HW#4: due Fri, 11/3/2017

The numbering refers to online version of the textbook

Problem 1: Exercise 1.6.9, Page 30

Problem 2: $N$ married couples go out dancing. At some point, the men get re-assigned randomly to the women (meaning, technically, that the men are assigned to the women using a uniformly-chosen permutation of $N$ elements). Compute the $N \to \infty$ limit of the probability that no husband is assigned to his wife.

Problem 3: Exercise 1.6.10, Page 31

Problem 4: [Key lemma for Jensen’s inequality] Let $\phi: \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function and denote

$$L := \{(a, b) \in \mathbb{R}^2: \phi(x) \geq ax + b \ \forall x \in \mathbb{R}\}.$$ 

Prove that $L \neq \emptyset$ and show that

$$\phi(x) = \sup_{(a, b) \in L} (ax + b), \ x \in \mathbb{R}$$

Problem 5: For every $n \geq 1$, prove that $B(\mathbb{R}^n) = \bigotimes_{i=1}^n B(\mathbb{R})$. In particular, conclude that if $X_1, \ldots, X_n$ are random variables and $f: \mathbb{R}^n \to \mathbb{R}$ a Borel measurable function, then $Y := f(X_1, \ldots, X_n)$ obeys

$$\sigma(Y) \subseteq \sigma(X_1, \ldots, X_n)$$

where $\sigma(X_1, \ldots, X_n) := \sigma\left(\{\{X_i \in B\}: i = 1, \ldots, n, B \in B(\mathbb{R})\}\right)$.

Problem 6: [Uncorrelated vs independent] Let $X$ and $Y$ be random variables on the same sample space. Do as follows:

(1) Prove that if $X$ and $Y$ are independent and (for simplicity) bounded, then they are uncorrelated in the sense that $E(XY) = (EX)(EY)$.

(2) Yet (prove by way of an example that) there are uncorrelated random variables that are not independent.

(3) Still, prove that $f(X)$ and $g(Y)$ are uncorrelated for every pair of bounded Borel-measurable functions $f, g: \mathbb{R} \to \mathbb{R}$ if, and only if, $X$ and $Y$ are independent.

Problem 7: Exercise 2.1.11, Page 46

Problem 8: Exercise 2.1.13, Page 47

Problem 9: Exercise 2.1.14, Page 47

Problem 10: Let $X$ be a non-negative random variable. Show that

$$EX = \int_0^\infty P(X > t)dt$$

where the second integral is either in (generalized) Riemann or Lebesgue sense.