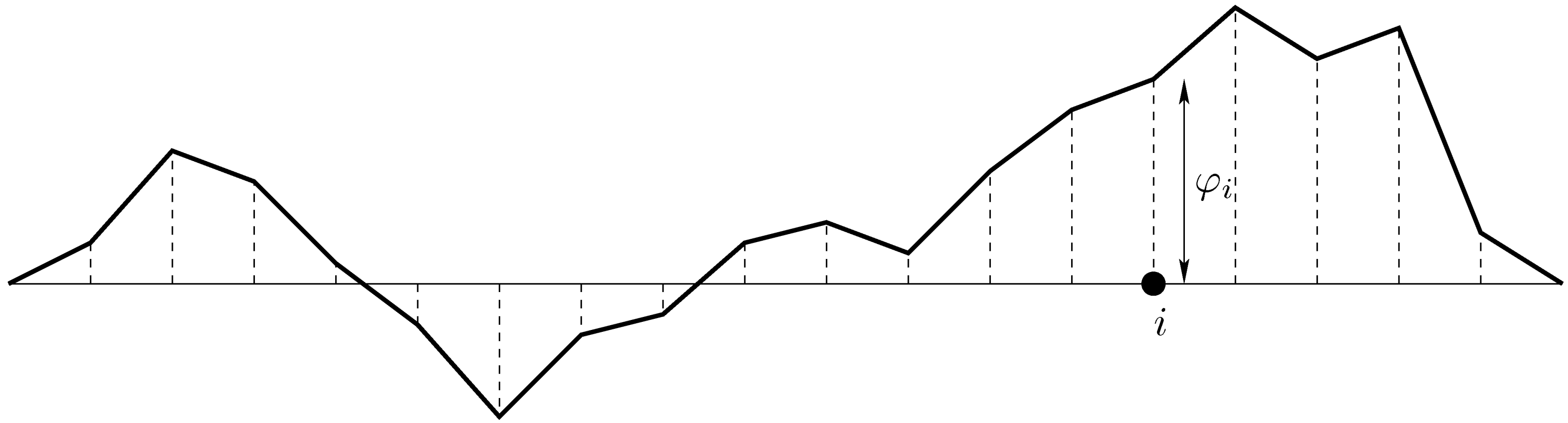
Uniqueness of random gradient states

Codina Cotar

(Based on joint work with Christof Külske)

