6. Abstract Measure Theory

Having developed the Lebesgue integral with respect to the general measures, we now have a general concept with few specific examples to actually test it on. (Indeed, so far we have constructed only one measure; namely, the Lebesgue measure on \( \mathbb{R}^{d} \).) In this chapter we will mend this by providing tools how to construct measures on abstract measurable spaces. As for the Lebesgue measure, the key tool will be outer measures.

6.1 From semialgebras to outer measures.

Consider an arbitrary set \( X \). Our goal is to develop structures needed for putting a measure on \( X \).

For this, let us recall how the construction of the Lebesgue measure went: We began by defining the measure of half-open boxes. Then we extended this to elementary sets, which were finite unions of half-open boxes. Next, we constructed the Lebesgue outer measure by coverings by countable collections of half-open boxes. Finally, we used topology to define measurable sets and checked that the outer measure restricted to these is actually a measure.

We will proceed along this plan even for a general set \( X \) except, in the absence of topology, we will define measurability differently. As is easy to check, the class of half-open boxes in \( \mathbb{R}^{d} \) forms the following structure:

**Definition 6.1** A class \( \mathcal{S} \subset 2^{X} \) is a semialgebra on \( X \) if it satisfies the following requirements:

1. \( \emptyset, X \in \mathcal{S} \),
2. \( A, B \in \mathcal{S} \) implies \( A \cap B \in \mathcal{S} \),
3. if \( A \in \mathcal{S} \), then there are disjoint \( B_{1}, \ldots, B_{n} \in \mathcal{S} \) such that

\[
A^{c} = \bigcup_{i=1}^{n} B_{i}. \quad (6.1)
\]

Recall that an algebra is a subset of \( 2^{X} \) that contains \( \emptyset \) and \( X \) and is closed under finite unions and complements. Moving up to our generalization of elementary sets, we observe:

**Lemma 6.2** If \( \mathcal{S} \) is a semialgebra, then

\[
\mathcal{A} := \left\{ \bigcup_{i=1}^{n} A_{i} : \{A_{i}\} \subset \mathcal{S} \right\} \quad (6.2)
\]

is an algebra. The same object is obtained when the union in required to be disjoint.

**Proof.** Let \( A_{1}, \ldots, A_{n} \subset \mathcal{S} \). By the property (3) of the semialgebra, for each \( i = 1, \ldots, n \) there are \( \{B_{i,j} : j = 1, \ldots, m\} \subset \mathcal{S} \) such that

\[
A_{i}^{c} = \bigcup_{j=1}^{m} B_{i,j}. \quad (6.3)
\]

(We can always add empty sets to make the union the same size for all \( i \).) From de Morgan’s laws and the distributive property of union around intersection we then get

\[
\left( \bigcup_{i=1}^{n} A_{i} \right)^{c} = \bigcap_{i=1}^{n} A_{i}^{c} = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m} B_{i,j} = \bigcup_{j_{1}=1}^{m} \cdots \bigcup_{j_{n}=1}^{m} \left( \bigcap_{i=1}^{n} B_{i,j_{i}} \right). \quad (6.4)
\]
But \( \bigcap_{i=1}^{n} B_{i,j} \in \mathcal{S} \) by property (2) of the semialgebra and so \( \bigcup_{i=1}^{n} A_i \in \mathcal{A} \). It follows that \( \mathcal{A} \) is closed under complements. As \( \mathcal{A} \) is obviously closed under finite unions and contains \( \emptyset \) and \( X \) (because it includes \( \mathcal{S} \) which contains these) it is an algebra.

To see that we can define the algebra using disjoint unions, let \( A = \bigcup_{i=1}^{n} A_i \) with \( \{ A_i \} \subset \mathcal{S} \) disjoint and let \( B \in \mathcal{S} \). By property (3) of the semialgebra, there are disjoint sets \( \{ B_j \} \subset \mathcal{S} \) such that \( B^c = \bigcup_{j=1}^{m} B_j \). Then

\[
B \cup A = B \cup (A \cap B^c) = B \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap B_j) = B \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap B_j).
\]

(6.5)

Now \( \{ A_i \cap B_j \} \) are disjoint and they are (being subsets of \( B^c \)) disjoint from \( B \). Hence, if any union of \( n \) sets from \( \mathcal{S} \) can be disjointified, also any union of \( n + 1 \) sets from \( \mathcal{S} \) can be disjointified. By induction, \( \mathcal{A} \) can be defined by disjoint unions as well. \( \square \)

Now consider a set function \( \mu : \mathcal{A} \to \mathbb{R} \cup \{ \pm \infty \} \), which we permit, for the time being, to take both positive and negative (and even infinite) values. Our goal is to extend it to a finitely additive set function on \( \mathcal{A} \) in accord with the following definition:

**Definition 6.3** A function \( \mu : \mathcal{A} \to \mathbb{R} \cup \{ \pm \infty \} \) on an algebra \( \mathcal{A} \) is finitely additive if for any disjoint \( \{ A_i \} \subset \mathcal{A} \),

\[
\mu \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu (A_i)
\]

(6.6)

and the sum on the right-hand side makes sense.

The only way the sum would not make sense is that both \( +\infty \) and \( -\infty \) appear among some of the terms. If this happens for disjoint sets then, indeed, we have a problem indeed. That disjointness does not provide enough protection for both infinities to coexists is the content of:

**Lemma 6.4** If \( \mu \) is a finitely additive set function on an algebra, then \( \mu \) cannot take both \( +\infty \) and \( -\infty \) there.

