2. REVERSIBLE MARKOV CHAINS AND RESISTOR NETWORKS

The goal of this chapter is to explore harmonic analysis of Markov chains when the chain admits a reversible measure. The key novelty is the ensuing connection to resistor networks which allows us to characterize transience by finiteness of the effective resistance to infinity. This leads to monotonicity of transience in the associated conductances as well as convenient criteria for recurrence and transience in specific network.

2.1 Reversible measures.

Consider an irreducible Markov chain on a finite or countably infinite state space $S$. We will write \( \{X_n: n \geq 0\} \) for the sample paths of the Markov chain and denote by $P$ the transition probabilities and by $P_x$ the law of the paths with $P_x(X_0 = x) = 1$. A key concept of this section is:

**Definition 2.1** A measure $\pi$ on $S$ is said to be reversible if

$$\forall x, y \in S: \quad \pi(x)P(x, y) = \pi(y)P(y, x) \tag{2.1}$$

As is easy to check, a reversible measure is automatically invariant, or stationary (just sum (2.1) over $y \in S$). A question thus arises to characterize the chains that admit a reversible measure. This is the content of:

**Theorem 2.2** (Kolmogorov cycle condition) An irreducible Markov chain admits a reversible measure if and only if

$$\forall n \geq 1, \forall x_0, x_1, \ldots, x_n = x_0 \in S: \quad \prod_{i=1}^n P(x_{i-1}, x_i) = \prod_{i=1}^n P(x_i, x_{i-1}) \tag{2.2}$$

In particular, a reversible measure, if it exists, is unique up to rescaling.

**Proof.** Suppose first that there is a $\pi$ such that (2.1) holds. By irreducibility, $\pi(x) > 0$ for all $x \in S$. Then $P(x, y) > 0$ is equivalent to $P(y, x) > 0$ and so (2.2) needs to be only checked when all the terms in the products are positive. Given any $x_0, \ldots, x_n = x_0$ with $P(x_i, x_{i-1}) > 0$, we then have

$$\pi(x_i) = \pi(x_{i-1}) \frac{P(x_{i-1}, x_i)}{P(x_i, x_{i-1})}. \tag{2.3}$$

Iterating and using that $x_n = x_0$, we get (2.2) as desired.

Suppose in turn that (2.2) holds. Pick any $z_0, \ldots, z_n$ and $y_0, \ldots, y_m$ with $z_0 = y_0$ and $z_n = y_m$ and $P(z_i, z_{i-1}) > 0$ and $P(y_i, y_{i-1}) > 0$ for all $i$. We claim that then

$$\prod_{i=1}^n P(z_{i-1}, z_i) = \prod_{j=1}^m P(y_{j-1}, y_j) \tag{2.4}$$

To see this define

$$x_i := \begin{cases} 
  z_i, & \text{if } i = 0, \ldots, n, \\
  y_{n+m-i}, & \text{if } i = n, \ldots, n + m. 
\end{cases} \tag{2.5}$$
This is consistent because $z_n = y_m = y_{n+m-n}$. Also observe that $x_{n+m} = x_0$ because $z_0 = y_0$. Applying (2.2) to this path, we get
\[
\left( \prod_{i=1}^{n} P(z_{i-1}, z_i) \right) \left( \prod_{j=1}^{m} P(y_j, y_{j-1}) \right) = \left( \prod_{i=1}^{n} P(z_i, z_{i-1}) \right) \left( \prod_{j=1}^{m} P(y_{j-1}, y_j) \right) \tag{2.6}
\]
which in light of positivity of all terms yields (2.4).

We now pick any $x_0 \in S$ and pick a value for $\pi(x_0)$. For any $x \in S$ we then pick a sequence of states $x_0, x_1, \ldots, x_n = x \in S$ with $P(x_{i-1}, x_i) > 0$ for all $i = 1, \ldots, n$ — such a path exists by irreducibility — and set
\[
\pi(x) := \pi(x_0) \prod_{i=1}^{n} \frac{P(x_{i-1}, x_i)}{P(x_i, x_{i-1})}. \tag{2.7}
\]
The condition (2.4) ensures that the value of $\pi(x)$ is the same regardless of the path chosen. If $y \in S$ is such that $P(x, y) > 0$ then concatenating the step $(x, y)$ to the path from $x_0$ to $x$ shows
\[
\pi(y) = \pi(x) \frac{P(x, y)}{P(y, x)} \tag{2.8}
\]
thus proving that $\pi$ is a reversible measure. The argument also shows that the reversible measure is unique up to the choice of the value at one point.

Although each Markov chain has at most one reversible measure up to rescaling, it can still admit (at least when $S$ is infinite) multiple invariant measures. Since invariant measures satisfy a very similar equation as harmonic functions, we can in fact use harmonic functions to characterize all invariant measures:

**Lemma 2.3** (Invariant measures from harmonic functions) Consider an irreducible Markov chain on $S$ and assume the chain admits a reversible measure $\pi$. Let $\mathcal{M}(S)$ denote the class of measures on $S$ and let us write
\[
\mathcal{I} := \{ v \in \mathcal{M}(S) : vP = v \}, \tag{2.9}
\]
for the class of invariant measures for the chain. Then $\mathcal{I}$ in one-to-one correspondence with the class of all non-negative harmonic functions via
\[
\mathcal{I} = \{ h(\cdot) : h \geq 0, Ph = h \}. \tag{2.10}
\]

**Proof.** Let $h \geq 0$ obey $Ph = h$. Then
\[
\sum_{y \in S} P(x, y) h(y) = h(x), \quad x \in S. \tag{2.11}
\]
Multiplying both sides by the value $\pi(x)$ of the reversible measure at $x$ and invoking (2.1) shows
\[
\sum_{y \in S} \pi(y) h(y) P(y, x) = \pi(x) h(x), \quad x \in S \tag{2.12}
\]
Hence $\nu(x) := \pi(x)h(x)$ is an invariant measure. Conversely, if $\nu$ is an invariant measure, then

$$\sum_{y \in S} \nu(y)P(y, x) = \nu(x), \quad x \in S. \tag{2.13}$$

Dividing both sides of the equation by $\pi(x)$ and invoking (2.1) we now readily check that $h(x) := \nu(x)/\pi(x)$ is a non-negative harmonic function. \hfill \square

We note that, even in the absence of a reversible measure, once there is an invariant measure, a connection to harmonic functions can be made. Indeed, if $\nu$ is invariant, then the chain admits a reversal, which is a Markov chain with the transition kernel

$$Q_\nu(x, y) := \frac{\nu(y)}{\nu(x)}P(y, x). \tag{2.14}$$

As is readily checked, if $h \geq 0$ is $P$-harmonic, then $h(\cdot)v(\cdot)$ is $Q_\nu$-invariant, and vice versa. For $\pi$ this reduces to the above because $Q_\pi = P$.

An interesting characterization of reversibility comes in:

**Lemma 2.4** For any invariant measure $\nu$, $P$ is a bounded linear operator on $\ell^2(\nu)$ with operator norm at most one. An invariant measure $\pi$ is reversible if and only if $P$ is self-adjoint on $\ell^2(\pi)$.

**Proof.** Consider first $f : S \to [0, \infty)$. Then $Pf$ is well defined by (1.2), albeit possibly infinite. Invoking the Cauchy-Schwarz inequality along with the fact that $y \mapsto P(x, y)$ and that $\nu$ obeys $\nu P = \nu$ shows that, for any $g : S \to [0, \infty)$,

$$\begin{align*}
( g, Pf )_{\ell^2(\nu)} &= \sum_{x,y \in S} \nu(x)g(x)P(x, y)f(y) \\
&\leq \left( \sum_{x,y \in S} \nu(x)g(x)^2 P(x, y) \right)^{1/2} \left( \sum_{x,y \in S} \nu(x)P(x, y)f(x)^2 \right)^{1/2} \\
&= \|g\|_{\ell^2(\nu)} \|f\|_{\ell^2(\nu)}
\end{align*}\tag{2.15}$$

By testing with various choices of $g$ we conclude that $Pf$ is finite everywhere whenever $f \in \ell^2(\nu)$. Moreover, choosing $g := Pf$ shows that

$$\|Pf\|_{\ell^2(\nu)} \leq \|f\|_{\ell^2(\nu)}. \tag{2.16}$$

Since $|Pf| \leq Pf$ and since the norm of $|f|$ is equal to the norm of $f$, the conclusions above apply to all $f \in \ell^2(\nu)$. Hence, $P$ is a linear operator $\ell^2(\nu) \to \ell^2(\nu)$ with operator norm at most one.

