Problem 1: Ex 3.8, Page 19

Problem 2: Prove that for any $A, B \subseteq \mathbb{R}$,

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

Problem 3: Let $E$ be a set and $\mathcal{P}(E)$ be the powerset of $E$ — namely, the set of all subsets of $E$. The relation $A \subseteq B$ defines a (partial) order on $\mathcal{P}(E)$. Prove that every nonempty set $F \subseteq \mathcal{P}(E)$ admits $\sup(F)$ and $\inf(F)$. In fact, prove the following:

$$\forall F \subseteq \mathcal{P}(E): \quad F \neq \emptyset \Rightarrow \sup(F) = \bigcup F \land \inf(F) = \bigcap F$$

Now take $F$ to be a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of $E$. Define the limes superior and limes inferior by the usual formulas and prove that

$$\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n$$

If equality is TRUE, then we say that $\lim_{n \to \infty} A_n$ exists.

Problem 4: Ex 7.5, Page 38

Problem 5: Ex 8.2(c), Page 44

Problem 6: Ex 8.2(e), Page 44

Problem 7: Ex 8.6, Page 44

Problem 8: Determine if

$$\lim_{n \to \infty} \left( \sqrt[n]{n^3 + n^2 + 1} - \sqrt[n]{n^3 + 1} \right)$$

exists or not. Prove your claim.

Problem 9: Define $a_n$ recursively by

$$a_0 := 1 \land \left( \forall n \in \mathbb{N}: a_{n+1} = \sqrt{2 + a_n} \right)$$

Determine if $\{a_n\}_{n \in \mathbb{N}}$ exists and if so, find its value. Prove your claim.