**Proof.** Let \( A, B \) be sets in the algebra such that \( \mu (A) = \infty \) and \( \mu (B) = -\infty \). Then

\[
\mu (A) = \mu (A \cap B) + \mu (A \setminus B) \quad \text{and} \quad \mu (B) = \mu (A \cap B) + \mu (B \setminus A)
\]

(6.7)

with both sums on the right making sense. This implies that \( \mu (A \cap B) \) must be finite while \( \mu (A \setminus B) = \infty \) and \( \mu (B \setminus A) = -\infty \). But then \( \mu \) is certainly not additive on the disjoint union of \( A \setminus B \) and \( B \setminus A \) as that would lead to a meaningless expression. \( \square \)

Thus we may assume from the start that \( \mu \) takes at most one of the infinities. There is, however, still another problem for the extension from \( \mathcal{S} \) to \( \mathcal{A} \): Finite additivity may be violated already on \( \mathcal{S} \). That this is the only obstruction is the content of:

**Lemma 6.5** (Extension to algebra) Let \( \mathcal{S} \subset 2^X \) be a semialgebra and let \( \mu : \mathcal{S} \to \mathbb{R} \cup \{ \pm \infty \} \) be a set function that is finitely additive on \( \mathcal{S} \) in the sense that

\[
\forall \{ A_i \} \subset \mathcal{S} \text{ disjoint: } \bigcup_{i=1}^{n} A_i \in \mathcal{S} \implies \mu \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu (A_i).
\]

(6.8)
Then for any two disjoint families \( \{A_i\} \subset \mathcal{F} \) and \( \{B_j\} \subset \mathcal{F} \),
\[
\bigcup_{i=1}^{m} A_i = \bigcup_{j=1}^{n} B_j \implies \sum_{i=1}^{m} \mu(A_i) = \sum_{j=1}^{n} \mu(B_j). \tag{6.9}
\]

Under such conditions \( \mu \) extends uniquely to a finitely additive set function on \( \mathcal{A} \).

**Proof.** Let \( \{A_i\} \subset \mathcal{F} \) and \( \{B_j\} \subset \mathcal{F} \) be two disjoint families with \( \bigcup_{i=1}^{m} A_i = \bigcup_{j=1}^{n} B_j \). Note that \( \{A_i \cap B_j\} \) are disjoint. Then \( A_i = \bigcup_{j=1}^{n} (A_i \cap B_j) \) and \( B_j = \bigcup_{i=1}^{m} (A_i \cap B_j) \) and so, by the assumption,
\[
\mu(A_i) = \sum_{j=1}^{n} \mu(A_i \cap B_j) \quad \text{and} \quad \mu(B_j) = \sum_{i=1}^{m} \mu(A_i \cap B_j). \tag{6.10}
\]

Then
\[
\sum_{i=1}^{m} \mu(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} \mu(A_i \cap B_j) = \sum_{j=1}^{n} \mu(B_j) \tag{6.11}
\]
thus proving the first part of the claim.

By Lemma 5.2, every set \( A \in \mathcal{A} \) can be written as a disjoint union \( \bigcup_{i=1}^{m} A_i \) for \( \{A_i\} \subset \mathcal{F} \). We then define
\[
\mu(A) := \sum_{i=1}^{m} \mu(A_i) \tag{6.12}
\]
and note that, by the first part of the claim, this does not depend on the representation of \( A \). If now \( B_1, \ldots, B_m \in \mathcal{A} \) are disjoint sets and \( B_i = \bigcup_{j=1}^{n} B_{ij} \) is their representation as a union of a disjoint family \( \{B_{ij} : j = 1, \ldots, n\} \subset \mathcal{F} \) (again, we use empty sets to make all these unions have the same number of terms), we have
\[
\mu \left( \bigcup_{i=1}^{m} B_i \right) = \mu \left( \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} B_{ij} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(B_{ij}) = \sum_{i=1}^{m} \mu \left( \bigcup_{j=1}^{n} B_{ij} \right) = \sum_{i=1}^{m} \mu(B_i). \tag{6.13}
\]
Here in the first equality we used that (by disjointness of \( \{B_i\} \)) the whole collection \( \{B_{ij} : i = 1, \ldots, m, j = 1, \ldots, n\} \) consists of pairwise disjoint sets, then (in the second equality) we applied the definition (6.12) and then (in the third equality) we used (6.12) again to contract the second sum into a union. As (6.13) is a statement of finite additivity of \( \mu \) on \( \mathcal{A} \), we are done.

Keeping in mind our plan from the construction of the Lebesgue measure, we now go back to non-negative set functions only and axiomatize the properties from Proposition 3.2:

**Definition 6.6 (Outer measure)** Given a set \( X \), an outer measure on \( X \) is a map \( \mu^* : 2^X \rightarrow [0, \infty] \) satisfying the following properties:

1. \( \mu^*(\emptyset) = 0 \),
2. (monotonicity) if \( A \subset B \), then \( \mu^*(A) \leq \mu^*(B) \),
3. (countable subadditivity) for any sets \( \{A_i : i \in \mathbb{N}\} \),
\[
\mu^* \left( \bigcup_{i \in \mathbb{N}} A_i \right) \leq \sum_{i \in \mathbb{N}} \mu^*(A_i). \tag{6.14}
\]

An imitation of the construction of the Lebesgue outer measure yields:
Proposition 6.7 Let \( \mathcal{C} \subset 2^X \) obey \( \emptyset \in \mathcal{C} \) and let \( \mu_0 : \mathcal{C} \to [0, \infty] \) satisfy \( \mu_0(\mathcal{C}) = 0 \). Then

\[
\mu^*(A) := \inf \left\{ \sum_{i \in \mathbb{N}} \mu_0(A_i) : \{A_i\} \subset \mathcal{C}, \bigcup_{i \in \mathbb{N}} A_i \supseteq A \right\}
\]  

(6.15)

is an outer measure on \( X \).