Consider now a reversible measure $\pi$. Then (2.1) yields

$$\begin{align*}
( g, Pf )_{\ell^2(\pi)} &= \sum_{x,y \in S} \pi(x)g(x)P(x, y)f(y) \\
&= \sum_{x,y \in S} \pi(y)g(x)P(y, x)f(y) = (Pg, f)_{\ell^2(\pi)}
\end{align*}\tag{2.17}$$

Hence, $P$ is symmetric (and being bounded) thus self-adjoint. Setting $g := \delta_x$ and $f := \delta_y$ in turn shows that self-adjointness of $P$ on $\ell^2(\pi)$ implies (2.1). \hfill \square
We note that the characterization of reversibility by self-adjointness of the transition operator is in fact the way reversibility is defined in the context of Markov chains on continuum state spaces.

### 2.2 Resistor networks and their laws.

The prime benefit of reversibility is that it allows us to cast harmonic analysis of the Markov chain in the language of resistor networks. Although somewhat incidental, this connection has been quite useful in many specific applications and so we will now spend time to develop its salient aspects more closely. A reader should bear patiently with these developments as we need a number of definitions and lemmas before a link back to Markov chains can productively be made.

A resistor network is a graph \( \mathcal{G} = (V, E) \) with a finite or countably-infinite set of vertices \( V \) and a set of unoriented edges \( E \). (In general networks, multiple edges may be allowed between the same pair of vertices but we will not do this here.) To simplify writing sums later we assume that each edge appears in \( E \) only once, meaning that both orientations are identified into one. Each edge \( e \in E \) is endowed with a resistance \( r(e) \) which we take to be positive and finite for all \( e \in E \). The reciprocal value \( c(e) \) of the resistance \( r(e) \) is called conductance. We thus have

\[
    c(e) := \frac{1}{r(e)}, \quad e \in E. \tag{2.18}
\]

To reveal, albeit with no explanation, the connection of resistor networks with reversible Markov chains, for the conductance of edge \((x, y)\) we can always take

\[
    c(x, y) := \pi(x)P(x, y) \tag{2.19}
\]

which is symmetric in \( x \) and \( y \) by (2.1). Conversely, if a resistor network on \( V \) is given, we can consider a Markov chain on \( S := V \) with the transition probabilities

\[
    P(x, y) := \frac{1}{\pi(x)}c(x, y) \quad \text{where} \quad \pi(x) := \sum_{y: (x, y) \in E} c(x, y). \tag{2.20}
\]

For this we only need to assume that \( \pi(\cdot) \), which is then a reversible measure, is positive and finite at every vertex of \( V \).

A classical problem in resistor network theory is to determine how much current flows through the network when a unit voltage is applied to two given nodes. A classical exposition of this problem requires introduction of Kirchhoff’s laws of currents and linking these to voltage differences via Ohm’s law. Instead, we postulate the solution in terms of a variational problem:

**Definition 2.5 (Dirichlet’s principle)** Consider a finite resistor network \( \mathcal{G} = (V, E) \) and let \( u, v \in V \) be distinct vertices. Then, upon a unit voltage difference applied to the pair \((u, v)\), the voltage at vertex \( x \) is obtained as the value of a minimizer of

\[
    E(f) := \sum_{(x, y) \in E} c(x, y)[f(y) - f(x)]^2 \tag{2.21}
\]
over  \[ \mathcal{F}(u, v) := \{ f : V \to \mathbb{R} : f(u) = 1, f(v) = 0 \}. \]  \hspace{1cm} (2.22)

Moreover, the value of the minimum,

\[ C_{\text{eff}}(u, v) := \inf \{ \mathcal{E}(f) : f \in \mathcal{F}(u, v) \} \]  \hspace{1cm} (2.23)
defines the effective conductances \( C_{\text{eff}}(u, v) \) between \( u \) and \( v \).

Under the condition that the underlying chain is reversible, the resistor network is connected as a graph (this is why we require that an edge be present if and only if the associate conductance is strictly positive). Since the network is assumed to be finite, we readily conclude that \( C_{\text{eff}}(u, v) \in (0, \infty) \) for any two distinct \( u, v \in V \). (As we will see, this may fail in infinite networks.) Our next task is to characterize the infimum more explicitly (still within a context of finite networks):

**Lemma 2.6 (Connection to Markov chain)**  For any distinct \( u, v \in V \), a minimizer in (2.23) exists and is unique. Moreover, if the network is related to a Markov chain with reversible measure \( \pi \) via (2.19), then the unique minimizer in (2.23) is given by

\[ f^*(x) := P^x(\hat{\tau}_u < \hat{\tau}_v) \]  \hspace{1cm} (2.24)

where, we recall, \( \hat{\tau}_x := \inf\{n \geq 1 : X_n = x\} \). In this case,

\[ C_{\text{eff}}(u, v) = \pi(u) P^u(\hat{\tau}_v < \hat{\tau}_u). \]  \hspace{1cm} (2.25)

**Proof.** We start by proving the existence of a minimizer. Let \( \{f_n : n \geq 1\} \in \mathcal{F}(u, v) \) be a sequence such that \( \mathcal{E}(f_n) \downarrow C_{\text{eff}}(u, v) \). The fact that \( \mathcal{F}(u, v) \) is convex and \( f \mapsto \mathcal{E}(f) \) quadratic implies

\[
0 \leq \mathcal{E}(f_n - f_m) = \mathcal{E}(f_n) + \mathcal{E}(f_m) - 2\mathcal{E}\left(\frac{f_n + f_m}{2}\right) \\
\leq \mathcal{E}(f_n) + \mathcal{E}(f_m) - 2C_{\text{eff}}(u, v). \]  \hspace{1cm} (2.26)

It follows that

\[
\lim_{n,m \to \infty} \mathcal{E}(f_n - f_m) = 0. \]  \hspace{1cm} (2.27)

Since the conductances are positive on each \( e \in E \), we get

\[
\lim_{n \to \infty} [f_n(x) - f_n(y)] = 0, \quad (x,y) \in E. \]  \hspace{1cm} (2.28)

But \( f_n(u) = 1 \) and so, by connectivity, \( f^*(x) := \lim_{n \to \infty} f_n(x) \) exists for all \( x \in V \). Clearly, \( f^* \in \mathcal{F}(u, v) \) and \( \mathcal{E}(f^*) = C_{\text{eff}}(u, v) \).

To see that the minimizer is unique, suppose that \( f_1^* \) and \( f_2^* \) are minimizers and invoke, once again, the calculation in (2.26) to get

\[
0 \leq \mathcal{E}(f_1^* - f_2^*) \leq \mathcal{E}(f_1^*) + \mathcal{E}(f_2^*) - 2C_{\text{eff}}(u, v) = 0. \]  \hspace{1cm} (2.29)

As \( f_1^*(u) = f_2^*(u) \), we then get that \( f_1^* = f_2^* \).

Finally, let us establish the stated connections to the underlying Markov chain. First, we note that since edges in \( E \) are unoriented with both orientations identified into one,
the connection (2.20) between conductances, the transition kernel $P$ and the reversible measure $\pi$ shows
\[
\mathcal{E}(f) = \sum_{(x,y) \in E} c(x,y)f(x)[f(x) - f(y)] + \sum_{(x,y) \in E} c(x,y)f(y)[f(y) - f(x)]
\]
\[
= \sum_{x \in V} \sum_{y: (x,y) \in E} c(x,y)f(x)[f(x) - f(y)]
\]
\[
= \sum_{x \in V} f(x)\pi(x)[f(x) - Pf(x)] = (f, (1 - P)f)_{\ell^2(\pi)}.
\] (2.30)

Since $\mathcal{E}(f^* + \epsilon \delta_x) \geq \mathcal{E}(f^*)$ for all $\epsilon \in \mathbb{R}$ and all $x \neq u, v$, we readily derive
\[
f^*(x) = Pf^*(x), \quad x \neq u, v.
\] (2.31)

This means that $f^*$ is harmonic on $V \setminus \{u, v\}$ with “boundary” values $f^*(u) = 1$ and $f^*(v) = 0$. The representation (1.5) of the solution to the Dirichlet problem (cf Theorem 1.1) then forces (2.24).

