HW#1: due Fri 5/2/2018

The following exercises on Martin boundary assume an irreducible transient Markov chain

**Problem 1:** Consider the simple random walk on a homogeneous tree of degree $d \geq 3$ (every vertex has the same degree). Pick a vertex and call it a root. Define the Martin boundary using the root as the “origin.” Do as follows:

1. Prove that each point on the Martin boundary $\partial S$ is equivalent to the sequence of points on the unique self-avoiding path from the root to (necessarily) infinity.
2. Describe explicitly the limit Martin kernel $K(\cdot, a)$ for each $a \in \partial S$ and prove that $\partial_MS = \partial S$.
3. Show that $x \mapsto K(x, a)$ diverges as $x \to a$ so all boundary kernels are unbounded.

**Problem 2:** For $h > 0$ harmonic on $S$ and $G \subset S$, recall that $h_G(x) := P^x(h(X_t)1_{\{T_G < \infty\}})$. Do as follows:

1. Prove that, if $G$ finite then $h_G$ is a potential.
2. Give an example of an infinite $G \subset S$ (and bounded $h$) such that $h_G$ is not a potential.

**Problem 3:** Let $h > 0$ be harmonic and let

$$h_A(x) := \inf_{G \text{ open}} h_G(x) \quad \text{where} \quad h_G(x) := E^x(h(X_t)1_{\{T_G < \infty\}}).$$

(1) Prove that the restriction of $A \mapsto h_A(x)$ to the Borel sets in $\partial S$ is a measure with

$$h_A(x) = h(x)P^x_0(X_{\infty} \in A).$$

**Hint:** Prove that $A \mapsto h_A(x)$ is a metric outer measure.

**Problem 4:** Consider a random walk on $\mathbb{Z}^2$ with transition kernel $P(0, x)$ that obeys:

1. $P(0, x) > 0$ if and only if $x$ is a neighbor of zero,
2. $\sum_x xP(0, x)$ is a non-zero vector pointing in the negative $x$-direction.

Describe the set $\{ \theta \in \mathbb{R}^2 : \sum_x P(0, x)e^{\theta \cdot x} = 1 \}$, which by the Choquet-Deny theorem parametrizes the minimal Martin boundary, as a curve in $\mathbb{R}^2$.

Next, consider a resistor network on a finite unoriented graph $(\mathcal{X}, E(\mathcal{X}))$ of finite degree (multiple edges are allowed). Assume the conductance $c(e)$ of each edge $e \in E(\mathcal{X})$ is positive. Given $\emptyset \neq A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$, recall that

$$\mathcal{F}(A, B) := \{ f : \mathcal{X} \to \mathbb{R} : f|_A = 0, f|_B = 1 \} \quad \text{and} \quad \mathcal{E}(f) := \frac{1}{2} \sum_{e=xy} c(e)[f(y) - f(x)]^2$$

For each $e \in E(\mathcal{X})$, denote $r(e) := 1/c(e)$ and recall that $\tilde{\mathcal{E}}(i) := \frac{1}{2} \sum_{e \in E(\mathcal{X})} r(e)i(e)^2$ and that $\mathcal{F}(A, B)$ denotes the set of unit flows from $A$ to $B$.

**Problem 5:** Recall that $C_{\text{eff}}(A, B) := \inf_{f \in \mathcal{F}(A, B)} \mathcal{E}(f)$ and $R_{\text{eff}}(A, B) := \inf_{i \in \mathcal{F}(A, B)} \tilde{\mathcal{E}}(i)$.

1. Prove that there is a unique $i^* \in \mathcal{F}(A, B)$ such that $R_{\text{eff}}(A, B) = \tilde{\mathcal{E}}(i^*)$. 
(2) Show that \( j(e) := \frac{1}{R_{\text{eff}}(A,B)} r(e) t(e) \) satisfies Kirchhoff’s cycle law: For each \( n \geq 1 \) and each \( x_1, \ldots, x_{n+1} \in (A \cup B)^c \) with \( x_{n+1} := x_1 \) and \( (x_i, x_{i+1}) \in E(\mathcal{X}) \) for each \( i = 1, \ldots, n \), we have
\[
\sum_{i=1}^{n} j((x_i, x_{i+1})) = 0
\]

(3) Conclude that there is \( f^* : \mathcal{X} \to \mathbb{R} \) such that
\[
j((x, y)) = f^*(y) - f^*(x), \quad (x, y) \in E(\mathcal{X}) \setminus (A \times A \cup B \times B)
\]
This means that the minimizer of the Thomson problem — i.e., the one leading to the definition of \( R_{\text{eff}}(A,B) \) — obeys Ohm’s Law for some choice of the potentials.

(4) Show that \( f^* \in \mathcal{F}(A,B) \) and use this to give an independent proof of the strong duality, \( C_{\text{eff}}(A,B) = R_{\text{eff}}(A,B)^{-1} \).

**Problem 6:** Let \( \mathcal{X}' \subset \mathcal{X} \) have at least two elements and consider the weights
\[
c'(x, y) := \pi(x) P^x(X_{T_{\mathcal{X}'}} = y), \quad x, y \in \mathcal{X}'
\]
where \( \{X_n : n \geq 0\} \) is the Markov chain associated with the conductance network \( \mathcal{X} \), \( T_A := \inf\{n \geq 1 : X_n \in A\} \) and \( \pi(x) := \sum_{(x,y) \in E(\mathcal{X})} c(x,y) \). Do as follows:

(1) Show that \( c'(x, y) = c'(y, x) \) for each \( x, y \in \mathcal{X}' \) and the weights thus serve as conductances on \( \mathcal{X}' \).

(2) Denoting \( \mathcal{E}'(f) := \frac{1}{2} \sum_{x,y \in \mathcal{X}'} c'(x, y) [f(y) - f(x)]^2 \) for each \( f : \mathcal{X}' \to \mathbb{R} \), show that
\[
\mathcal{E}'(f) = \inf \{ \mathcal{E}(g) : g|_{\mathcal{X}'} = f \}. \tag{3}
\]
This is an example of network reduction.

**Problem 7:** Use ideas from the previous exercise to state and prove the Parallel Law and the Series Law.

**Problem 8:** Consider the resistor network on \( \mathbb{Z}^d \) where an edge of unit conductance is placed between each pair of nearest neighbor vertices. For any pair of vertices \( x, y \in \mathbb{Z}^d \), compute \( C_{\text{eff}}(x, y) \) explicitly (e.g., as an integral). Hint: Use Fourier transform to solve the associated Dirichlet problem. (Dirichlet boundary conditions at infinity need to be imposed or one has to work in finite volume — a torus — and take a limit.)

**Problem 9:** (Optional) Find a direct proof that \( C_{\text{eff}}(x, y) = 1/2 \) when \( d = 2 \) and \( x, y \) are nearest neighbors.