Blow-up in multidimensional aggregation equations with mildly singular interaction kernels*

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Abstract

We consider the multidimensional aggregation equation $u_t - \nabla \cdot (u \nabla K * u) = 0$ in which the radially symmetric attractive interaction kernel has a mild singularity at the origin (Lipschitz or better). In the case of bounded initial data, finite time singularity has been proved for kernels with a Lipschitz point at the origin (Bertozzi and Laurent 2007 *Commun. Math. Sci.* **274** 717–35), whereas for C^2 kernels there is no finite-time blow-up. We prove, under mild monotonicity assumptions on the kernel K, that the Osgood condition for well-posedness of the ODE characteristics determines global in time well-posedness of the PDE with compactly supported bounded nonnegative initial data. When the Osgood condition is violated, we present a new proof of finite time blow-up that extends previous results, requiring radially symmetric data, to general bounded, compactly supported nonnegative initial data without symmetry. We also present a new analysis of radially symmetric solutions under less strict monotonicity conditions. Finally, we conclude with a discussion of similarity solutions for the case $K(x) = |x|$ and some open problems.

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1. Introduction

The analysis of the long-time behaviour of a collection of self-interacting individuals via pairwise potentials arises in the modelling of animal collective behaviour: flocks, schools or

swarms formed by insects, fishes and birds. The simplest models based on ODE systems [11, 17, 20, 31, 32] led to continuum descriptions [10, 13, 14, 26, 30, 35, 36] for the evolution of densities of individuals. The aggregation equation

$$u_t - \nabla \cdot (u \nabla K * u) = 0, \qquad u_{t=0} = u_0 \geqslant 0, \qquad (1.1)$$

shares some features with the classical Patlak–Keller–Segel model for chemotaxis [22, 34] without diffusion, see [8, 9, 12, 18] for the state of the art in this problem. Here, the main similarity is the possible formation of a finite-time point concentration and the main difference the strong singularity of the potential in the PKS system. Equation (1.1) with additional fractional diffusion also has some prior and recent study in the literature, namely [7, 24, 25]. In the case of fractional diffusion and Lipschitz kernels, there is a critical diffusion exponent for which the solution no longer blows up in finite time.

In this paper, we focus on the case involving only attractive forces [5, 6, 23, 35] and no diffusion. Individuals attract each other under the action of a radially symmetric Lipschitz interaction potential $K(x) = k(|x|)$ with $k(r)$ increasing in r, smooth away from zero and bounded below. Since potentials are defined up to constant, we assume without loss of generality that $k(0) = 0$. Some examples appearing in applications are $K_1(x) = 1 - e^{-|x|}$, $K_2(x) = 1 - e^{-|x|^2}$ and $K_\alpha(x) \simeq |x|^\alpha$ locally near 0 with $\alpha \geqslant 1$.

This class of equations belongs to the same family of nonlinear friction equations that appear in the modelling of granular media [4, 15, 16, 27, 37]. In those references, several results regarding the long time asymptotics and rates of equilibration were obtained in cases in which the potential $K(x)$ is smooth and convex. In our typical cases, convexity fails. In fact, equation (1.1) can be formally considered as a gradient flow of the energy functional:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x - y) u(x) u(y) \, dx \, dy \qquad (1.2)$$

with respect to the Euclidean Wasserstein distance as introduced in [33] and generalized to a large family of PDEs in [2, 16]. Its connection to optimal transport theory comes from the convexity properties of the energy functional with respect to geodesic convexity in this distance, called displacement convexity [29]. A nice introduction to this different point of view can be seen in [38]. However, in this paper we do not take advantage of this approach which will be developed elsewhere.

In previous works [5, 6, 23], the local-in-time well-posedness of solutions with pointy interaction potential K_1 has been shown for initial data in Sobolev spaces [6, 23] and for integrable and bounded initial data [5] inspired by ideas from 2D-Euler equations in fluid mechanics [28, 39]. Finite time blow up has also been addressed in these works showing that for certain radial initial data the solution blows-up in finite time in the sense that the solution ceases to be in the corresponding class, i.e. blow-up in Sobolev norm or blow-up in L^∞ norm. Let us mention that in the literature of granular media models, potentials of the form $K(x) = |x|^{1+\gamma}$ with $0 < \gamma < 1$ were considered in [27] in one dimension proving that the support of the solutions shrinks to a single point in finite time.