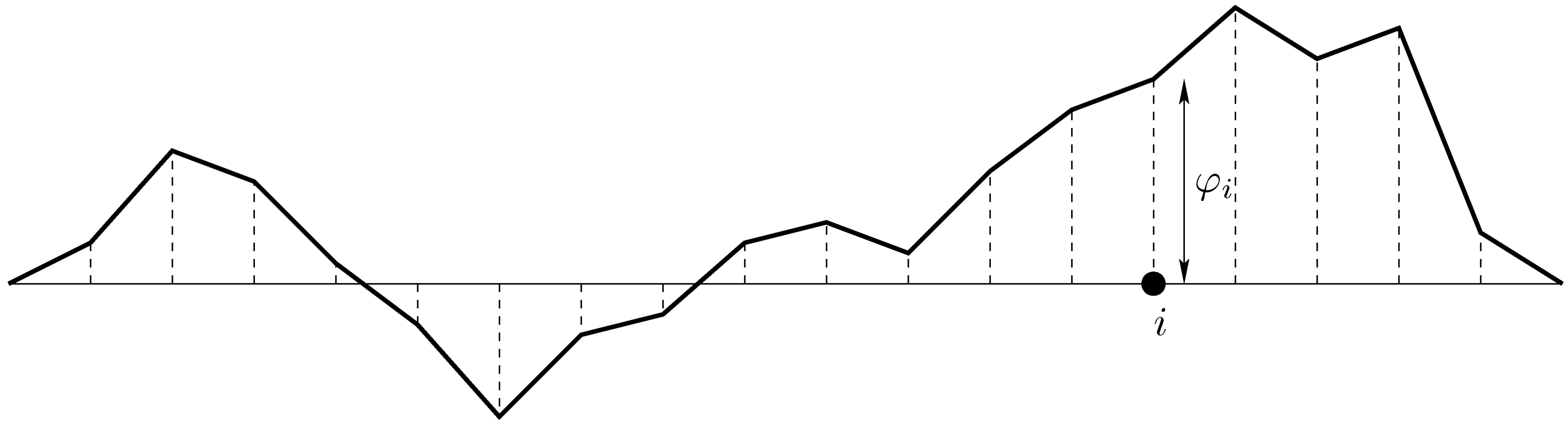
Model

- Interface — transition region that separates different phases



Model

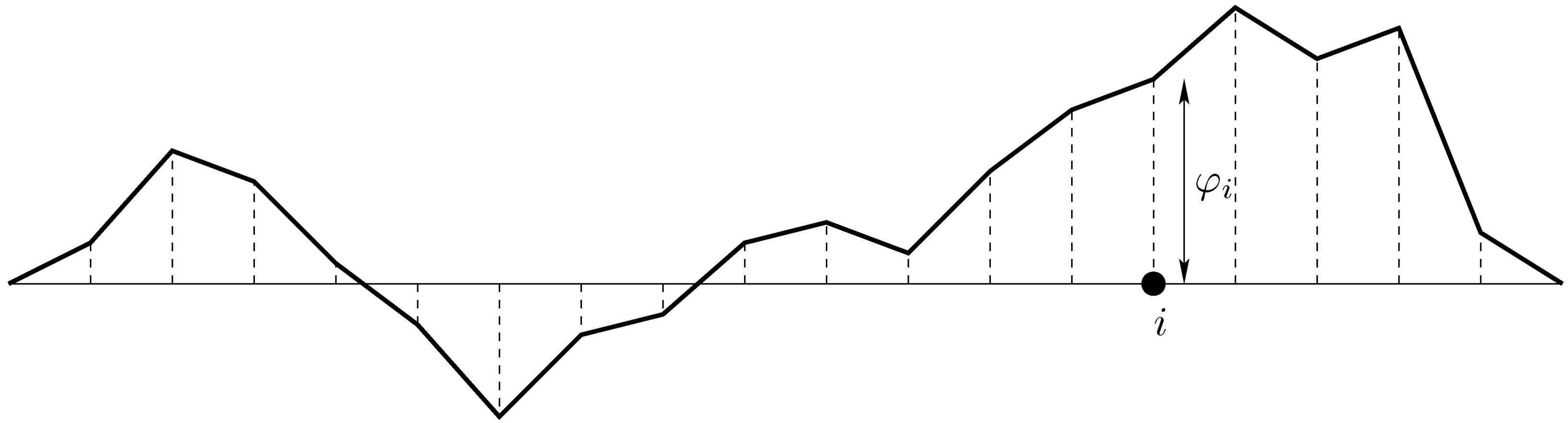
- Interface — transition region that separates different phases



- Finite set $\Lambda \subset \mathbb{Z}^d$

Model

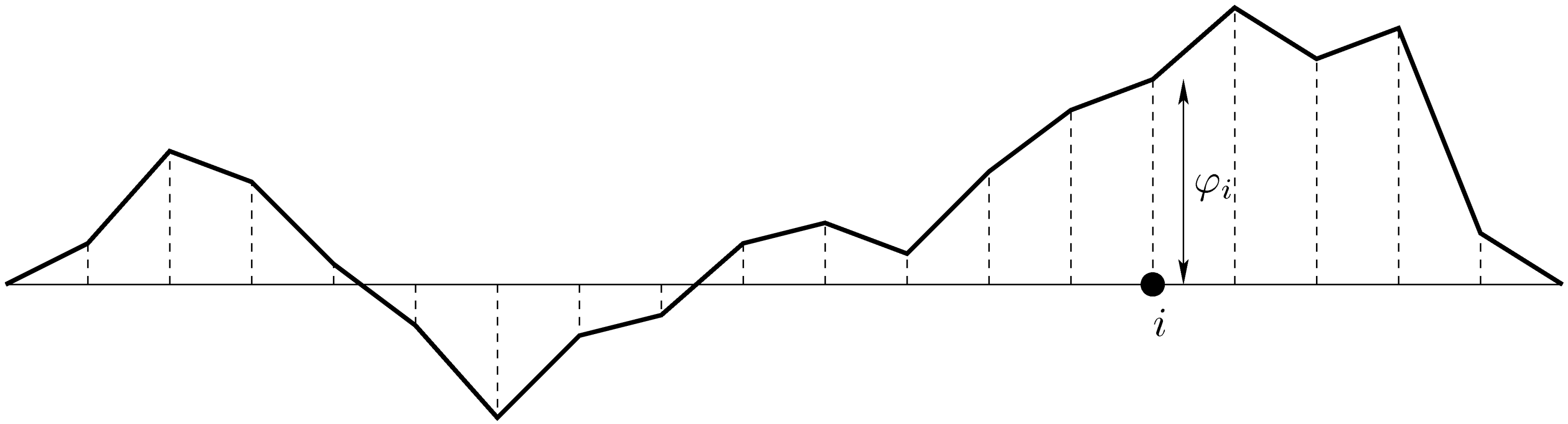
- Interface — transition region that separates different phases