We now move to the formulation using currents. For this we let $\bar{E}$ denote the set of oriented edges with both orientation included for each edge $e \in E$.

**Definition 2.7 (Current)** Let $u, v \in V$ be distinct vertices. A current (or a flow) from $u$ to $v$ is a map $i: \bar{E} \to \mathbb{R}$ such that
\[
i(x, y) = -i(y, x), \quad (x, y) \in \bar{E},
\] (2.32)
and
\[
\sum_{y: (x,y) \in \bar{E}} i(x, y) = 0, \quad x \in V \setminus \{u, v\}.
\] (2.33)

The latter condition is sometimes referred to as Kirchhoff’s node law.

We note that condition (2.32) expresses the natural requirement that the current flowing from $x$ to $y$ along edge $(x, y)$ equals the negative of the current flowing backwards along this edge. The condition (2.33) in turn says that the current is conserved at all vertices except $u$ and $v$. Next we observe:

**Lemma 2.8** Suppose $(V, E)$ is a finite network and let $i$ be a current from $u$ to $v$. Then
\[
\sum_{y: (u,y) \in \bar{E}} i(u, y) = \sum_{y: (x,v) \in \bar{E}} i(x, v)
\] (2.34)

**Proof.** Using (2.32) and then (2.33) shows
\[
0 = \sum_{(x,y) \in \bar{E}} i(x, y) = \sum_{x \in V} \sum_{y: (x,y) \in \bar{E}} i(x, y) = \sum_{y: (u,y) \in \bar{E}} i(u, y) + \sum_{y: (v,y) \in \bar{E}} i(v, y)
\] (2.35)
Invoking (2.32) once again, this is now readily converted into (2.34). 

We note that the conclusion (2.34) fails in general for infinite networks because there one can arrange currents to flow “to/from infinity.” In finite networks, we call the common value in (2.34) the value of current with the notation
\[ \text{val}(i) := \sum_{y: (u,y) \in E} i(u,y). \] (2.36)

We say that “a unit current flows from \( u \) to \( v \)” if \( i \) is a current with \( \text{val}(i) = 1 \). To characterize the “physical” currents we postulate another variational problem:

**Definition 2.9 (Thomson’s principle)**  Consider a finite resistor network \( \mathcal{G} = (V, E) \) and let \( u, v \in V \) be distinct vertices. Then upon imposing that a unit current flows from \( u \) to \( v \), the value of the current across any edge is obtained as the value of a minimizer of
\[ \mathcal{E}(i) := \sum_{e \in E} r(e) i(e)^2 \] (2.37)
over
\[ \mathcal{I}(u,v) := \{i : \text{current from } u \text{ to } v, \text{val}(i) = 1\}. \] (2.38)
Moreover, the value of the minimum,
\[ R_{\text{eff}}(u,v) := \inf \{ \mathcal{E}(i) : i \in \mathcal{I}(u,v) \} \] (2.39)
defines the effective resistance from \( u \) to \( v \).

As an exercise, we ask the reader to prove:

**Lemma 2.10 (Kirchhoff’s cycle law & Ohm’s law)**  Assuming that the network is connected as a graph (remember that \( c(e) \in (0, \infty) \) for each \( e \in E \)), for any distinct \( u, v \in V \) a minimizing current \( i^* \in \mathcal{I}(u,v) \) in (2.39) exists and is unique. Moreover, it obeys the Kirchhoff cycle law,
\[ \forall n \geq 1 \forall x_0, \ldots, x_n = x_0 \in V : \sum_{i=1}^{n} r(x_{i-1}, x_i) i^*(x_{i-1}, x_i) = 0. \] (2.40)

In particular, there exists \( f : V \to \mathbb{R} \) with \( f(v) = 0 \) such that Ohm’s law holds
\[ \forall (x,y) \in \mathcal{E} : \hspace{1em} r(x,y) i^*(x,y) = f(y) - f(x) \] (2.41)
and \( f \) is harmonic on \( V \setminus \{u,v\} \).

The previous lemma touches on the fact that the variational problems for the effective conductance and effective resistance are closely related. This is articulated in:

**Theorem 2.11 (Electrostatic duality)**  Consider a finite and connected resistor network \( \mathcal{G} = (V, E) \). Then for all distinct \( u, v \in V \),
\[ R_{\text{eff}}(u,v) = \frac{1}{C_{\text{eff}}(u,v)}. \] (2.42)

We begin by proving the easier direction:
Proof of “$\geq$” in (2.42). Let $f \in \mathcal{F}(u,v)$ and $i \in \mathcal{I}(u,v)$. Then (2.33) and a straightforward (anti)symmetrization implies
\begin{equation}
1 = \sum_{x \in V} \sum_{(x,y) \in E} f(x)i(x,y) = \sum_{(x,y) \in E} [f(x) - f(y)] i(x,y). 
\end{equation}
Multiplying the expression on the right by the square root of $c(x,y)r(x,y)$ (which does not do anything as this expression equals one) and invoking the Cauchy-Schwarz inequality bounds the right-hand side by $\mathcal{E}(f)\bar{\mathcal{E}}(i)$. We thus get
\begin{equation}
\mathcal{E}(f)\bar{\mathcal{E}}(i) \geq 1, \quad f \in \mathcal{F}(u,v), \quad i \in \mathcal{I}(u,v).
\end{equation}
Optimizing over $f$ and $i$ then shows $C_{\text{eff}}(u,v)R_{\text{eff}}(u,v) \geq 1$. \hfill $\square$

The key is thus to show that the inequality in (2.44) can be saturated. This comes in:

Proof of “$\leq$” in (2.42). Let $f^* \in \mathcal{F}(u,v)$ be the minimizer of $f \mapsto \mathcal{E}(f)$; see Lemma 2.6. For each $(x,y) \in \bar{E}$ define (inspired by Ohm’s law)
\begin{equation}
i(x,y) := c(x,y)\left[f^*(y) - f^*(x)\right].
\end{equation}
Then (2.32) holds and, thanks to the harmonicity of $f^*$ on $V \setminus \{u,v\}$, also (2.33) is satisfied. Hence $i$ is a current from $u$ to $v$. Concerning its value, we re-use the second equality in (2.43) to get
\begin{equation}
\text{val}(i) = \sum_{(x,y) \in E} [f^*(x) - f^*(y)] i(x,y) = \mathcal{E}(f^*) = C_{\text{eff}}(u,v).
\end{equation}
Hence,
\begin{equation}
i^*(e) := \frac{1}{C_{\text{eff}}(u,v)} i(e), \quad e \in \bar{E},
\end{equation}
obeycs $i^* \in \mathcal{I}(u,v)$. But then we get
\begin{equation}
R_{\text{eff}}(u,v) \leq \bar{\mathcal{E}}(i^*) = \frac{1}{C_{\text{eff}}(u,v)^2} \bar{\mathcal{E}}(i) = \frac{1}{C_{\text{eff}}(u,v)^2} \mathcal{E}(f^*) = \frac{1}{C_{\text{eff}}(u,v)}.
\end{equation}
This proves the desired complementary inequality in (2.42). \hfill $\square$

Notice that the proof actually shows that $i^*$ from (2.47) is the minimizing current from (2.39). We thus get the following additional interpretation of the effective conductance and resistance, whose detailed proof we leave to the reader:

**Corollary 2.12** $R_{\text{eff}}(u,v)$ is the value of the current flowing from $u$ to $v$ when a unit voltage difference is imposed between these vertices. Equivalently, $C_{\text{eff}}(u,v)$ is the voltage difference between $u$ and $v$ that induces a unit current through the network.

Note that the electrostatic duality is also important for approximation. Suppose we want to estimate the effective resistance between two vertices. Plugging a suitable trial current into the variational characterization of the effective resistance then gives us an upper bound. Using a suitable trial function in the variational characterization of the...
Section 2.3: Infinite networks.