Proof. The proof is almost identical to that for the Lebesgue measure. First off, \( \mu^* \geq 0 \) because \( \mu_0 \geq 0 \). Hence, \( \mu^*(\emptyset) = \mu_0(\emptyset) = 0 \). Next if \( \{B_i\} \subset \mathcal{C} \) covers \( B \) then it also covers any \( A \subset B \) and so \( \mu^*(A) \leq \mu^*(B) \). Finally, if \( \{A_i\} \) is a collection of sets with \( \mu^*(A_i) \leq \infty \) (otherwise there is nothing to prove), then for each \( \varepsilon > 0 \) we may find \( \{B_{ij}\} \) with

\[
A_i \subset \bigcup_{j \in \mathbb{N}} B_{ij} \quad \text{and} \quad \mu^*(A_i) + \varepsilon 2^{-i} \geq \sum_{j \in \mathbb{N}} \mu_0(B_{ij})
\]

(6.16)

valid for all \( i \). But then \( \bigcup_{i,j \in \mathbb{N}} B_{ij} \supseteq \bigcup_{i \in \mathbb{N}} A_i \) and thus

\[
\mu^*(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mu_0(B_{ij}) = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \mu(B_{ij})
\]

\[
\leq \sum_{i \in \mathbb{N}} (\mu^*(A_i) + \varepsilon 2^{-i}) = \varepsilon + \sum_{i \in \mathbb{N}} \mu^*(A_i)
\]

(6.17)

Since this holds for all \( \varepsilon > 0 \), \( \mu^* \) is countably subadditive and thus an outer measure. \( \square \)

The class \( \mathcal{C} \) class of sets in the above construction will sometimes be referred to as the covering class. (Thus, for the Lebesgue measure, the covering class where half-open boxes.) There are three natural questions that surround this construction:

1. Is \( \mu^*(A) = \mu_0(A) \) for sets in the covering class?
2. Is there a restriction of \( \mu^* \) to a \( \sigma \)-algebra where it becomes a measure?
3. If (2) is answered affirmatively, are the sets in the covering class measurable?

In the forthcoming sections we will answer all these at various level of detail.

6.2 Carathéodory measurability and extension theorem.

When \( \mu^*(X) < \infty \), a natural idea how to define measurable sets goes via the inner measure. Indeed, we could set

\[
\mu_*(A) := \mu^*(X) - \mu^*(X \setminus A)
\]

(6.18)

and try to define \( A \) to be measurable when \( \mu^*(A) = \mu_*(A) \). However, this is quickly found to run into a dead end. Indeed, consider the example of \( X := \{1, 2, 3\} \) with \( \mu^* \) given by

\[
\mu^*(\emptyset) = 0, \quad \mu^*(X) = 3, \quad \mu(A) = 1 \quad \text{for} \quad A \neq \emptyset, X.
\]

(6.19)

Then \( \mu_*(A) = \mu^*(A) \) for all sets \( A \subset X \) and yet \( \mu^* \) is not a measure on \( 2^X \) — e.g., for \( B := \{1, 2\} \) and \( A := \{1\} \) we have \( \mu^*(B) = 1 \) and \( \mu^*(B \cap A) + \mu^*(B \setminus A) = 2 \).

The moral of this example is that, for \( A \) to be measurable, it has to satisfy more than just the condition \( \mu^*(X) = \mu^*(X \cap A) + \mu^*(X \setminus A) \). As it turns out, we have to require that \( \{A, A^c\} \) partitions not just \( X \) additively, but in fact all subsets thereof:
Definition 6.8 (Carathéodory measurability) Let \( \mu^* \) be an outer measure on a set \( X \). We call \( A \subset X \) (Carathéodory) measurable if

\[
\forall E \subset X: \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A). \tag{6.20}
\]

We will write \( \Sigma(\mu^*) \) to denote the class of all (Carathéodory) measurable sets.

Note that, thanks to the subadditivity of \( \mu^* \), the nontrivial condition to check is that

\[
\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A). \tag{6.21}
\]

The principal result of this section is then:

Theorem 6.9 (Carathéodory) Let \( \mu^* \) be an outer measure on a set \( X \). Then:

1. \( \Sigma(\mu^*) \) is a \( \sigma \)-algebra, and
2. \( \mu^* \) is a measure on \( \Sigma(\mu^*) \).

Proof. The proof comes in four steps.

STEP 1: \( \Sigma(\mu^*) \) is an algebra

Obviously, \( \emptyset, X \in \Sigma(\mu^*) \) and if \( A \in \Sigma(\mu^*) \) then also \( A^c \in \Sigma(\mu^*) \). Now let \( A, B \in \Sigma(\mu^*) \) and pick an arbitrary set \( E \subset X \). Then

\[
E \setminus (A \cup B) = E \cap A^c \cap B^c \quad \text{and} \quad E \cap (A \cup B) = (E \cap A) \cup (E \cap A^c \cap B) \tag{6.22}
\]

and so by subadditivity

\[
\mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)) \\
\leq \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B) \\
= \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E) \tag{6.23}
\]

where the first equality is because \( A \in \Sigma(\mu^*) \) and the second is due to \( B \in \Sigma(\mu^*) \). This shows (6.21) and thus (6.20) for \( A \cup B \). Hence, \( \Sigma(\mu^*) \) is an algebra.

STEP 2: For any \( A_1, \ldots, A_n \in \Sigma(\mu^*) \) disjoint and any \( E \subset X \),

\[
\mu^*(E \cap \bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu^*(E \cap A_i). \tag{6.24}
\]

In particular, \( \mu^* \) is finitely additive on \( \Sigma(\mu^*) \).

We observe that, since \( A_{n+1} \in \Sigma(\mu^*) \) and \( A_{n+1} \) is disjoint from \( \bigcup_{i=1}^n A_i \),

\[
\mu^*(E \cap \bigcup_{i=1}^{n+1} A_i) = \mu^*(E \cap \bigcup_{i=1}^n A_i) \cap A_{n+1} + \mu^*(E \cap A_{n+1}) - \mu^*(E \cap A_{n+1}) \\
= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap \bigcup_{i=1}^n A_i) \tag{6.25}
\]

Since the claim holds trivially for \( n = 1 \), the identity (6.24) follows by induction on \( n \). Setting \( E := X \) then shows that \( \mu^* \) is finitely additive.