- Finite set $\Lambda \subset \mathbb{Z}^d$
- Height Variables $\phi \colon \Lambda \to \mathbb{R}$

Model

- Interface — transition region that separates different phases



- Finite set $\Lambda \subset \mathbb{Z}^d$
- Height Variables $\phi \colon \Lambda \to \mathbb{R}$
- Boundary $\partial\Lambda$ with boundary condition ψ, such that

$$\phi_x = \psi_x, \quad \text{when} \quad x \in \partial\Lambda.$$

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$
- **Hamiltonian**

$$H_\Lambda^\psi(\phi) = \sum_{\substack{x\in\Lambda, y\in\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y) + 2 \sum_{\substack{x\in\Lambda, y\in\partial\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y)$$

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$
- **Hamiltonian**

$$H_\Lambda^\psi(\phi) = \sum_{\substack{x\in\Lambda, y\in\Lambda,\\ |x-y|=1}} V(\phi_x - \phi_y) + 2 \sum_{\substack{x\in\Lambda, y\in\partial\Lambda,\\ |x-y|=1}} V(\phi_x - \phi_y)$$

- **Bonds** $b = (x, y)$, $x \in \Lambda, y \in \Lambda \cup \partial\Lambda$, $|x - y| = 1$

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$
- **Hamiltonian**

$$H_\Lambda^\psi(\phi) = \sum_{\substack{x\in\Lambda, y\in\Lambda,\\ |x-y|=1}} V(\phi_x - \phi_y) + 2\sum_{\substack{x\in\Lambda, y\in\partial\Lambda,\\ |x-y|=1}} V(\phi_x - \phi_y)$$

- **Bonds** $b = (x, y)$, $x \in \Lambda, y \in \Lambda \cup \partial\Lambda$, $|x - y| = 1$
- **Gradients** $\eta_b = \nabla\phi_b = \phi_x - \phi_y$ for $b = (x, y)$

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$
- **Hamiltonian**

$$H_\Lambda^\psi(\phi) = \sum_{\substack{x\in\Lambda, y\in\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y) + 2 \sum_{\substack{x\in\Lambda, y\in\partial\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y)$$

- **Bonds** $b = (x, y)$, $x \in \Lambda, y \in \Lambda \cup \partial\Lambda$, $|x - y| = 1$
- **Gradients** $\eta_b = \nabla\phi_b = \phi_x - \phi_y$ for $b = (x, y)$
- $\beta = \frac{1}{T} > 0$, where T is the temperature

- **Potentials** $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$
  Example: Gaussian case: $V(s) = s^2$
- **Hamiltonian**

$$H_\Lambda^\psi(\phi) = \sum_{\substack{x\in\Lambda, y\in\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y) + 2 \sum_{\substack{x\in\Lambda, y\in\partial\Lambda, \\ |x-y|=1}} V(\phi_x - \phi_y)$$

- **Bonds** $b = (x, y)$, $x \in \Lambda, y \in \Lambda \cup \partial\Lambda$, $|x - y| = 1$
- **Gradients** $\eta_b = \nabla\phi_b = \phi_x - \phi_y$ for $b = (x, y)$
- $\beta = \frac{1}{T} > 0$, where T is the temperature
- Finite volume **surface tension** $\sigma_\Lambda(u)$: macroscopic energy of a surface with tilt $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$.