We have so far restricted our considerations to finite networks. This was dictated by the fact that only there we can unconditionally guarantee the existence and uniqueness of the solution to the Dirichlet problem. For Markov chains (and resulting infinite networks) with infinite state space, we have to restrict considerations to suitably defined finite subnetworks.

Lemma 2.13 (Network reduction) Consider a finite connected resistor network \( G = (V, E) \) and assume that \( \pi, \) defined from the conductances via (2.20), obeys \( \pi(x) \in (0, \infty) \) for each \( x \in V. \) Then for any \( V' \subseteq V \) and any \( g: V' \rightarrow \mathbb{R}, \)

\[
\inf \{ \mathcal{E}(f): f |_{V'} = g \} = \frac{1}{2} \sum_{x,y \in V'} c'(x,y) \left[ g(y) - g(x) \right]^2, \tag{2.49}
\]

where

\[
c'(x,y) := \pi(x) P^x( X_{\tau_{V'}} = y), \quad x,y \in V'. \tag{2.50}
\]

We leave the proof to the homework. The upshot of this lemma is that reduction of the network to a subset of vertices still produces a resistor network, albeit for derived conductances. An example of such a reduction is when \( V' \) has just two points; in this case we readily check

\[
V' = \{u,v\} \quad \Rightarrow \quad c'(u,v) = C_{\text{eff}}(u,v). \tag{2.51}
\]

A simple consequence of the above lemma is:

Lemma 2.14 (Series law) Suppose that a resistor network \( G = (V, E) \) contains a path of \( n+1 \) distinct vertices \( x_0, \ldots, x_n \) such that \( x_i, \) for \( i = 1, \ldots, n-1, \) is incident to no other edges than \( (x_{i-1}, x_i) \) and \( (x_i, x_{i+1}). \) Then, in the reduced network on \( V' := V \setminus \{x_1, \ldots, x_{n-1}\}, \) these edges are replaced by a single edge between \( x_0 \) and \( x_n \) with resistance

\[
r'(e) := \sum_{i=1}^{n} r(x_{i-1}, x_i). \tag{2.52}
\]

All other edges in \( E \) are retained including their original resistances.

We leave the proof to the reader. The operation of “reducing a path of edges to a single edge” is nicely complemented by “reducing a bundle of edges between same vertices to a single edge.” Although this is not cast within the framework that we have discussed above (where there is at most one edge between any pair of vertices), the reader will have no difficulty proving:

Lemma 2.15 (Parallel law) Suppose that a (generalized) resistor network contains \( n \) edges \( e_1, \ldots, e_n \) between vertices \( x \) and \( y \) with \( e_i \) having conductance \( c(e_i). \) Then replacing these by a
Infinite Networks

Fig. 1. The setting for the Series Law (left) and the Parallel law (right).

Single edge \( e \) with conductance
\[
c'(e) := \sum_{i=1}^{n} c(e_i).
\] (2.53)

preserves the value of \( \mathcal{E}(f) \) for every \( f \).

Although the connection of the derived resistances from (2.50) to the effective conductance is rather appealing, the reader should be cautioned that for reductions to networks with more than two vertices this connection is far from direct. For instance, we have:

**Lemma 2.16** Consider a connected resistor network with \( V \) containing at least three distinct vertices, \( u_1, u_2, u_3 \in V \). Let \( V' := \{ u_1, u_2, u_3 \} \). Then for any labeling \( i, j, k \) of \( \{1, 2, 3\} \), the corresponding conductances from (2.50) are given by
\[
c'(u_i, u_j) = \frac{2Q_{ij}}{Q_{ij}Q_{ik} + Q_{ij}Q_{jk} + Q_{ik}Q_{jk}}
\] (2.54)
where
\[
Q_{ij} := R_{\text{eff}}(u_i, u_k) + R_{\text{eff}}(u_j, u_k) - R_{\text{eff}}(u_i, u_j).
\] (2.55)

We again leave the proof to the reader with the hint to invoke the above Parallel and Series Laws.

Equipped with the above network reduction ideas, we can now address the situation of infinite state spaces/networks. We begin by noting that the notion of effective conductance can be extended from pairs of vertices to pairs of disjoint sets. Indeed, if \( A, B \subset V \) are non-empty sets with \( A \cap B = \emptyset \) and \( V \setminus (A \cup B) \) finite, then
\[
C_{\text{eff}}(A, B) := \inf \{ \mathcal{E}(f) : f|_A = 1, f|_B = 0 \}.
\] (2.56)

Obviously, we can then identify \( C_{\text{eff}}(\{u\}, \{v\}) \) with \( C_{\text{eff}}(u, v) \) defined previously.

We can also interpret this extension in the previous sense by considering a network where \( A \) is replaced by a single vertex \( \langle A \rangle \) and \( B \) by a vertex \( \langle B \rangle \) and all edges with both vertices in \( A \) or both vertices in \( B \) discarded; those from a vertex in \( A \) to some \( x \) are retained as an edge from \( \langle A \rangle \) to \( x \), etc. (This may lead to multiple edges between \( \langle A \rangle \) and a single vertex; these are subsequently reduced using the Parallel Law.) The reader will then verify that \( C_{\text{eff}}(A, B) \) then coincides with \( C_{\text{eff}}(\langle A \rangle, \langle B \rangle) \) in the reduced network.
The effective resistance \( R_{\text{eff}}(A, B) \) is then easier to define directly by \( R_{\text{eff}}((\langle A \rangle, \langle B \rangle)) \) in the reduced network.

In this context, we now observe:

**Lemma 2.17 (Resistance to infinity)** Consider a connected resistor network \( \mathcal{G} = (V, E) \), finite or infinite but subject to \( \pi(x) \in (0, \infty) \) for each \( x \in V \). Then for any finite sets \( \Lambda, \Delta \subset V \) and any \( x \in V \),

\[
x \in \Lambda \subset \Delta \quad \Rightarrow \quad R_{\text{eff}}(x, \Lambda^c) \leq R_{\text{eff}}(x, \Delta^c).
\] (2.57)

In particular, the effective resistance from \( x \) to infinity,

\[
R_{\text{eff}}(x, \infty) := \sup \{ R_{\text{eff}}(x, \Lambda^c) : x \in \Lambda \subset V, \text{ finite} \},
\] (2.58)

obeys, for any sequence of finite sets \( \{ \Lambda_n \} \) with \( \Lambda_n \uparrow V \),

\[
\lim_{n \to \infty} R_{\text{eff}}(x, \Lambda_n^c) = R_{\text{eff}}(x, \infty).
\] (2.59)

**Proof.** Thanks to the electrostatic duality, it suffices to prove (2.57) in the form

\[
x \in \Lambda \subset \Delta \quad \Rightarrow \quad C_{\text{eff}}(x, \Lambda^c) \geq C_{\text{eff}}(x, \Delta^c)
\] (2.60)

This follows by noting that every function contributing to the infimum in (2.56) defining \( C_{\text{eff}}(x, \Lambda^c) \) also contributes to the infimum defining \( C_{\text{eff}}(x, \Delta^c) \). The rest of the properties is checked from the definition of supremum. \( \square \)

Although the above is stated for the effective resistance, the electrostatic duality gives us the corresponding statement for the effective conductance from \( x \) to infinity:

\[
C_{\text{eff}}(x, \infty) := \inf \{ C_{\text{eff}}(x, \Lambda^c) : x \in \Lambda \subset V, \text{ finite} \},
\] (2.61)

which then trivially obeys

\[
C_{\text{eff}}(x, \infty) = \frac{1}{R_{\text{eff}}(x, \infty)}.
\] (2.62)

Note, however, that since limits are involved, \( C_{\text{eff}}(x, \infty) \) is no longer strictly positive and \( R_{\text{eff}}(x, \infty) \) finite. We also note that we could in fact use

\[
C_{\text{eff}}(x, \infty) = \inf \{ \mathcal{E}(f) : f(x) = 1, |\text{supp}(f)| < \infty \}
\] (2.63)

as an alternative definition. (We leave a proof of this to an exercise.) The idea of limiting the support of the trial function naturally leads to:

**Definition 2.18 (Effective conductance/resistance in infinite networks)** Consider a connected network \( \mathcal{G} = (V, E) \), finite or countably infinite but subject to \( \pi(x) \in (0, \infty) \) for each \( x \in V \). Then any pair of distinct vertices \( u, v \in V \),

\[
C_{\text{eff}}(u, v) := \inf \{ \mathcal{E}(f) : f(u) = 1, f(v) = 0, |\text{supp}(f)| < \infty \}.
\] (2.64)

Denoting by \( I_0(u, v) \) the set of currents from \( u \) to \( v \) that are supported only on a finite set of edges, we also set

\[
R_{\text{eff}}(u, v) := \inf \{ \tilde{\mathcal{E}}(i) : i \in I_0(u, v) \}
\] (2.65)
We note that the requirement that the current be supported on a finite set of edges ensures that the conclusion of Lemma 2.8 still holds and so the value of the current is well defined. By approximations from finite sets it is then readily checked that the electrostatic duality extends to this setting as well.