STEP 3: \( \Sigma(\mu^*) \) is a \( \sigma \)-algebra
Let \( \{A_n : n \in \mathbb{N}\} \subset \Sigma(\mu^*) \). Then \( \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A'_n \) where \( A'_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i \) are now disjoint from each other. But \( \Sigma(\mu^*) \) is an algebra and so \( \{A'_n\} \in \Sigma(\mu^*) \) as well. Hence, we may suppose that \( \{A_n\} \) are disjoint. Then (6.24) gives

\[
\mu^*(E) \overset{\text{STEP 1}}{=} \mu^*\left( E \cap \bigcup_{i=1}^n A_i \right) + \mu^*\left( E \setminus \bigcup_{i=1}^n A_i \right)
\geq \mu^*(E \cap \bigcup_{i=1}^n A_i) + \mu^*(E \setminus \bigcup_{i=1}^n A_i)
\overset{\text{STEP 2}}{=} \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(- \bigcup_{i=1}^n A_i)
\]

(6.26)

where we also used monotonicity of \( \mu^* \) to replace the finite union by infinite union in the second term on the right. Taking \( n \to \infty \) and using subadditivity of \( \mu^* \) now shows

\[
\mu^*(E) \geq \sum_{i \in \mathbb{N}} \mu^*(E \cap A_i) + \mu^*(E \setminus \bigcup_{i \in \mathbb{N}} A_i) \geq \mu^*(E \cap \bigcup_{i \in \mathbb{N}} A_i) + \mu^*(E \setminus \bigcup_{i \in \mathbb{N}} A_i)
\]

(6.27)

and so (6.21) holds for the countable union \( \bigcup_{i \in \mathbb{N}} A_i \). Hence \( \Sigma(\mu^*) \) is closed under countable unions and, being an algebra, it is a \( \sigma \)-algebra.

**STEP 4: \( \mu^* \) is a measure on \( \Sigma(\mu^*) \)**

Let \( \{A_n\} \subset \Sigma(\mu^*) \) be disjoint. Taking \( E := \bigcup_{i \in \mathbb{N}} A_i \) in the previous argument shows

\[
\mu^*(\bigcup_{i \in \mathbb{N}} A_i) \geq \sum_{i \in \mathbb{N}} \mu^*(A_i)
\geq \mu^*(\bigcup_{i \in \mathbb{N}} A_i)
\]

(6.28)

and, by countable subadditivity of \( \mu^* \), equality must in fact hold here. Hence \( \mu^* \) is countably additive on \( \Sigma(\mu^*) \) and, since \( \mu^*(\emptyset) = 0 \), it is a measure. \( \square \)

Note that any \( A \subset X \) with \( \mu^*(A) = 0 \) is automatically Carathéodory measurable. As a consequence, the measure space \( (X, \Sigma(\mu^*), \mu^*) \) is automatically complete (see Definition 4.16).

The Carathéodory theorem completely answers question (2) from the end of the class section. To address the remaining two questions, and to make practical use of Carathéodory extension easier, we state and prove:

**Theorem 6.10 (Hahn-Kolmogorov)** Let \( \mathcal{A} \subset 2^X \) be an algebra and assume that \( \mu_0 : \mathcal{A} \to [0, \infty] \) obeys \( \mu_0(\emptyset) = 0 \) and that the following holds:

1. (finite additivity) \( A, B \in \mathcal{A}, A \cap B = \emptyset \) implies \( \mu_0(A \cup B) = \mu_0(A) + \mu_0(B) \).
2. (countable subadditivity) \( \{A_i\} \subset \mathcal{A} \) and \( \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A} \) imply \( \mu_0(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mu_0(A_i) \).

Define the outer measure generated by the covering class \( \mathcal{A} \) and set function \( \mu_0 \) by

\[
\mu^*(A) := \inf\left\{ \sum_{i \in \mathbb{N}} \mu_0(A_i) : \{A_i\} \subset \mathcal{A}, \bigcup_{i \in \mathbb{N}} A_i \supseteq A \right\}.
\]

(6.29)

Then

\[ \mathcal{A} \subset \Sigma(\mu^*) \quad \text{and} \quad \mu^* = \mu_0 \text{ on } \mathcal{A}. \]

(6.30)

In particular, \( \mu^* \) extends \( \mu_0 \) to a measure on \( \Sigma(\mu^*) \).

**Proof.** The proof comes again in two steps.

**STEP 1: \( \mathcal{A} \subset \Sigma(\mu^*) \)**
Let \( A \in \mathscr{A} \) and pick any \( E \subset X \). We need to show (6.21). For this we may assume \( \mu^*(E) < \infty \). By (6.29), there are sets \( \{B_i\} \subset \mathscr{A} \) such that
\[
E \subset \bigcup_{i \in \mathbb{N}} B_i \quad \text{and} \quad \mu^*(E) + \varepsilon \geq \sum_{i \in \mathbb{N}} \mu_0(B_i). \tag{6.31}
\]
But \( \mu_0 \) is finitely additive and so \( \mu(B_i) = \mu(B_i \cap A) + \mu_0(B_i \setminus A) \) and so
\[
\mu^*(E) + \varepsilon \geq \sum_{i \in \mathbb{N}} \mu_0(B_i) = \sum_{i \in \mathbb{N}} \mu_0(B_i \cap A) + \sum_{i \in \mathbb{N}} \mu_0(B_i \setminus A)
\]
\[
\geq \mu^*(\bigcup_{i \in \mathbb{N}} B_i \cap A) + \mu^*(\bigcup_{i \in \mathbb{N}} B_i \setminus A) \geq \mu^*(E \cap A) + \mu^*(E \setminus A), \tag{6.32}
\]
where we used \( \{B_i \cap A\} \) and \( \{B_i \setminus A\} \) cover their own unions and then applied (6.29) and the monotonicity of \( \mu^* \). As \( \varepsilon \) was arbitrary, we have \( A \in \Sigma(\mu^*) \).