ϕ-Gibbs Measure

- The **finite volume Gibbs measure** in Λ

$$\nu_\Lambda^\psi(\mathrm{d}\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \prod_{x\in\Lambda} \mathrm{d}\phi_x .$$

ϕ-Gibbs Measure

- The **finite volume Gibbs measure** in Λ

$$\nu_\Lambda^\psi(\mathrm{d}\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \prod_{x\in\Lambda} \mathrm{d}\phi_x.$$

- The probability measure $\nu \in P(\mathbb{R}^{\mathbb{Z}^d})$ is called a **Gibbs measure** for the ϕ-field if

$$\nu(\cdot|\mathcal{F}_{\Lambda^c})(\psi) = \nu_\Lambda^\psi(\cdot), \quad \nu - \text{a.e. } \psi,$$

for every $\Lambda \subset \mathbb{Z}^d$, where $\mathcal{F}_{\Lambda^c}$ is the σ-field of $\mathbb{R}^{\mathbb{Z}^d}$ generated by $\{\phi(x) : x \notin \Lambda\}$.

$\nabla\phi$-Gibbs Measures

- Gibbs measures do not exist for $d = 1, 2$.

$\nabla\phi$-Gibbs Measures

- Gibbs measures do not exist for $d = 1, 2$.
- The Hamiltonian is a functional of the gradient fields $\nabla\phi$, so we can consider the distribution of the $\nabla\phi$-field under the Gibbs measure ν. We call this measure the **$\nabla\phi$-Gibbs measure** μ.

$\nabla\phi$-Gibbs Measures

- Gibbs measures do not exist for $d = 1, 2$.
- The Hamiltonian is a functional of the gradient fields $\nabla\phi$, so we can consider the distribution of the $\nabla\phi$-field under the Gibbs measure ν. We call this measure the **$\nabla\phi$-Gibbs measure** μ.
- $\nabla\phi$-Gibbs measures exist for $d \geq 1$.

Questions

1. Existence and (strict) convexity of the surface tension

$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$

Questions

1. Existence and (strict) convexity of the surface tension

$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$

2. Existence of shift-invariant $\nabla\phi$- Gibbs measure μ.

Questions

1. Existence and (strict) convexity of the surface tension
$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$
2. Existence of shift-invariant $\nabla\phi$- Gibbs measure μ.
3. Uniqueness of shift-invariant $\nabla\phi$- Gibbs measure μ under additional assumptions on μ.

Questions

1. Existence and (strict) convexity of the surface tension
$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$
2. Existence of shift-invariant $\nabla\phi$- Gibbs measure μ.
3. Uniqueness of shift-invariant $\nabla\phi$- Gibbs measure μ under additional assumptions on μ.
4. Decay of covariances with respect to μ.

Questions

1. Existence and (strict) convexity of the surface tension
$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$
2. Existence of shift-invariant $\nabla\phi$- Gibbs measure μ.
3. Uniqueness of shift-invariant $\nabla\phi$- Gibbs measure μ under additional assumptions on μ.
4. Decay of covariances with respect to μ.
5. Central limit theorem (CLT) results.

Questions

1. Existence and (strict) convexity of the surface tension
$$\sigma(u) = \lim_{|\Lambda|\to\infty} \sigma_\Lambda(u).$$
2. Existence of shift-invariant $\nabla\phi$- Gibbs measure μ.
3. Uniqueness of shift-invariant $\nabla\phi$- Gibbs measure μ under additional assumptions on μ.
4. Decay of covariances with respect to μ.
5. Central limit theorem (CLT) results.
6. Large deviations (LDP) results.

Results: Strictly Convex Potentials

1. **Funaki-Spohn (CMP, 1997):** Existence and convexity of the surface tension for $d \geq 1$.

Results: Strictly Convex Potentials

1. **Funaki-Spohn (CMP, 1997):** Existence and convexity of the surface tension for $d \geq 1$.
2. **Deuschel-Giacomin-Ioffe (PTRF, 2000)/Giacomin-Olla-Spohn (AOP, 2001):** Strict convexity of the surface tension for $d \geq 1$.

Results: Strictly Convex Potentials

1. **Funaki-Spohn (CMP, 1997):** Existence and convexity of the surface tension for $d \geq 1$.
2. **Deuschel-Giacomin-Ioffe (PTRF, 2000)/Giacomin-Olla-Spohn (AOP, 2001):** Strict convexity of the surface tension for $d \geq 1$.
3. **Funaki-Spohn (CMP, 1997):** Assume $C_1 \leq V'' \leq C_2$. For every $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ there exists a unique shift-invariant $\nabla\phi$- Gibbs measure μ with $E_\mu[\phi(e_i) - \phi(0)] = u_i$, for all $i = 1, \ldots, d$.

