Problem 1: Prove the (what the textbook calls) Doob-Dynkin lemma: Let $(\mathcal{X}, \Sigma)$ be a Standard Borel space and assume that $X$ and $Y$ are $\mathcal{X}$-valued random variables such that $Y$ is $\sigma(X)$-measurable, where we recall $\sigma(X) := \sigma(\{X \in B : B \in \Sigma\})$. Prove that there is a measurable function $f: \mathcal{X} \to \mathcal{X}$ such that $Y = f(X)$.

Problem 2: Let $T$ be a set and let $\{X_t : t \in T\}$ be a family of random variables. Prove $\sigma(\{X_t : t \in T\}) = \bigcup_{S \subseteq T, \text{countable}} \sigma(\{X_t : t \in S\})$.

Problem 3: (Gaussian Hilbert space) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ with the scalar product denoted by $\langle \cdot, \cdot \rangle$. Show that there exists a process $\{X(f) : f \in \mathcal{H}\}$ such that for any $f_1, \ldots, f_n \in \mathcal{H}$, the collection $(X(f_1), \ldots, X(f_n))$ is multivariate normal with $EX(f) = 0$ and $\text{Cov}(X(f), X(g)) = \langle f, g \rangle$ for all $f, g \in \mathcal{H}$. Prove that $f \mapsto X(f)$ is necessarily linear. Note: We may at times be able to interpret $f \mapsto X(f)$ as a continuous linear functional on $\mathcal{H}$. The Riesz lemma then yields the existence of an $\mathcal{H}$-valued random variable $X$ such that $X(f) = \langle X, f \rangle$. The law of $X$ (i.e., the push-forward of the underlying probability onto the image space of $X$) then defines a Gaussian measure on $\mathcal{H}$.

Problem 4: For the setting in the previous problem, let $\mathcal{H}$ be the completion of the set $\{f \in C^1([0, \infty]) : f(0) = 0, f' \in L^2([0, \infty])\}$ with respect to the norm-topology associated with the inner product $\langle f, g \rangle := \int_{[0, \infty)} dx f'(x)g'(x)$.

Do the following:

1. Show that $f_t(x) := \min\{x, t\}$ lies in $\mathcal{H}$ for each $t \geq 0$.
2. Prove that setting $B_t := X(f_t)$ defines a process $\{B_t : t \geq 0\}$ whose finite dimensional distributions are those of the standard Brownian motion.