**STEP 2: \( \mu^* = \mu_0 \) on \( \mathscr{A} \)**

Pick \( A \in \mathscr{A} \). We have \( \mu^*(A) \leq \mu_0(A) \) because \( \emptyset \in \mathscr{A} \) and \( \mu_0(\emptyset) = 0 \). For the opposite inequality let \( \{B_i\} \subset \mathscr{A} \) be any collection of sets such that \( A \subset \bigcup_{i \in \mathbb{N}} B_i \). Then \( \bigcup_{i \in \mathbb{N}} (B_i \cap A) = A \) and so, by the monotonicity and countable subadditivity of \( \mu_0 \) on \( \mathscr{A} \),
\[
\sum_{i \in \mathbb{N}} \mu_0(B_i) \geq \sum_{i \in \mathbb{N}} \mu_0(B_i \cap A) \geq \mu_0(A). \tag{6.33}
\]
Taking infimum over all such \( \{B_i\} \) gives \( \mu^*(A) \geq \mu_0(A) \) as desired. \( \square \)

To demonstrate the power of this theorem, let us consider a couple of examples. The first one will be quite familiar:
\[
\mathscr{A} := \{ A \subset \mathbb{R}^d : \text{Peano-Jordan measurable} \} \quad \text{and} \quad \mu_0(A) := c(A). \tag{6.34}
\]
Then (6.29) yields \( \mu^* = \lambda^* \) and so we have a definition of the Lebesgue measure on the class \( \Sigma(\lambda^*) \) of Carathéodory measurable sets. However, the relation between \( \Sigma(\lambda^*) \) and \( \mathscr{L}(\mathbb{R}^d) \), and the values of \( \lambda^* \) thereupon, is unclear at this point (we will return to this later).

As another example, pick any set \( X \) and consider the collection of sets
\[
\mathscr{A} := \{ A \subset X : \text{finite or co-finite} \}. \tag{6.35}
\]
This is easily verified to be an algebra and so we set
\[
\mu_0(A) := \begin{cases} 
0, & \text{if } A \text{ finite}, \\
1, & \text{if } A \text{ co-finite}. 
\end{cases} \tag{6.36}
\]
Then \( \mu_0 \) is additive and countably additive on \( \mathscr{A} \). The definition (6.29) yields
\[
\mu^*(A) := \begin{cases} 
0, & \text{if } A \text{ countably infinite}, \\
1, & \text{else} 
\end{cases} \tag{6.37}
\]
and, as is again checked easily, \( \Sigma(\mu^*) = \{ A \subset X : \text{countable or co-countable} \} \). Further examples will be produced in the sections on Lebesgue-Stieltjes measures and product spaces.

In some cases verifying the conditions of the Hahn-Kolmogorov theorem in an algebra can be quite a nuisance. Invariably (as we did in the context of the Lebesgue measure in Lemma 3.4) this is usually easier at the level of semialgebras. We then need the following general fact:
Lemma 6.11 Let \( \mathcal{S} \) be a semialgebra and \( \mu_0 : \mathcal{S} \to [0, \infty) \) obey \( \mu_0(\emptyset) = 0 \). If \( \mu_0 \) is finitely additive and countably subadditive on \( \mathcal{S} \), then so is also its unique extension (guaranteed by Lemma 6.5) to the algebra \( \mathcal{A} \) generated by taking all finite unions of sets from \( \mathcal{S} \).

Proof. That additivity of \( \mu_0 \) on \( \mathcal{S} \) implies additivity on \( \mathcal{A} \) was the subject of Lemma 6.5. To show that countable additivity on \( \mathcal{S} \) implies countable additivity on \( \mathcal{S} \), we note that, since \( \mathcal{A} \) is an algebra, it suffices to check this for disjoint unions.

Let \( \{ A_n : n \in \mathbb{N} \} \subset \mathcal{A} \) be disjoint sets with \( A := \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A} \). By Lemma 6.2, each \( A_n \) is a finite disjoint union of sets \( \{ B_{nj} : j = 1, \ldots, k(n) \} \) and \( A \) is a finite disjoint union of sets \( \{ C_i : i = 1, \ldots, m \} \). By the disjointness assumptions,

\[
C_i \cap B_{nj} \neq \emptyset \quad \Rightarrow \quad B_{nj} \subset C_i
\]  

(6.38)

and setting

\[
S_i := \{ (n, j) : n \in \mathbb{N}, j = 1, \ldots, k(n), B_{nj} \subset C_i \}
\]  

(6.39)

we thus have

\[
C_i = \bigcup_{(n, j) \in S_i} B_{nj} \quad \text{and} \quad \mu_0(C_i) \leq \sum_{(n, j) \in S_i} \mu_0(B_{nj})
\]  

(6.40)

where the second conclusion follows by countable subadditivity of \( \mu_0 \) on \( \mathcal{S} \). On the other hand, \( A_n \) can be written as the finite disjoint union

\[
A_n = \bigcup_{i=1}^{m} \bigcup_{j : (n, j) \in S_i} B_{nj}
\]  

(6.41)

and so finite additivity of \( \mu_0 \) implies

\[
\mu(A_n) = \sum_{i=1}^{m} \sum_{j : (n, j) \in S_i} \mu_0(B_{nj}).
\]  

(6.42)

Putting this together, we conclude

\[
\mu_0(A) = \sum_{i=1}^{m} \mu_0(C_i) \leq \sum_{i=1}^{m} \sum_{(n, j) \in S_i} \mu_0(B_{nj})
\]

\[
= \sum_{n \in \mathbb{N}} \sum_{i=1}^{m} \sum_{j : (n, j) \in S_i} \mu_0(B_{nj}) = \sum_{n \in \mathbb{N}} \mu_0(A_n)
\]  

(6.43)

as desired. \( \square \)

6.3 Uniqueness of extension.

Having constructed extensions of (non-negative) additive set functions on algebras to measures on \( \sigma \)-algebras, we need to address the question of uniqueness. As already noted, at this point we have two (possibly equivalent) definitions of the Lebesgue measure: one on the Lebesgue \( \sigma \)-algebra \( \mathcal{L}(\mathbb{R}^d) \) of sets that can be approximated, in the sense of outer measure, by open sets from outside, and the other by way of Carathéodory measurability. A question is then how to relate these to each other.