Results: Strictly Convex Potentials

4. **Naddaf-Spencer (CMP, 1997)/Giacomin-Olla-Spohn (AOP, 2001):** CLT

$$\epsilon^{\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} f(\epsilon x)[\phi_{x+e_i} - \phi_x - u_i] \xrightarrow{\epsilon \to 0} N(0, \sigma_u^2(f)).$$

Results: Strictly Convex Potentials

4. **Naddaf-Spencer (CMP, 1997)/Giacomin-Olla-Spohn (AOP, 2001):** CLT

$$\epsilon^{\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} f(\epsilon x)[\phi_{x+e_i} - \phi_x - u_i] \xrightarrow{\epsilon \to 0} N(0, \sigma_u^2(f)).$$

5. **Deuschel-Giacomin-Ioffe (PTRF, 2000):** LDP

Techniques

- Brascamp-Lieb Inequality: for all $x \in \Lambda$

$$\text{var}_{\nu_\Lambda^\psi}(\phi_x) \leq \text{var}_{\tilde{\nu}_\Lambda^\psi}(\phi_x),$$

$\tilde{\nu}_\Lambda^\psi$ is the Gibbs measure with potential $\tilde{V}(s) = C_1 s^2$.

Techniques

- Brascamp-Lieb Inequality: for all $x \in \Lambda$

$$\text{var}\,_{\nu_\Lambda^\psi}(\phi_x) \le \text{var}\,_{\tilde{\nu}_\Lambda^\psi}(\phi_x),$$

  $\tilde{\nu}_\Lambda^\psi$ is the Gibbs measure with potential $\tilde{V}(s) = C_1 s^2$.

- Random Walk Representation: Representation of the Covariance Matrix in terms of the Green function of a particular random walk.

Results: Non-convex potentials

- **Funaki-Spohn (CMP, 1997):** Convexity of the surface tension for all potentials V.

Results: Non-convex potentials

- **Funaki-Spohn (CMP, 1997):** Convexity of the surface tension for all potentials V.
- **Cotar-Deuschel-Müller (CMP, 2009):** Surface tension is strictly convex if

$$\begin{aligned} V &= V_0 + g,\ C_1 \le V_0'' \le C_2, \\ -C_0 &\le g'' < 0, \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \text{ small } (C_0, C_1, C_2). \end{aligned}$$

Results: Non-convex potentials

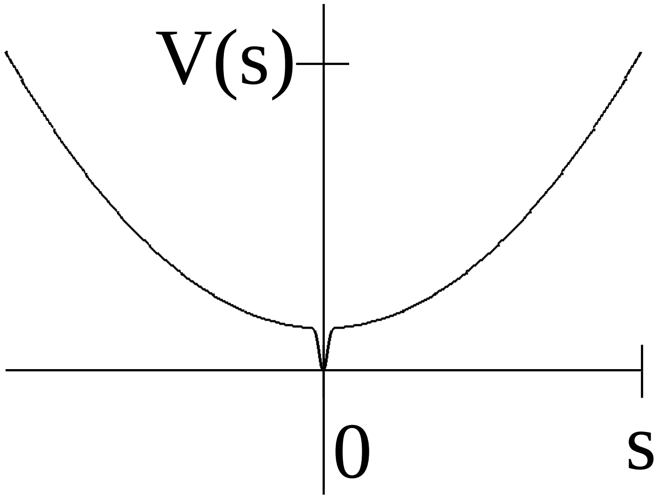
- **Funaki-Spohn (CMP, 1997):** Convexity of the surface tension for all potentials V.
- **Cotar-Deuschel-Müller (CMP, 2009):** Surface tension is strictly convex if

$$\begin{aligned} V &= V_0 + g,\ C_1 \leq V_0'' \leq C_2, \\ -C_0 &\leq g'' < 0, \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \text{ small } (C_0, C_1, C_2). \end{aligned}$$

- **Adams-Kotecký-Müller (preprint, 2011):** Strict convexity of the surface tension for small tilts u and large β.

- For the potential

$$e^{-V(s)} = pe^{-k_1 \frac{s^2}{2}} + (1-p)e^{-k_2 \frac{s^2}{2}}$$



- **Biskup-Kotecký (PTRF, 2007):** Non-uniqueness of several $\nabla\phi$-Gibbs measures with $E_\mu[\phi(e_i) - \phi(0)] = 0$, but with different variances.