In our main result, we give natural conditions on the potential K such that all solutions corresponding to bounded, compactly supported initial data either blow up in finite time or exist for all time. Under mild monotonicity conditions, we show that the Osgood condition for well-posedness of the ODE characteristics is sufficient to guarantee a bounded global-in-time solution of the PDE. Moreover, this condition is sharp and thus necessary for global-in-time existence. Actually, for kernels violating the Osgood condition, the solution blows up in finite time regardless of the symmetry of the initial data. Moreover, the blow-up time is bounded uniformly for all initial data with given mass M compactly supported inside a ball of fixed radius R_0 by a constant depending only on M and R_0.

More specifically, for bounded, compactly supported, nonnegative initial data, if the potential $K(x) = k(|x|)$ is monotone decreasing in $k'(r)/r$ and has

$$\int_0^1 \frac{1}{k'(r)} \, dr < \infty, \tag{1.3}$$

then the solution blows up in finite time. Moreover, the bound on the time of blow-up depends only on the radius of the support of the initial data and the total mass of the solution. The integral condition on the kernel is none other than the well-known Osgood condition for finite-time vanishing of the ODE $dR/dt = -k'(R)$, which implies non-uniqueness of solutions, see for instance [3] and the recent preprint for linear continuity equations [1]. This case includes kernels with a Lipschitz point at the origin, for which we already have blow-up results with symmetric data. Our proof includes initial data without symmetry.

Furthermore, when the Osgood condition gives a global-in-time solution of the ODE, i.e.

$$\int_0^1 \frac{1}{k'(r)} \, dr = \infty, \tag{1.4}$$

we derive a global-in-time bound on the solution, generalizing previous results for C^2 kernels. The proof requires the monotonicity assumption on $k'(r)/r$ along with some lower bound, e.g. $0 < k''$.

Throughout this paper, we say that K is an Osgood potential if (1.4) is satisfied and non-Osgood when (1.3) holds. We note that if K is Osgood but not C^2 at the origin, then k'' is typically positive and decreasing from infinity near zero, and thus k' is superlinear near the origin, consistent with the monotonicity assumption on $k'(r)/r$ and with the lower bound required on k''. While the global-in-time bound implies global existence of bounded solutions, nevertheless the solution is proved to concentrate onto a single Dirac delta as $t \to \infty$.

Our results can be more finely tuned and improved in the case of radial solutions. In the case of finite-time blow-up, we are able to show a bound from below in the blow-up time and to weaken the additional hypotheses on the potential. More precisely, we can show that for $k'(r)$ monotone and (1.3), then finite-time blow-up of radial solutions happens. Using the radially symmetric formulation, we also examine the problem of existence of similarity solutions blowing up in finite time. In the case of the Lipschitz potential $K(x) = |x|$, we prove in odd space dimensions larger than 1 (i.e. 3, 5, 7, ...,) that no similarity solution exists with support containing an open set. We consider both L^p similarity solutions, $p > 1$, and measure-valued similarity solutions with compact support. This result is in contrast to the one-dimensional case where there exists a simple compactly supported L^∞ similarity solution.

Note that in the case of power-law potentials K_α, condition (1.3) is satisfied if and only if $\alpha < 2$. If $\alpha \geqslant 2$, then (1.4) holds and it is known that solutions exist globally. Our results include all power-law potentials and extend the $\alpha \geqslant 2$ case to additional potentials that fail to be C^2 by a marginal amount (e.g. logarithmic correction) yet still satisfy (1.4).

This paper is organized as follows. In section 2 we review the ODE problem of interacting particles and present a simple estimate for the collapse of all particles to their centre of mass. We then show how to extend the ODE result to the continuum problem, using continuity of particle paths for the case of $L^1 \cap L^\infty(\mathbb{R}^N)$ data. A comparison principle is established that provides a proof of blow-up in the continuum problem for general initial data without symmetry and for a general class of symmetric kernels with mild singularity. At the end of section 2 we prove that the Osgood condition for blow-up is sharp, by extending previously known global existence results for C^2 kernels to the general Osgood case, provided they satisfy mild monotonicity conditions. In section 3 we study the radially symmetric case showing that, for the blow-up proof, monotonicity assumptions can be weakened. Also in that section we present a discussion of self-similar blowing-up solutions for the special case of the $|x|$ potential.