That the extension may not be unique is in fact fairly easy to check by the following example. Let \( \mathcal{A} \) be the algebra consisting of finite unions of all half-open intervals in \( \mathbb{R} \) and consider the set...
function \( \mu_0 \) defined by \( \mu_0(A) = \infty \) if \( A \neq \emptyset \) and \( \mu_0(\emptyset) = 0 \). Now consider three measures on \( 2^\mathbb{R} \):

first, the counting measure

\[
\mu_1(A) := \begin{cases} 
\text{card}(A), & \text{if } A \text{ finite,} \\
\infty, & \text{else,}
\end{cases}
\]

second, the “countability-detection” measure

\[
\mu_2(A) := \begin{cases} 
0, & \text{if } A \text{ finite or countably infinite,} \\
\infty, & \text{else,}
\end{cases}
\]

and, third, the identically-infinite measure

\[
\mu_3(A) := \begin{cases} 
0, & \text{if } A = \emptyset, \\
\infty, & \text{else.}
\end{cases}
\]

Since every non-empty set in \( \mathfrak{A} \) is automatically infinite and uncountable, we have

\[
\mu_1 = \mu_2 = \mu_3 = \mu_0 \text{ on } \mathfrak{A}.
\]

Yet, clearly, all three measures are different. (Note also that all sets are of course Carathéodory measurable with respect to these measures, regarded as outer measures.)

It is important to note the highlight the issues that underpin the conclusion of this example. First, either the class \( \mathfrak{A} \) is too small or the class \( 2^\mathbb{R} \) is too large to make all its sets “reachable” by approximations using sets from \( \mathfrak{A} \). So the relative “size” of the extension \( \sigma \)-algebra compared to that of the generating algebra must somehow matter. Also, the fact that the measures are infinite must play a role because the examples rely on the fact that removal of a finite-measure set has no effect on the measure of an infinite-measure set.

We will first address the fact that some restrictions on the “size” of the target \( \sigma \)-algebra must be made should the extension be unique. Noting the easy fact that an intersection of any number of \( \sigma \)-algebras on \( X \) is a \( \sigma \)-algebra on \( X \), we introduce:

**Definition 6.12** For \( \mathcal{M} \subset 2^\mathbb{R} \), let

\[
\sigma(\mathcal{M}) := \bigcap_{\mathcal{F} \text{-}\sigma\text{-algebra}} \mathcal{F} \ni \mathcal{M}
\]

be the least \( \sigma \)-algebra containing \( \mathcal{M} \).

As an example, define

\[
\mathcal{B}(\mathbb{R}^d) := \sigma(\{O \subset \mathbb{R}^d : \text{ open}\})
\]

to be the class of Borel sets in \( \mathbb{R}^d \). (Here the open sets are defined using the Euclidean metric.) As is not hard to check, the same object will be obtained if we replace open sets by half-open boxes. Thus,

\[
\mathcal{B}(\mathbb{R}^d) \subset \mathcal{L}(\mathbb{R}^d) \cap \Sigma(\lambda^*)
\]

and, in fact, \( \mathcal{B}(\mathbb{R}^d) \) is the least \( \sigma \)-algebra on which the Lebesgue measure can be defined.

It is obvious that, if we have a measure on a \( \sigma \)-algebra, then it naturally restricts to a measure on any sub-\( \sigma \)-algebra thereof. So if there is a hope to prove uniqueness at all, it should be primarily attempted on \( \sigma(\mathfrak{A}) \). This is the conclusion of:
Theorem 6.13 (Uniqueness criterion) Let $\mathcal{A} \subset 2^X$ be an algebra and let $\mu$ and $\nu$ be finite measures on $\sigma(\mathcal{A})$. Then

$$\mu = \nu \text{ on } \mathcal{A} \quad \Rightarrow \quad \mu = \nu \text{ on } \sigma(\mathcal{A}). \quad (6.51)$$

The same is true even when $\mu$ and $\nu$ are infinite provided that there are sets $\{X_n\} \subset \mathcal{A}$ such that

$$\bigcup_{n \in \mathbb{N}} X_n = X \quad \text{and} \quad \mu(X_n) = \nu(X_n) < \infty, \quad \forall n \in \mathbb{N}. \quad (6.52)$$

The proof is abstract but also quite instructive as its ideas show up in many other reincarnations elsewhere. First we define:

Definition 6.14 (Monotone class) A class of sets $\mathcal{M} \subset 2^X$ is a monotone class if

1. $\forall \{A_n\} \subset \mathcal{M}: A_n \uparrow A$ implies $A \in \mathcal{M}$,
2. $\forall \{A_n\} \subset \mathcal{M}: A_n \downarrow A$ implies $A \in \mathcal{M}$.

In words, a monotone class is class that is closed under increasing and decreasing limits.

It is easy to check that any $\sigma$-algebra is a monotone class. The class of all intervals in $\mathbb{R}$ — open, half-open and closed including degenerate ones — is a monotone class but not a $\sigma$-algebra. However, once a monotone class contains an algebra, it contains also the least $\sigma$-algebra generated thereby:

Lemma 6.15 (Monotone Class Lemma) Let $X$ be any set. If $\mathcal{M} \subset 2^X$ is a monotone class and $\mathcal{A} \subset 2^X$ an algebra, then

$$\mathcal{A} \subset \mathcal{M} \quad \Rightarrow \quad \sigma(\mathcal{A}) \subset \mathcal{M}. \quad (6.53)$$

Proof. As is easy to check, the intersection of any number of monotone classes is a monotone class, so it makes sense to define

$$\mathcal{M}(\mathcal{A}) := \bigcap_{\mathcal{M}: \text{monotone class}} \mathcal{M} \quad (6.54)$$

Now $\sigma(\mathcal{A})$ is a monotone class containing $\mathcal{A}$ and so $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. Our goal is to show that $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ because that, by the minimality of $\mathcal{M}(\mathcal{A})$, implies the claim.