2.4 Criteria for recurrence and transience.

The main benefit of the above formalism is the following characterization of recurrence/transience of discrete-time Markov chains:

**Theorem 2.19 (Transience via effective resistivity)** Consider an irreducible Markov chain \( X \) on a countably-infinite state space \( V \) and assume that the chain admits a non-zero reversible measure \( \pi \). Let \( \mathcal{G} = (V, E) \) be the associated infinite resistor network. Then

\[
X \text{ is transient } \iff R_{\text{eff}}(x, \infty) < \infty, \quad \forall x \in S. \tag{2.66}
\]

In particular, a reversible Markov chain is transient if and only if there exists a unit current to infinity of finite total energy.

**Proof.** Pick \( x \in V \) and let \( \{\Lambda_n\} \) be a sequence of finite sets containing \( x \) such that \( \Lambda_n \uparrow S \). Thanks to irreducibility,

\[
X \text{ is transient } \iff \lim_{n \to \infty} P^X(\hat{\tau}_{\Lambda_n} < \hat{\tau}_x) > 0 \tag{2.67}
\]

But the above network reduction ideas along with Lemma 2.6 show

\[
C_{\text{eff}}(x, \Lambda_n^c) = \pi(x) P^X(\hat{\tau}_{\Lambda_n} < \hat{\tau}_x) \tag{2.68}
\]

and so transience of \( X \) is equivalent to \( C_{\text{eff}}(x, \infty) > 0 \). Employing the electrostatic duality (or writing the above in terms of resistances) yields the first part of the claim.

For the second part we note that, if there is a unit current \( i \) from some \( u \) to infinity with \( \tilde{E}(i) < \infty \), then its restriction to \( \Lambda_n \) constitutes a current \( i_n \) from \( u \) to \( \Lambda_n^c \) with \( \tilde{E}(i_n) \leq \tilde{E}(i) \). Hence \( R_{\text{eff}}(u, \infty) \leq \tilde{E}(i) < \infty \), implying transience. On the other hand, if the chain is transient, setting

\[
i(x, y) := c(x, y)[f^*(y) - f^*(x)] \quad \text{for} \quad f^*(x) := P^X(\hat{\tau}_u < \infty) \tag{2.69}
\]

defines a current from infinity with \( \tilde{E}(i) = \mathcal{E}(f) = \pi(u) P^u(\hat{\tau}_u = \infty) < \infty \).

For the above criterion to be useful, we need criteria for finiteness of the effective resistance. There are two kinds of criteria that are generally used: first, those based on Rayleigh Monotonicity Principle to be discussed next, and the other one based on network reduction ideas and representations using paths and cuts.

**Lemma 2.20 (Rayleigh Monotonicity Principle)** Suppose \( X \) and \( Y \) are irreducible reversible Markov chains on the same state-space \( V \) and assume that the conductances \( c_X \) and \( c_Y \) in the resistor networks associated with \( X \) and \( Y \), respectively, obey

\[
c_X(u, v) \leq c_Y(u, v), \quad u, v \in V, \ u \neq v. \tag{2.70}
\]
Then

\[ X \text{ is transient } \Rightarrow Y \text{ is transient} \quad \text{(2.71)} \]

**Proof.** Let \( \mathcal{E}_X(f) \), resp., \( \mathcal{E}_Y(f) \) denote the electrostatic energy of \( f : V \to \mathbb{R} \) in the network associated with \( X \), resp., \( Y \). Then (2.70) implies \( \mathcal{E}_X(f) \leq \mathcal{E}_Y(f) \). It follows that the effective resistance from \( u \) to infinity in the \( X \)-network is bounded by that in the \( Y \)-network. The former is positive if and only if \( X \) is transient and so \( X \) being transient implies that \( Y \) is transient too. \( \square \)

The key point is that, if the state-space of a Markov chain, viewed as a weighted graph, embeds the state-space of another Markov chain with lesser or equal conductances, then the embedded chain is recurrent if the original one is. Thanks to Polya's Theorem (see Theorem 1.4) shows that the random walk on any graph that embeds a copy of \( \mathbb{Z}^3 \) (or a rooted binary tree) is transient. On the other hand, the random walk on any connected subgraph of \( \mathbb{Z}^2 \) is recurrent. (A natural way to generate such subgraphs is to dilute edges by way of a low-density Bernoulli process; this is the subject of percolation theory.)

Having discussed monotonicity criteria, let us now move to discussing criteria based on network reduction ideas. We start by recalling a standard notion from graph theory:

**Definition 2.21 (Cutset)** Given a connected graph \( (V, E) \) and distinct vertices \( u, v \in V \), a set of edges \( E \subset E \) is a cutset separating \( u \) from \( v \), if every edge-simple path connecting \( u \) and \( v \) uses an edge in \( E \). Similarly, in a locally-finite, infinite connected graph, \( E \subset E \) is a cutset separating \( u \) from infinity, if every infinite edge-simple path from \( u \) uses an edge in \( E \).

Using this notion, we then have the following criterion for recurrence:

**Lemma 2.22 (Nash-Williams criterion for recurrence)** Suppose that a connected resistor network \( \mathcal{G} = (V, E) \) contains a family \( \Pi \) of finite (edge-)disjoint cutsets separating \( u \in V \) from \( v \in V \). Then

\[ C_{\text{eff}}(u, v) \leq \left[ \frac{1}{\sum_{E \in \Pi} \sum_{e \in E} c(e)} \right]^{-1} \quad \text{(2.72)} \]

In particular, if the state space of an irreducible Markov chain contains a family of disjoint cutsets of some state \( u \) from infinity and the reciprocal conductances of these cutsets are non-summable, then the Markov chain is recurrent.

If the cutsets are nested in the sense that they can the sense that they can be enumerated to a sequence \( \{E_n\} \) such that \( E_{n+1} \) separates, for each \( n \geq 0 \), the endpoints of every edge in \( E_n \) from infinity, then the proof can be reduced to an application of the Series Law. Indeed, denoting by \( V_n \), for \( n \geq 1 \), the set of vertices that are separated from infinity by \( E_n \) but not by \( E_{n-1} \) (where \( E_0 := \emptyset \)), we can use

\[ f(x) := \sum_{n \geq 1} a_n 1_{ \{ x \in V_n \} } \quad \text{(2.73)} \]
Moreover, in each edge in resistor networks has the same underlying graph structure as on Proof of Theorem 1.4, d unction of the above criterion, we can give a new proof of one half of Polya’s Theorem: of the definition of the effective conductance from a state to infinity. trostatic duality and the definition of where the last step follows by Lemma 2.23. The inequality (2.72) follows from the elec-

Proof of Lemma 2.22. Suppose \( E_n : n \geq 1 \) is a sequence of disjoint cutsets separating \( u \in V \) from \( v \in V \). Let \( i^\ast \) be the current minimizing \( i \mapsto \bar{E}(i) \). Then for any cutset \( E \subset E \) separating \( u \) from \( v \),

\[
\sum_{e \in E} \|i^\ast(e)\| \geq 1.
\]

(2.76)

We leave the proof of this lemma to the reader and proceed to give:

**Lemma 2.23** Consider a finite network \( \Phi = (V, E) \) and let \( i^\ast \in \mathcal{I}(u, v) \) be the current minimizing \( i \mapsto \bar{E}(i) \). Then for any cutset \( E \subset E \) separating \( u \) from \( v \),

\[
\sum_{e \in E} |i^\ast(e)| \geq 1.
\]

By the Series Law, the effective resistance of this network is \( \sum_{n \geq 1} c'(n - 1, n)^{-1} \). Notwithstanding, the claim applies even for non-nested families of cutsets. As we shall see, all we need is the following inequality:

**Lemma 2.23** Consider a finite network \( \Phi = (V, E) \) and let \( i^\ast \in \mathcal{I}(u, v) \) be the current minimizing \( i \mapsto \bar{E}(i) \). Then for any cutset \( E \subset E \) separating \( u \) from \( v \),

\[
\sum_{e \in E} |i^\ast(e)| \geq 1.
\]

(2.76)

We leave the proof of this lemma to the reader and proceed to give:

**Proof of Lemma 2.22.** Suppose \( \{E_n : n \geq 1\} \) is a sequence of disjoint cutsets separating \( u \in V \) from \( v \in V \). Let \( i^\ast \) be the current minimizing \( \mathcal{I}(u, v) \). Since the cutsets \( \{E_n : n \geq 1\} \) are disjoint, the Cauchy-Schwarz inequality and the relation \( c(e)r(e) = 1 \) yield

\[
R_{\text{eff}}(u, v) = \bar{E}(i^\ast) \geq \sum_{n \geq 1} \sum_{e \in E_n} r(e)i^\ast(e)^2 \geq \sum_{n \geq 1} \frac{\sum_{e \in E_n} |i^\ast(e)|}{\sum_{e \in E_n} c(e)^2} \geq \sum_{n \geq 1} \frac{1}{\sum_{e \in E_n} c(e)^2}.
\]

(2.77)

where the last step follows by Lemma 2.23. The inequality (2.72) follows from the electrostatic duality and the definition of \( R_{\text{eff}}(u, v) \). The second part is now a consequence of the definition of the effective conductance from a state to infinity.

We note that the requirement that the cutsets be finite can be dropped. As a consequence of the above criterion, we can give a new proof of one half of Polya’s Theorem:

**Proof of Theorem 1.4, \( d = 1, 2 \).** The simple random walk corresponds to the Markov chain on \( S := \mathbb{Z}^d \) with transition probabilities \( P(x, y) = (2d)^{-1} \) whenever \( x \) and \( y \) are nearest neighbors. As observed before, \( \pi(x) := 2d \) is a reversible measure. The associated resistor networks has the same underlying graph structure as \( \mathbb{Z}^d \) with conductance of each edge in \( \mathbb{Z}^d \) equal to one. Our goal is to show that \( R_{\text{eff}}(0, \infty) = \infty \) in \( d = 1, 2 \).

Let \( \Lambda_n := \{x \in \mathbb{Z}^d : |x|_\infty \leq n\} \) and let \( E_n \) be the set of edges with exactly one endpoint in \( \Lambda_n \). Then \( E_n \) is a cutset separating \( 0 \) from infinity and \( E_n \cap E_m = \emptyset \) for all \( n \neq m \). Moreover,

\[
\sum_{e \in E_n} c(e) = 2d(2n + 1)^{d-1}
\]

(2.78)
Hence, by Lemma 2.22 and electrostatic duality
\begin{equation}
R_{\text{eff}}(0, A_n^c) \geq c' \sum_{k=1}^{n} \frac{1}{k^{d-1}}
\end{equation}
for some constant $c' > 0$ independent of $n$. The recurrence claim now follows from the fact that $\sum_{n \geq 1} \frac{1}{n^{d-1}} = \infty$ in $d = 1, 2$. \hfill \Box

Unlike the proof based on Fourier transform representation of the Green function we gave earlier, this proof is quite robust with respect to perturbations. Note that the Nash-Williams bound is still monotonic in the underlying conductances.

Let us move to a criterion for transience. First we note the following simple fact:

**Lemma 2.24** Consider a finite network $G = (V, E)$ and let $u, v \in V$ be distinct. Then for any collection $\mathcal{P}$ of edge-disjoint paths from $u$ to $v$,
\begin{equation}
R_{\text{eff}}(u, v) \leq \left[ \sum_{P \in \mathcal{P}} \frac{1}{\sum_{e \in P} r(e)} \right]^{-1}
\end{equation}

*Proof.* Since dropping of loops from paths decreases the right-hand side, we may assume that all paths in $\mathcal{P}$ are simple. Let $R_*$ denote the sum on the right-hand side of (2.80) and, for any edge $e \in E$ whose orientation agrees with that along which it is traversed on the way from $u$ to $v$ along the (unique!) path $P \in \mathcal{P}$ containing $e$, let
\begin{equation}
i(e) := \frac{R_*}{\sum_{e' \in P} r(e')}.
\end{equation}
(We set $i(-e) := -i(e)$ for the opposite orientation of $e$ and $i(e) := 0$ for all the edges not belonging to any path in $\mathcal{P}$.) The definition of $R_*$ ensures $i \in \mathcal{I}(u, v)$ and so
\begin{equation}
R_{\text{eff}}(u, v) \leq \tilde{\mathcal{E}}(i) = \sum_{P \in \mathcal{P}} \sum_{e \in P} r(e) i(e) = R_*^2 \sum_{P \in \mathcal{P}} \frac{1}{\sum_{e' \in P} r(e')}.
\end{equation}
The claim follows by invoking the definition of $R_*$ one more time. \hfill \Box

Notice that our proof above essentially boils down to the Parallel Law with each path designating one generalized edge between $u$ and $v$. Unfortunately, the previous lemma is useful only if $u$ and $v$ represent (large) sets in a reduced network; only then we can expect (assuming local finiteness) to contain a significant number of paths from $u$ to $v$. This is often too restrictive in locally finite graphs. Fortunately, one can generalize the above lemma as follows:

**Lemma 2.25** (Terry Lyons’ random path method) Consider a finite network $G = (V, E)$ with distinct vertices $u, v \in V$. Then for any probability law on random paths $P$ from $u$ to $v$, setting
\begin{equation}
i(e) := E(1_{\{e \in P\}} - 1_{\{-e \in P\}}), \quad e \in \tilde{E},
\end{equation}
defines a unit current \( i \in \mathcal{I}(u, v) \). In particular, \( R_{\text{eff}}(u, v) \leq \mathcal{E}(i) \). Moreover, the unit current arising via the random-path law defined by the very paths of the Markov chain,

\[
i^*(x, y) = \mathbb{E}^u \left( \sum_{n=1}^{\tau_{n-1}} 1_{\{X_{n-1} = x, X_n = y\}} - \sum_{n=1}^{\tau_{n-1}} 1_{\{X_{n-1} = y, X_n = x\}} \right), \quad (x, y) \in \mathcal{E},
\]

minimizes \( i \mapsto \mathcal{E}(i) \). In particular, \( \mathcal{E}(i^*) = R_{\text{eff}}(u, v) \).

We remark that the set of paths from \( u \) to \( v \) is a countable set. So a probability law on such paths is simply an assignment to a non-negative number to each path (with these numbers adding up to one when summed over all paths).

**Proof.** For each path \( P \), setting

\[
i_P(e) := 1_{\{e \in P\}} - 1_{\{-e \in P\}}, \quad e \in \mathcal{E},
\]

defines a unit current from \( u \) to \( v \). Hence, so does the expectation over \( P \).

