Problem 1: Let \( d \geq 1 \) be an integer. Prove that
\[
S := \left\{ \prod_{i=1}^{d} (a_i, b_i) : -\infty \leq a_i \leq b_i \leq \infty \right\}
\]
with the convention \((a, a] := \emptyset\) and \((a, \infty] := \{x \in \mathbb{R} : x > a\}\), is a semialgebra on \(\mathbb{R}^d\) with \(\sigma(S) = \mathcal{B}(\mathbb{R}^d)\) — with the latter being the Borel sets in \(\mathbb{R}^d\).

Problem 2: Given two measurable spaces \((\Omega_1, \mathcal{F}_1)\) and \((\Omega_2, \mathcal{F}_2)\) and a map \(f : \Omega_1 \to \Omega_2\), prove the following:
1. \(\{f^{-1}(A) : A \in \mathcal{F}_2\}\) is a \(\sigma\)-algebra over \(\Omega_1\),
2. \(\{A \subseteq \Omega_2 : f^{-1}(A) \in \mathcal{F}_1\}\) is a \(\sigma\)-algebra over \(\Omega_2\).

Use (2) to prove that if \(\mathcal{A}_2\) generates \(\mathcal{F}_2\) in the sense that \(\mathcal{F}_2 = \sigma(\mathcal{A}_2)\) and \(f^{-1}(A) \in \mathcal{F}_1\) for each \(A \in \mathcal{A}_2\), then \(f\) is \(\mathcal{F}_1 / \mathcal{F}_2\)-measurable. Use also (1) to solve the next problem.

Problem 3: EXERCISE 1.3.1, PAGE 14

Problem 4: EXERCISE 1.3.5, PAGE 15

Problem 5: EXERCISE 1.3.6, PAGE 15

Problem 6: EXERCISE 1.3.7, PAGE 15

Problem 7: EXERCISE 1.3.8/1.3.9, PAGE 15

Problem 8: An outer measure \(\mu^*\) on \(\Omega\) is called regular if for each \(A \subseteq \Omega\) there is \(B \in \Sigma(\mu^*)\) — with \(\Sigma(\mu^*)\) denoting the class of \(\mu^*\)-measurable sets in the sense of Carathéodory — such that \(A \subseteq B\) and \(\mu^*(A) = \mu^*(B)\). Assuming \(\mu^*(\Omega) < \infty\), prove
\[
A \in \Sigma(\mu^*) \iff \mu^*(\Omega) = \mu^*(A) + \mu^*(\Omega \setminus A)
\]
In other words, for finite regular outer measures, Carathéodory measurability is equivalent to equality of the inner and the outer measure.

Problem 9: EXERCISE 1.4.1, PAGE 20

Problem 10: EXERCISE 1.4.3, PAGE 20

Problem 11: EXERCISE 1.4.4, PAGE 20