First we note that $\emptyset, X \in \mathcal{M}(\mathcal{A})$ by their containment in $\mathcal{A}$. Also, since

$$\mathcal{M}' := \{A^c: A \in \mathcal{M} \} \quad (6.55)$$

is a monotone class if $\mathcal{M}$ is a monotone class, we have $A \in \mathcal{M}(\mathcal{A})$ if and only if $A^c \in \mathcal{M}(\mathcal{A})$. We will now show that $\mathcal{M}(\mathcal{A})$ is closed under finite intersection. For this we let

$$\mathcal{M}_A := \{C \in \mathcal{M}(\mathcal{A}): C \cap A \in \mathcal{M}(\mathcal{A})\} \quad (6.56)$$

Now we observe that, since $\mathcal{A}$ is itself closed under intersections and $\mathcal{M}(\mathcal{A})$ is a minimal monotone class containing $\mathcal{A}$,

$$\forall A \in \mathcal{A}: \quad \mathcal{A} \subset \mathcal{M}_A \quad \text{and so} \quad \mathcal{M}(\mathcal{A}) \subset \mathcal{M}_A. \quad (6.57)$$

But the conclusion can be rewritten as

$$\forall B \in \mathcal{M}(\mathcal{A}): \quad \mathcal{A} \subset \mathcal{M}_B \quad \text{and so} \quad \mathcal{M}(\mathcal{A}) \subset \mathcal{M}_B. \quad (6.58)$$
This means that if \( A, B \in \mathcal{m}(\mathcal{A}) \), then also \( A \cap B \in \mathcal{m}(\mathcal{A}) \), i.e., \( \mathcal{m}(\mathcal{A}) \) is closed under finite intersections and, in light of the above, it is an algebra. But \( \mathcal{m}(\mathcal{A}) \) is also a monotone class and so it is closed under countable intersections as well. It follows that \( \mathcal{m}(\mathcal{A}) \) is a \( \sigma \)-algebra containing \( \mathcal{A} \) and so \( \mathcal{m}(\mathcal{A}) \supseteq \sigma(\mathcal{A}) \) by the minimality of \( \sigma(\mathcal{A}) \). \( \square \)

**Proof of Theorem 6.13.** Let \( \mu \) and \( \nu \) be finite measures on a \( \sigma \)-algebra \( \mathcal{F} \). Then, by the Monotone Convergence Theorem for sets,

\[
\mathcal{M} := \{ A \in \mathcal{F} : \mu(A) = \nu(A) \}
\]  

is a monotone class. So if \( \mathcal{A} \) is an algebra and \( \mu = \nu \) on \( \mathcal{A} \), we have \( \mathcal{A} \subseteq \mathcal{M} \) and, by the Monotone Class Lemma, \( \sigma(\mathcal{A}) \subseteq \mathcal{M} \). Hence, \( \mu = \nu \) on \( \sigma(\mathcal{A}) \).

To address the infinite-measure case, consider the sets \( \{ X_n \in \mathcal{A} \} \) such that \( \mu(X_n) = \nu(X_n) < \infty \). Without loss of generality (take finite unions), we may assume that \( X_n \uparrow X \). Define measures

\[
\mu_n(A) := \mu(A \cap X_n) \quad \text{and} \quad \nu_n(A) := \nu(A \cap X_n).
\]  

These are finite measures and \( \mu_n = \nu_n \) on \( \mathcal{A} \). Hence, \( \mu_n = \nu_n \) on \( \sigma(\mathcal{A}) \) by the first part of the claim. But \( \mu_n(A) \uparrow \mu(A) \) and \( \nu_n(A) \uparrow \nu(A) \) by the Monotone Convergence Theorem for sets, and so \( \mu = \nu \) on \( \sigma(\mathcal{A}) \) as well. \( \square \)

### 6.4 Lebesgue-Stieltjes measures.

The Hahn-Kolmogorov Theorem gives us a way to construct a large class of measures on the real line. Recall that \( \mathcal{B}(\mathbb{R}) \) denotes the class of Borel sets in \( \mathbb{R} \) (see (6.49)). Then we have:

**Theorem 6.16** Let \( F : \mathbb{R} \to \mathbb{R} \) be a non-decreasing, right-continuous function. Then there exists a unique measure \( \mu_F \) on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) such that

\[
\forall a < b: \quad \mu_F((a, b]) = F(b) - F(a).
\]  

The key point is to show:

**Lemma 6.17** Let \( \mathcal{I} := \{(a, b] : -\infty \leq a \leq b \leq \infty \} \). Then \( \mathcal{I} \) is a semialgebra on \( \mathbb{R} \) and

\[
\mu_0((a, b]) := F(b) - F(a)
\]  

is finitely additive and countably subadditive on \( \mathcal{I} \).

**Proof.** That \( \mathcal{I} \) is a semialgebra is verified easily. It is also immediate that \( \mu_0 \) is finitely additive (see the first part of the proof of Lemma 2.4). To prove countable additivity, thanks to Lemma 6.11, we need to verify that for any disjoint collection \( \{(a_n, b_n] : n \in \mathbb{N}\} \subseteq \mathcal{I} \) and any \( (a, b) \in \mathcal{I} \),

\[
(a, b) = \bigcup_{n \in \mathbb{N}} (a_n, b_n] \quad \Rightarrow \quad F(b) - F(a) \leq \sum_{n \in \mathbb{N}} (F(b_n) - F(a_n)).
\]  

By intersection with a finite interval we may assume that \( (a, b] \) is finite. Then \( F(b) - F(a) < \infty \) and so, by the right-continuity of \( F \), given \( \varepsilon > 0 \) there is \( a' > a \) so that \( F(a') \leq F(a) + \varepsilon \). Similarly, for each \( n \in \mathbb{N} \), we can find \( b'_n > b_n \) so that \( F(b'_n) \leq F(b_n) + \varepsilon 2^{-n} \). Since

\[
[a', b] \subseteq (a, b] = \bigcup_{n \in \mathbb{N}} (a_n, b_n] \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b'_n)
\]  

(6.64)
the open intervals \( \{(a_n, b'_n)\} \) constitute a cover of a compact interval \([a', b]\). By the Heine-Borel Theorem, there is a finite subcover.

Let \((\alpha_1, \beta_1'], \ldots, (\alpha_N, \beta_N']\) denote the intervals in a finite subcover so that each interval contains a point that is not contain by the other intervals. We may assume that the intervals are labeled so that \(\alpha_1 < \alpha_2 < \cdots < \alpha_N\), where the strict inequality must hold by disjointness of \(\{(a_n, b_n]\}\). Since each \((\alpha_i, \beta_i']\) contains a point which does not lie in any other interval, we must have
\[
\alpha_1 < a', \quad \beta_N > b \quad \text{and} \quad \beta_i' > \alpha_{i+1}, \quad i = 1, \ldots, N - 1. \tag{6.65}
\]
By the monotonicity of \(F\), our choices of \(a'\) and \(b'_n\) and (6.65) we thus have
\[
F(b) - F(a) - \epsilon \leq F(b) - F(a') \leq F(\beta_N') - F(\alpha_1) \leq \sum_{i=1}^{N-1} (F(\alpha_{i+1}) - F(\alpha_i)) \leq \sum_{i=1}^{N} (F(\beta_i') - F(\alpha_i)) \leq \sum_{n \in \mathbb{N}} (F(b_n') - F(a_n)) \leq \epsilon + \sum_{n \in \mathbb{N}} (F(b_n) - F(a_n)).
\]
Since \(\epsilon\) was arbitrary, we get (6.63).

