B^{\pm}_{k_<}|}{|B_{k_<}|} = \frac{1}{2} + O(|B_{k_<}|^{-\frac{1}{2}})$ so Local Field Term has uncertainty of

$$\beta\epsilon|B_{k_<}|^{\frac{1}{2}}.$$



Best we can hope for:

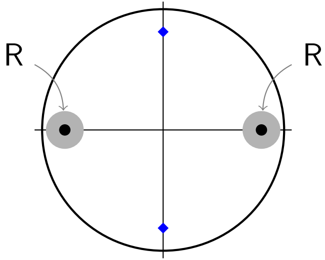
$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2}.$$

Recall $\frac{|B^{\pm}_{k_<}|}{|B_{k_<}|} = \frac{1}{2} + O(|B_{k_<}|^{-\frac{1}{2}})$ so Local Field Term has uncertainty of

$$\beta\epsilon|B_{k_<}|^{\frac{1}{2}}.$$

Leads to correction:

$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2 + O(\beta\epsilon|B_{k_<}|^{\frac{1}{2}})}.$$



Best we can hope for:

$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2}.$$

Recall $\frac{|B^{\pm}_{k_<}|}{|B_{k_<}|} = \frac{1}{2} + O(|B_{k_<}|^{-\frac{1}{2}})$ so Local Field Term has uncertainty of

$$\beta\epsilon|B_{k_<}|^{\frac{1}{2}}.$$

Leads to correction:

$$\langle \mathbf{1}_{\{M^{\pm}\text{not in R }\}} \rangle \sim e^{-c\beta|B_{k_<}|\epsilon^2 + O(\beta\epsilon|B_{k_<}|^{\frac{1}{2}})}.$$

To suppress this need

$$|B_{k_<}| >> \epsilon^{-2}$$