Next, given an edge \( (x, y) \in \mathcal{E} \), consider the expectation

\[
j(x, y) := \mathbb{E}^u \left( \sum_{n=1}^{\tau_{n-1}} 1_{\{X_{n-1} = x, X_n = y\}} \right).
\]

Using the Markov property and the connection between the transition probability and the conductances, we can write this as

\[
j(x, y) = \sum_{n \geq 0} P^u(X_n = x, \tau_v > n)P(x, y)
\]

\[
= c(x, y) \sum_{n \geq 0} \pi(x)^{-1}P^u(X_n = x, \tau_v > n)
\]

\[
= \pi(u)^{-1}c(x, y) \sum_{n \geq 0} P^x(X_n = u, \tau_v > n)
\]

\[
= \pi(u)^{-1}c(x, y)P^x(\tau_u < \tau_v)\mathbb{E}^u \left( \sum_{n=0}^{\tau_{n-1}} 1_{\{X_n = y\}} \right),
\]

where we used reversibility to get the third equality and the strong Markov property to get the last equality. Recall our notation \( f^*(x) := P^x(\tau_u < \tau_v) \) for the minimizer of \( f \mapsto \mathcal{E}(f) \) over \( f \in \mathcal{F}(u, v) \). Since (2.25) gives us

\[
\pi(u)^{-1}\mathbb{E}^u \left( \sum_{n=0}^{\tau_{n-1}} 1_{\{X_n = u\}} \right) = \frac{1}{\pi(u)P^u(\tau_u < \tau_v)} = \frac{1}{C_{\text{eff}}(u, v)},
\]

we thus get

\[
j(x, y) = \frac{1}{C_{\text{eff}}(u, v)}c(x, y)f^*(x).
\]

In particular, \( j(x, y) < \infty \). Comparing this with (2.45) and (2.47) then shows that \( e \mapsto j(e) - j(-e) \) is the minimizing current as claimed.

We will now use the random-path method to give the proof of the second half of Polya’s Theorem:
Proof of Theorem 1.4, \(d \geq 3\). Let \(\Lambda_n := \{x \in \mathbb{Z}^d: |x|_\infty \leq n\}\) and consider the problem of estimating \(R_{\text{eff}}(0, \Lambda_n^c)\) via random paths defined as follows. Let \(\theta\) be a uniform sample from the unit sphere in \(\mathbb{R}^d\). Resolving ties using a predefined order, let \(P\) be the path from zero to infinity that is closest to the straight line with tangent vector \(\theta\). Now define a unit current \(i = i_{\text{eff}}(0, \cdot)\) via (2.83). Denoting by \(E_k\) the set of edges with at least one endpoint in \(\Lambda_n\) but no end-point in \(\Lambda_{n-1}\), it is now easy to check that, for some \(c > 0\) and all \(k = 1, \ldots, n\),

\[
e \in E_{k-1} \implies |i(e)| \leq \frac{c}{k^{d-1}}
\]

because for \(i_P(e)\) to be non-zero we need \(\theta\) lie in a cone with aperture of order \(1/k^{d-1}\). Since \(|E_k|\) is at most a constant times \(k^{d-1}\) and \(r(e) = 1\), we thus get

\[
R_{\text{eff}}(0, \Lambda_n^c) \leq \tilde{\mathcal{E}}(i) = \sum_{k=0}^{n-1} \sum_{e \in E_k} r(e) i(e)^2 \leq c' \sum_{k=1}^{n} \frac{1}{k^{d-1}}. \tag{2.91}
\]

It follows that

\[
R_{\text{eff}}(0, \infty) \leq c' \sum_{k=1}^{\infty} \frac{1}{k^{d-1}}. \tag{2.92}
\]

This is finite in all \(d \geq 3\), implying transience via Theorem 2.19.

We remark that the inequalities (2.72) and (2.80) are generally not sharp due to the requirement that the collections of cuts/paths be edge disjoint. This requirement may be relaxed by splitting, with the help of the Parallel/Series Law, each edge into a number of edges whose resistances/conductances are then subsequently optimized as well. This leads to variational characterizations of effective resistance/conductance (including existence and uniqueness of minimizers) using geometric objects only; namely, collections of paths/cuts and conductances/resistances thereupon. Details can be found in Chapter 14 of my PIMS notes (arXiv:1712.09972).

## 2.5 Green function and hitting/commute time identities.

The notion of the Green function has been very important in our earlier study of transient Markov chains. Notwithstanding, the Green function can be defined for recurrent chains, or even finite-state chains by (effectively) killing the chain once it hits a given set:

**Definition 2.26 (Green function in a set)** Given a non-empty set \(A \subset S\),

\[
G^A(x, y) := E_x \left( \sum_{n=0}^{\tau_{Ac}^{-1}} 1_{\{X_n=y\}} \right). \tag{2.93}
\]

defines the Green function in \(A\) from \(x\) to \(y\).

This object is quite important in many parts of probability. As it turns out, we have been talking about this quantity all along:
Lemma 2.27 (Effective resistance and diagonal Green function) Consider a finite network $\mathcal{G} = (V, E)$ and let $u \in A \subseteq V$. Then

$$G^A(u, u) = \pi(u) R_{\text{eff}}(u, A)$$  \hspace{1cm} (2.94)

Proof. Using the network reduction ideas, it suffices to prove this for $A^c := \{v\}$ for some $v \neq u$. The claim then follows by inspection from (2.88). \qed

Using these we get:

Lemma 2.28 (Green function for simple random walk) Consider the simple symmetric random walk on $\mathbb{Z}^d$ and denote, as before, $\Lambda_n := \{x \in \mathbb{Z}^d : |x|_\infty \leq n\}$. Then, as $n \to \infty$,

$$G^{\Lambda_n}(0, 0) \asymp \begin{cases} n, & \text{if } d = 1, \\ \log(n), & \text{if } d = 2, \\ 1, & \text{if } d \geq 3, \end{cases} \hspace{1cm} (2.95)$$

where “$\asymp$” means that the ratio of the left and right-hand sides are bounded by positive and finite quantities in the stated limit.

Proof. By Lemma 2.27, it suffices to prove the stated asymptotics for $R_{\text{eff}}(0, \Lambda^c_n)$. These follow from (2.79) and (2.91). \qed

We remark that the expressions in (2.95) are actually asymptotic, meaning that the ratios of the two sides of the expression converge to a definite limit as $n \to \infty$. The derivation of these is not hard, but we will not delve into that as that would take us on a path that we have no time to follow.

For the sake of later derivations, we also note the following property of the Green function:

Lemma 2.29 Consider an irreducible, reversible and transient Markov chain on finite or infinite set $S$ and let $A \subseteq S$ be non-empty and such that $G^A(x, y) < \infty$ for all $x, y \in A$. Then

$$\left\{ \frac{G^A(x, y)}{\pi(y)} \right\}_{x, y \in A}$$

regarded as a matrix, is symmetric and positive semi-definite.

We leave a proof of this lemma to homework. As a consequence,

$$C^A(x, y) := \frac{G^A(x, y)}{\pi(y)}$$ \hspace{1cm} (2.97)

is a covariance of a multivariate Gaussian random variable (or a random field) indexed by $A$ called the (discrete) Gaussian Free Field. Note that the diagonal value of the covariance, $C^A(x, x)$, whose overall growth rate for the simple random walk was determined in Lemma 2.28, is then the variance of the field at $x$.

We have so far used electrostatic quantities to control only the recurrence/transience of the underlying Markov chain. However, the framework gives us access to other important quantities as well. This is the subject of:
Lemma 2.30 (Hitting time identity) Consider an irreducible finite-state Markov chain with reversible measure \( \pi \) and recall that \( \tau_v := \inf \{ n \geq 0 : X_n = v \} \). Then for any distinct \( u, v \in S \),
\[
E^u(\tau_v) = R_{\text{eff}}(u, v) \sum_{x \in S} \pi(x) f^*(x),
\]
where \( f^*(x) := P^x(\tau_u < \tau_v) \).