**Proof of Theorem 6.16.** By Lemma 6.11 and the Hahn-Kolmogorov Theorem, \(\mu_0\) from the previous lemma extends to a measure on \(\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})\) which agrees with \(\mu_0\) on \(\mathcal{S}\), i.e., (6.61) holds. Since \(\mu_0((n, n+1]) = F(n+1) - F(n) < \infty\) for all \(n \in \mathbb{N}\), the second part of Theorem 6.13 applies and so \(\mu_F\) is unique. □

We remark that the construction actually gives an extension to the \(\sigma\)-algebra of Carathéodory measurable sets \(\Sigma(\mu_F^+)\). This \(\sigma\)-algebra may vary quite dramatically depending on \(F\). Here are some natural examples of measures that can be identified via the above construction:

1. If \(F(x) := x\) then \(\mu_F = \lambda\), the Lebesgue measure on \(\mathcal{B}(\mathbb{R})\),
2. If \(F(x) := 1_{\{x \geq a\}}\) then \(\mu_F\) is the Dirac point mass at \(a\), often denoted by \(\delta_a\), such that
\[
\delta_a(A) := \begin{cases} 
1, & \text{if } a \in A, \\
0, & \text{else}.
\end{cases} \tag{6.67}
\]
\(\Sigma(\mu_F^+) = 2^\mathbb{R}\) in this case.
3. If \(F(x) := \sum_{n \in \mathbb{N}} p_n 1_{\{x \geq a_n\}}\) for some \(\{a_i\} \subset \mathbb{R}\) and \(p_i \in (0, \infty)\) with \(\sum_{i \in \mathbb{N}} p_i < \infty\), then
\[
\mu_F(A) = \sum_{i \in \mathbb{N}} p_i \delta_{a_i}(A), \tag{6.68}
\]
i.e., \(\mu_F\) is a collection of point masses with mass at \(a_i\) equal to \(p_1\). (Note that, in general, \(\mu_F\) will charge a singleton \(\{a\}\) if and only if \(F\) is discontinuous at \(a\).) Again \(\Sigma(\mu_F^+) = 2^\mathbb{R}\), i.e., all subsets of \(\mathbb{R}\) are measurable.
4. If \(F\) is continuously differentiable, then \(F(b) - F(a) = \int_a^b F'(x)dx = \int_{(a,b]} F' d\lambda\) and \(\mu_F\) is then the measure given by
\[
\mu_F(A) = \int_A F' d\lambda. \tag{6.69}
\]
Note that, reversing this process gives us a way to define a meaning of \( f \, d\lambda \), i.e., multiplying the Lebesgue measure by an integrable function.

(5) \( F \) is the Cantor-Lebesgue function that is defined as follows: Let \( C_n \) be the \( n \)-th iteration of the process of “removal of middle-third intervals” defining to the middle-third Cantor set in \( C_0 := [0,1] \). Then \( C_{n-1} \cap C_n \) is a collection of \( 2^n \) open intervals of length \( 3^{-n} \). Examining these intervals “left to right”, \( F \) is constant on each of these and takes values in increasing non-negative odd multiples of \( 2^{-n} \). See Fig. 6.4

Let \( C := \bigcap_{n \geq 1} C_n \) be the middle-third Cantor set. Then

\[
\mu_F([0,1] \cap C) = 0
\]  

because this set is a union of open intervals on which \( F \) is piecewise constant. Yet

\[
\mu_F([0,1]) = F(1) - F(0) = 1
\]

Since \( F \) is also continuous, \( \mu_F(\{a\}) = 0 \) for all \( a \). Note that \( \lambda(C) = 0 \).

The defining property of the measure \( \mu_F \) suggests a natural connection to the Riemann-Stieltjes integration. And, indeed, the Lebesgue integral

\[
\int_{[a,b]} f \, d\mu_g
\]

bears the same connection to the Riemann-Stieltjes integral

\[
\int_a^b f \, dg
\]
as the Lebesgue integral (with respect to the Lebesgue measure) bears to the Riemann integral. Thus, for instance, it can be checked that, for a non-decreasing, right-continuous function, the function \( f \) is Riemann-Stieltjes integrable with respect to \( dg \) on \([a,b] \) if and only if \( f \) is \( \mu_g \)-a.e. continuous. Thanks to this connection, we have:

**Definition 6.18** A measure \( \mu \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is a Lebesgue-Stieltjes measure if there is a non-decreasing, right-continuous function \( F \) such that (6.61) holds for \( \mu \). (Then, of course, \( \mu = \mu_F \).)

The class of Lebesgue-Stieltjes measures is quite rich. Indeed, every measure \( \mu \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R})) \) that assigns finite values to compact sets — such Borel measures are called *Radon measures* — is a Lebesgue-Stieltjes measure. The function \( F \) such that that \( \mu = \mu_F \) is then given by

\[
F(x) := \begin{cases} 
\mu([0,x]), & \text{if } x \geq 0, \\
-\mu((x,0)), & \text{if } x < 0. 
\end{cases}
\]  
(6.74)

An example of a measure that is not of this kind is the counting measure of the rationals,

\[
\mu(A) := \begin{cases} 
\text{card}(A \cap \mathbb{Q}), & \text{if } A \cap \mathbb{Q} \text{ finite}, \\
\infty, & \text{else}, 
\end{cases}
\]  
(6.75)

Obviously, there is no \( F : \mathbb{R} \to \mathbb{R} \) that could make a unit upward jump at every rational.