2. Blow-up estimates and the connection to Osgood potentials

In this section we derive the main results for general (non-symmetric) compactly supported nonnegative bounded data. In section 2.1 we present a discussion of the ODE system formed by L interacting point masses. We derive an estimate for the size of the support of the solution that can be used to prove collapse to a single point mass in finite time for kernels that violate the Osgood condition (1.4) yet have simple monotonicity conditions. Under the same monotonicity conditions, if the kernel satisfies the Osgood condition, the particles never collide in finite time, yet they still collapse to a point in infinite time. Sections 2.2 and 2.3 prove the analogue of the ODE results for the PDE problem with bounded, compactly supported initial data. Under precisely the same conditions on the potential as in the ODE case, the Osgood condition yields global existence of bounded solutions that nevertheless collapse to a point in infinite time. On the other hand, if the potential violates the Osgood condition, then finite-time blow-up occurs for any compactly supported and bounded initial condition. Thus the Osgood condition is sharp for finite time versus infinite time blow-up of bounded, nonnegative, compactly supported solutions to the aggregation equation. These results generalize previous results on (a) global existence of solutions with C^2 kernels and (b) finite-time blow-up of bounded solutions with radially symmetric data and kernels with a Lipschitz point at the origin. Section 2.2 proves the finite time blow-up results and section 2.3 proves the global existence and infinite time blow-up results.

2.1. The discrete particle problem

When the solution is represented by L particles $\{x_1, \ldots, x_L\}$ of respective mass $\{m_1, \ldots, m_L\}$ the evolution equation reduces (at least formally) to a coupled set of ODEs for the particle paths:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\sum_{j \neq i} m_j \nabla K(x_i - x_j) = -\sum_{j \neq i} m_j \frac{x_i - x_j}{|x_i - x_j|} k'(|x_i - x_j|), \qquad i = 1, \ldots, L, \quad (2.1)$$

with $x_i(t) \in \mathbb{R}^N$ for all $t \geqslant 0$. Note that these equations preserve the total mass $M := \sum_j m_j$ of the system and the centre of mass $c_\mathrm{M} := \sum_j x_j m_j / (\sum_j m_j)$. The latter is true because of the symmetry of K. Assume that the L-particles with total mass M and zero centre of mass are initially inside the ball of radius R_0. Denote by $R(t)$ the distance to the centre of mass of the particle situated further apart from the centre of mass $c_\mathrm{M} = 0$, i.e. $R(t) = |x_i(t)|$ with i being its label. Thus, due to (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t)^2 = \frac{\mathrm{d}}{\mathrm{d}t} |x_i|^2 = -2 \sum_{j \neq i} m_j \frac{(x_i - x_j) \cdot x_i}{|x_i - x_j|} k'(|x_i - x_j|).$$

Since the ith particle is the one farthest away from the centre of mass, we have that $(x_i - x_j) \cdot x_i \geqslant 0$ and that $|x_i - x_j| \leqslant 2R(t)$ for $j \neq i$. Assume that

$$\frac{k'(r)}{r} \text{ is decreasing for } r > 0. \tag{2.2}$$

Putting together the previous information, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t)^2 \leqslant -\frac{k'(2R(t))}{R(t)} \sum_{j \neq i} m_j (x_i - x_j) \cdot x_i.$$

Due to conservation of mass and centre of mass, we get

$$\sum_{j \neq i} (x_i - x_j) \cdot x_i m_j = \sum (x_i - x_j) \cdot x_i m_j = M |x_i|^2 = M R(t)^2,$$

and thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t) \leqslant -\frac{M}{2} k'(2R(t)). \tag{2.3}$$

Therefore, if the potential is such that the ODE $\mathrm{d}R/\mathrm{d}t = -M\, k'(2R)/2$ with initial data $R = R_0$ touches down to zero in finite time, then the particles aggregate in a single particle with the total mass M located at the centre of mass before the touch-down time of the ODE (2.3). This time is uniform for particles inside a fixed ball of radius R_0 initially with total mass M. This argument is inspired by and extends previous work in the control theory literature on cooperative motion with first order control laws involving pairwise interaction potentials (see [17] for the case of attractive–repulsive potentials and [19] for quadratic potentials). We make the argument rigorous in the following theorem.

