**Problem 1:** Prove that UI families of random variables are sequentially compact in weak-$L^1$: For each $\{X_n \in n \in \mathbb{N}\}$ be UI, there is $n_k \to \infty$ and $X \in L^1$ such that

$$\forall Z \in L^\infty : \quad E(ZX_{n_k}) \xrightarrow{k \to \infty} E(ZX).$$

Show also that boundedness of $L^1$-norms does not suffice by constructing $\{X_n : n \in \mathbb{N}\}$ with $\sup_{n \geq 1} E|X_n| < \infty$ for which the above conclusion fails.

**Problem 2:** (Localization trick) Let $\{M_t, \mathcal{F}_t : t \geq 0\}$ be a continuous local martingale. Prove that there is a unique continuous, non-decreasing, adapted process $A_t$ with $A_0 = 0$ such that $\{M_t^2 - A_t, \mathcal{F}_t : t \geq 0\}$ is a local martingale.

*Note:* We still call this process the variance process, keeping the notation $\langle M \rangle_t$, even though $M_t$ need not be in $L^2$ for any $t \geq 0$.

**Problem 3:** (Covariance process) Let $X$ and $Y$ be continuous local martingales. Prove there is a unique continuous, adapted process $\{A_t : t \geq 0\}$ of finite (first) variation and $A_0 = 0$ such that

$$M_t := X_tY_t - A_t \quad \text{is a continuous martingale}$$

We call $A$ the *cross-variation* or *covariance process* of $X$ and $Y$ with the notation $\langle X, Y \rangle_t$. *Hint:* Note that $\langle X, X \rangle_t = \langle X \rangle_t$.

**Problem 4:** (Developing the previous problem) Suppose $X$ and $Y$ are both continuous local martingales. Given a partition $\Pi := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of $[0, t]$, consider the midpoint Riemann sum

$$\sum_{i=1}^n \frac{Y_{t_{i-1}} + Y_{t_i}}{2} (M_{t_i} - M_{t_{i-1}}).$$

Prove that for any sequence of partitions $\{\Pi_n\}$ with $|\Pi_n| \to 0$, these sums converge in probability to

$$\int_0^t Y_s \circ dM_s := \int_0^t Y_s \, dM_s + \frac{1}{2} \langle Y, M \rangle_t.$$  

This is the *Stratonovich* integral.