Proof. Using the above notion of the Green function in a set and its reversibility with respect to \( \pi \), we compute
\[
E^u(\tau_v) = \sum_{x \in S} G^{(v)^c}(u, x)
\]
\[
= \frac{1}{\pi(u)} \sum_{x \in S} \pi(x) G^{(v)^c}(x, u)
\]
\[
= \frac{G^{(v)^c}(u, u)}{\pi(u)} \sum_{x \in S} \pi(x) \frac{G^{(v)^c}(x, u)}{G^{(v)^c}(u, u)}
\]
Lemma 2.27 now equates the prefactor with \( R_{\text{eff}}(u, v) \) and the fact that \( x \mapsto G^{(v)^c}(x, u) \) is harmonic on \( \{ u, v \}^c \) and vanishing at \( v \) shows (thanks to the uniqueness of the solution to the Dirichlet problem) that the ratio of the Green function equals \( f^*(x) \). \( \square \)

A consequence we get:

Corollary 2.31 (Hitting time estimate) For the above setting,
\[
E^u(\tau_v) \leq R_{\text{eff}}(u, v) \pi(S).
\]

Proof. This follows directly from \( f^*(x) \leq 1 \). \( \square \)

A more elegant, albeit somewhat less useful, consequence is:

Corollary 2.32 (Commute time identity) For the above setting,
\[
E^u(\tau_u) + E^v(\tau_v) = R_{\text{eff}}(u, v) \pi(S).
\]

Proof. This follows from (2.98) and the fact that \( P^x(\tau_u < \tau_v) + P^x(\tau_v < \tau_u) = 1 \) thanks to irreducibility and the finite state-space property of the Markov chain. \( \square \)

2.6 Capacity.

As our last item of business, let us briefly discuss the concept of capacity. This concept is ubiquitous in harmonic analysis (apparently introduced by G. Choquet himself) although its origins are in electrostatics. We caution the reader that, although we will use the same framework and notation, the actual physics problem is different. Let us actually start by describing the physics context first.

One of the basic components of electric circuits are capacitors. In engineering-lab conditions, these usually consist of two metal (and thus nearly ideally conductive) plates
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separated by a dielectric material of essentially zero resistivity. The role of the capacitor in the circuit is to use the difference in voltage on the metal plates to store charge (and thus modulate the current which is particularly useful when the voltages/currents evolve with time). The capacitor will be in a steady state if the charge is just enough to compensate for the voltage difference, and thus no current is required. Of course, the charge cannot be arbitrarily large; otherwise, a spark passes (and burns) through the dielectric between the plates while rendering the capacitor useless.

A standard formulation of this problem in harmonic analysis of Markov chains is as follows. Consider a reversible Markov chain on an infinite state-space and let \( A \subset S \) be a finite set. We may think of \( c(x,y) \) as the capacitance of edge \((x,y)\). We again define the physical configuration by way of a variational problem:

**Definition 2.33 (Newtonian capacity)** Let \( \mathcal{E}(f) \) be the energy defined from \( f \) via the capacitances \( c(e) \). Let \( A \subset S \) be non-empty and finite. Then

\[
\text{cap}(A) := \inf \{ \mathcal{E}(f) : f \in L^2(\pi), f\big|_A = 1 \}. \tag{2.102}
\]

is the capacity of \( A \).

To make a connection to the aforementioned “stored charge,” we give:

**Lemma 2.34 (Capacity and charge distribution)** For each non-empty finite \( A \subset S \),

\[
f^*(x) := P^x(\tau_A < \infty) \tag{2.103}
\]

is the unique minimizer in (2.102). In particular, \( \text{cap}(A) = C_{\text{eff}}(A, \infty) > 0 \) if and only if the chain is transient. Moreover, for the transient case, \( f^* \) is a potential for the charge distribution \( q(x) := (1 - P)f^*(x) \) and

\[
\text{cap}(A) = \sum_{x \in A} \pi(x)q(x). \tag{2.104}
\]

**Proof.** Let \( f : S \to \mathbb{R} \) and let \( f \lor 0 \land 1 \) be its truncation to interval \([0,1]\). As is readily checked, \( \mathcal{E}(f \lor 0 \land 1) \leq \mathcal{E}(f) \), with the inequality strict if the truncation kicks in at least one place. Hence, it suffices to look for minimizers taking values in \([0,1]\). The argument in (2.26–2.28) and (2.30–2.31) now shows that a minimizer exists, is in \( L^2(\pi) \) (by Fatou’s lemma) and is harmonic in \( S \setminus A \). Since \( f^* \) from (2.103) is harmonic in \( A^c \) and equal to one on \( A \), the minimizer takes the form

\[
f(x) = f^*(x) + h(x) \tag{2.105}
\]

where \( h \) is a bounded function that is harmonic on \( A^c \) and vanishes on \( A \). But the representation (2.30) then gives \( \mathcal{E}(f) = \mathcal{E}(f^*) + \mathcal{E}(h) \) and so \( h = 0 \). We then get a formal identification of \( \text{cap}(A) \) with \( C_{\text{eff}}(A, \infty) \), although physically these represent different problems.

To get the second part of the claim we assume that the chain is transient. Then (2.103) is a potential by the observations in the previous chapter and so we get

\[
f^*(x) = \sum_{y \in A} G(x,y)q(y). \tag{2.106}
\]
But then (2.30) again gives
\[ \text{cap}(A) = \mathcal{E}(f^*) = (f^*, (1 - P)f^*)_{\rho(\pi)} = (f^*, q)_{\rho(\pi)}. \] (2.107)

The formula (2.104) now follows by noting that \( q \) is supported on \( A \) and \( f^*|_A = 1 \).

In our description of a physical capacitor we noted that, in a steady state, the charge on the capacitor is just enough to compensate for the imposed electric field. Somehow, this should mean that the physical charge should be distributed so that the electrostatic interaction of the charge with itself (note that a charge \( q(y) \) at \( y \) results in the electrostatic potential \( G(x, y)q(y) \) at \( x \), and so the energy of placing charge \( q(x) \) at \( x \) is \( q(x)q(y)G(x, y) \)) is minimal. This intuition is correct is the subject of:

**Lemma 2.35** Let \( \mathcal{M}_1(A) \) denote the class of probability measures on \( A \). For a transient irreducible and reversible Markov chain on \( S \) and any finite non-empty \( A \subset S \), we then have
\[
\frac{1}{\text{cap}(A)} = \inf \left\{ \sum_{x,y \in A} \frac{G(x, y)}{\pi(y)} \mu(x)\mu(y) : \mu \in \mathcal{M}_1(A) \right\}.
\] (2.108)

Moreover, a minimizer exists and is given by \( \mu(x) := \pi(x)q(x) \) where \( q \) is as in Lemma 2.34.

**Proof.** The properties stated in Lemma 2.29 imply that the functional in (2.108) is positive definite and strictly convex. As \( \mathcal{M}_1(A) \) is convex, the arguments used before show that a minimizer \( \mu^* \) exists and is non-vanishing everywhere on \( A \). Examining perturbations shows that
\[
f(x) := \sum_{y \in A} \frac{G(x, y)}{\pi(y)} \mu^*(y)
\] (2.109)
must be constant on \( A \), say \( f(x) = \lambda \), where \( \lambda \) is the Lagrange multiplier arising from the constraint that \( \mu^* \) is a probability measure. Then \( \lambda = \text{cap}(A) \) and so
\[
f'(x) := \frac{1}{\text{cap}(A)} f(x)
\] (2.110)
obeys \( f'|_A = 1 \). Moreover, since \( x \mapsto G(x, y) \) is in \( \ell^2(\pi) \) for each \( y \), then so is \( f \). Hence,
\[
\text{cap}(A) \leq \mathcal{E}(f') = \text{cap}(A)^2 \mathcal{E}(f)
= \text{cap}(A)^2 (f, (1 - P)f)_{\rho(\pi)}
= \text{cap}(A)^2 \sum_{x,y \in A} \frac{G(x, y)}{\pi(y)} \mu^*(x)\mu^*(y).
\] (2.111)
where we used that \( (1 - P)f(x) = \mu^*(x)/\pi(x) \). This proves \( \leq \) in (2.108), but in fact yields the full equality thanks to the fact that of the minimizer in (2.102) (as proved in Lemma 2.34) is uniquely determined by its values on \( A \) and the fact that is harmonic outside of \( A \).

We remark that the above representation of capacity generalizes this concept beyond the narrow context of electrostatic theory. Indeed, for any positive-definite kernel \( K: A \times
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$A \rightarrow \mathbb{R}$ we may set

$$\frac{1}{\text{cap}_K(A)} := \inf \left\{ \sum_{x,y \in A} K(x,y)\mu(x)\mu(y) : \mu \in \mathcal{M}_1(A) \right\}. \quad (2.112)$$

This is then the $K$-capacity of $A$ (while referring to the object in (2.108) using the adjective “Newtonian”). These generalized notions of capacity are quite useful in harmonic analysis, as well as analysis of fractal sets (e.g., in the so called Frostman lemma).