Theorem 2.1 (Collapse of the ODEs). *Consider the ODE system (2.1) satisfying $k'(r)/r$ monotone decreasing, with $k''(r)$ defined and nonnegative on $(0, \infty)$. If K satisfies the Osgood condition (1.4) then there exists a unique global-in-time forward solution with no collisions, in which the particles converge to their centre of mass in infinite time. If K satisfies the non-Osgood condition (1.3) then there exists a unique global-in-time forward solution with collisions, in which the particles all merge at their centre of mass in finite time. In the latter case, for a given potential, the merger time is a function of the radius of support of the initial data and the total mass only.*

Proof. If K is an Osgood potential, then the modulus of continuity of the vector field on the right-hand side of the full ODE system satisfies the classical Osgood condition for local existence and uniqueness of solutions. In lemma A.15 postponed to the appendix, we estimate the modulus of continuity of the vector field $v = \nabla k(|x|)$ and show it is bounded by $2k'(2|x|)$, which means that local existence and uniqueness of solutions of the ODE system is satisfied if the Osgood condition (1.4) holds. This is a sufficient condition for the ODE problem to have a unique solution on compact sets in $\mathbb{R}^N$, see for instance [1, 3]. By the calculation above, the monotonicity condition implies that the particles remain inside a bounded set going forward in time, thereby implying that maximal C^1-solutions of the ODE can be extended globally in time, or in simple words, implying global existence and uniqueness of a smooth solution. Particle merger is not possible in finite time because otherwise we violate the uniqueness result for the ODE system, i.e. by interchanging the initial data for the particles that eventually merge, we obtain two solutions with different initial conditions that arrive at the same state in finite time, thereby violating backward-in-time uniqueness. Convergence to the centre of mass in infinite time results from the estimate (2.3).

If K is not an Osgood potential, then we are not guaranteed a unique solution of the ODE system. However, as long as particles are separated from each other, the ODE vector field is Lipschitz and implies unique solutions of the particle paths provided they remain separated and inside bounded sets of $\mathbb{R}^N$. So the only possibility to violate uniqueness of particle paths is collision of particles in finite time. We choose the unique solution going forward in time, corresponding to merger of any colliding particles into a single particle, at the time of collision. Note that this preserves both centre of mass and total mass at the time of collision. The system going forward in time, after merger, continues to preserve the centre of mass of the original ODE system. Moreover, since there is a finite number of initial particles, there can only be a finite number of merger times. Thus on any of the finite subintervals of time in which the particles stay separated, the ODE has a unique solution going forward in time. Moreover, by the estimate (2.3), all particles collapse to a single particle in finite time, with mass equal to the sum of the masses of the original L particles. □

Remark 2.2 (Non-uniqueness). We note that in the non-Osgood case, uniqueness of the ODE does not hold going backward in time because merger of particles destroys information. This is, in some sense, analogous to shocks in conservation laws, where information goes into the shock and is lost afterwards.

Remark 2.3 (The sign of k''). The proof does not require nonnegative curvature, but actually a lower bound on k'' if it goes negative. Specifically, we need to control $|k''(r)|$ by $|k'|/r$ in lemma A.15 in the appendix that estimates the modulus of continuity of $v = \nabla k(|x|)$. Following the details of the lemma, we see that the example $k(r) = 1 - e^{-r}$, although having negative curvature, still satisfies the estimates in lemma A.15 of the appendix, and thus satisfies the theorem.

In the next subsection we show how this collapsing support argument can be used to prove finite-time blow-up of the continuum problem in the case of non-Osgood potentials. We consider bounded initial data, therefore the characteristic paths are smoother than the point particle case considered in this subsection. However, we can still implement the estimate on the size of the support of the solution, proving finite-time blow-up of the continuum problem.

2.2. The continuum problem with bounded data and non-Osgood potentials

In this section we consider the well-posedness of the continuum problem with bounded data. We build primarily on the work of [5, 6, 23] and thus review the well-posedness results from those works. These papers establish the existence and uniqueness theory for (1.1) in dimensions two and higher, in the case of an *acceptable* potential satisfying the following criteria:

Definition 2.4. *The potential K on $\mathbb{R}^N$, $N \geqslant 2$ is acceptable if $\nabla K \in L^2(\mathbb{R}^N)$ and $\Delta K \in L^p(\mathbb{R}^N)$ for some $p \in [p^*, 2]$, where $\frac{1}{p^*} = \frac{1}{2} + \frac{1}{N}$. In the case of compactly supported initial data, we can take $\nabla K \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\Delta K \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$.*

We note that the typical kernels considered in this paper satisfy the acceptability condition. In particular, K Lipschitz satisfies ∇K bounded a.e. and thus in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, the most singular case at the origin is $\Delta K \sim \frac{1}{|x|}$, which satisfies the L^p condition above in dimensions two and higher. The case of one space dimension has special issues and we discuss that at the end of this section.