- **Biskup-Kotecký (PTRF, 2007):** Non-uniqueness of several $\nabla\phi$-Gibbs measures with $E_\mu[\phi(e_i) - \phi(0)] = 0$, but with different variances.
- **Biskup-Spohn (AOP, 2011):** CLT for every ergodic $\nabla\phi$-Gibbs measure μ with $E_\mu[\phi(e_i) - \phi(0)] = 0$.

- **Cotar-Deuschel (AIHP, to appear):** Let $u \in \mathbb{R}^d$. Assume $V = V_0 + g$, with $C_1 \leq V_0'' \leq C_2$ and $-\infty < g'' < 0$. If

$$\sqrt{\beta}||g''||_{L^1(\mathbb{R})} \text{ small } (C_1, C_2),$$

there exists exactly one shift-invariant $\nabla\phi$-Gibbs measure μ such that $E_\mu\left[\phi(e_i) - \phi(0)\right] = u_i$ for all $i = 1, 2, \ldots, d$.

- **Cotar-Deuschel (AIHP, to appear):** Let $u \in \mathbb{R}^d$. Assume $V = V_0 + g$, with $C_1 \leq V_0'' \leq C_2$ and $-\infty < g'' < 0$. If

$$\sqrt{\beta}||g''||_{L^1(\mathbb{R})} \text{ small } (C_1, C_2),$$

there exists exactly one shift-invariant $\nabla\phi$-Gibbs measure μ such that $E_\mu\left[\phi(e_i) - \phi(0)\right] = u_i$ for all $i = 1, 2, \ldots, d$.

Interfaces with Disorder

Two ways to add randomness

(a) Add to the Hamiltonian the random part

$$\sum_{x \in \Lambda} \xi_x \phi_x$$

$(\xi_x)_{x \in \mathbb{Z}^d}$ i.i.d with $E_{\mathbb{P}}(\xi_x^2) < \infty$.
Or

Interfaces with Disorder

Two ways to add randomness

(a) Add to the Hamiltonian the random part

$$\sum_{x\in\Lambda} \xi_x \phi_x$$

$(\xi_x)_{x\in\mathbb{Z}^d}$ i.i.d with $E_{\mathbb{P}}(\xi_x^2) < \infty$.
Or

(b) Make the potentials random: $V_{\langle x,y\rangle}$ i.i.d with $As^2 - B_{(x,y)} \le V_{(x,y)}(s) \le C_2 s^2$ and $\mathbb{E}|B_{\langle x,y\rangle}| < \infty$.

Results

- Translation-covariance:

$$\int \mu[\tau_v(\xi)](d\eta)F(\eta) = \int \mu[\xi](d\eta)F(\tau_v\eta),$$

for $(\tau_v\eta)(b) := \eta(b-v)$, for all bonds b and all $v \in \mathbb{Z}^d$.

Results

- Translation-covariance:

$$\int \mu[\tau_v(\xi)](d\eta)F(\eta) = \int \mu[\xi](d\eta)F(\tau_v\eta),$$

  for $(\tau_v\eta)(b) := \eta(b-v)$, for all bonds b and all $v \in \mathbb{Z}^d$.

- **For model (a), van Enter-Külske (2007):** For $d = 2$, non-existence of shift-covariant gradient Gibbs measures.

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$
2. for model (b), when $d \geq 2$.

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$
2. for model (b), when $d \geq 2$.

- **Cotar-Külske (in preparation):** Uniqueness of shift-covariant gradient Gibbs measures with tilt u when V uniformly strictly convex

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$
2. for model (b), when $d \geq 2$.

- **Cotar-Külske (in preparation):** Uniqueness of shift-covariant gradient Gibbs measures with tilt u when V uniformly strictly convex

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$

- **Cotar-Külske (2010):** Existence of shift-covariant gradient Gibbs measures

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$
2. for model (b), when $d \geq 2$.

- **Cotar-Külske (in preparation):** Uniqueness of shift-covariant gradient Gibbs measures with tilt u when V uniformly strictly convex

1. for model (a) when $\mathbb{E}_{\mathbb{P}}(\xi(x)) = 0$ and $d \geq 3$
2. for model (b), when $d \geq 2$.