The continuum model assumes a nonnegative density $u(t, x)$ at position $x \in \mathbb{R}^N$ and time $t > 0$ satisfying

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + \mathrm{div}\, [u(t, x)v(t, x)] = 0 & t > 0, \quad x \in \mathbb{R}^N, \\ \text{with velocity field } v(t, x) := -\nabla K * u(t, x) & t > 0, \quad x \in \mathbb{R}^N, \\ u(0, x) = u_0(x) \geqslant 0 & x \in \mathbb{R}^N. \end{cases} \tag{2.4}$$

where v is the velocity field under which individuals in the swarm are moving obtained through the 'averaging' of the pairwise potential by the distribution of mass. The continuity equation comes from the assumption that the mass density of individuals in a set is preserved by the flow map or characteristics associated with the ODE system (2.1) determined by

$$\begin{cases} \dfrac{\mathrm{d}X(t, \alpha)}{\mathrm{d}t} = v(t, X(t, \alpha)) & t \geqslant 0, \\ X(0, \alpha) = \alpha & \alpha \in \mathbb{R}^N. \end{cases}$$

More precisely, if $X(t) : \mathbb{R}^N \to \mathbb{R}^N$ is the flow map for all $t \geqslant 0$ associated with the velocity field $v(t, x)$, $X(t)(\alpha) := X(t, \alpha)$ for all $\alpha \in \mathbb{R}^N$, then

$$\int_B u(t, x) \, \mathrm{d}x = \int_{X(t)^{-1}(B)} u_0(x) \, \mathrm{d}x$$

for any measurable set $B \subset \mathbb{R}^N$. In the optimal transport terminology, this is equivalent to saying that $X(t)$ *transports* the measure u_0 onto $u(t)$ and we denote it by $u(t) = X(t)\#u_0$ defined by

$$\int_{\mathbb{R}^N} \zeta(X(t, x))u(t, x) \, \mathrm{d}x = \int_{\mathbb{R}^N} \zeta(y)u_0(y) \, \mathrm{d}y \qquad \forall \zeta \in \mathcal{C}_b^0(\mathbb{R}^N). \qquad (2.5)$$

Throughout the rest of this section, the initial data are assumed to be bounded, compactly supported and nonnegative,

$$u_0 \in L^\infty(\mathbb{R}^N), \qquad \text{compactly supported} \quad \text{and} \quad u_0 \geqslant 0. \qquad (2.6)$$

We remark that (2.5) is a weak formulation for equation (2.4) whenever the flow map or characteristics are well defined and give homeomorphisms $X(t)$ from $\mathbb{R}^N$ onto itself, for instance when the velocity field $v(t, x)$ is globally Lipschitz.

It is clear that solutions of (2.4) formally preserve the total mass of the system

$$\int_{\mathbb{R}^N} u(t, x) \, \mathrm{d}x = \int_{\mathbb{R}^N} u_0(y) \, \mathrm{d}y := M \qquad (2.7)$$

and the centre of mass

$$\int_{\mathbb{R}^N} x \, u(t, x) \, \mathrm{d}x = \int_{\mathbb{R}^N} x \, u_0(y) \, \mathrm{d}y := M \, c_{\mathrm{M}}, \qquad (2.8)$$

where for the last one, we use that ∇K is anti-symmetric, $\nabla K(-x) = -\nabla K(x)$. We now review the well-posedness theory for smooth solutions.

Theorem 2.5 (Continuation theorem for H^s data [5]). *Given initial data $u_0 \in H^s(\mathbb{R}^N)$, $N \geqslant 2$, for positive integer $s \geqslant 2$, there exists a unique solution $u(x, t)$ of (2.4) and a maximal time interval of existence $[0, T^*)$ such that either $T^* = \infty$ or $\lim_{t \to T^*} \sup_{0 \leqslant \tau \leqslant t} \|u(\cdot, \tau)\|_{L^q} = \infty$. The result holds for all $q \geqslant 2$ for $N > 2$ and $q > 2$ for $N = 2$.*

When the kernel K is C^2, one can derive an *a priori* bound for u in L^∞ (see [23, 35]) thereby guaranteeing global existence of an H^s solution. Moreover, when the kernel has a Lipschitz point at the origin, for example the Morse potential $K_1(x) = 1 - \mathrm{e}^{-|x|}$, one can have finite-time blow-up. The proof in [6] uses the energy (1.2) and provides an *a priori* lower bound for E while simultaneously proving an *a priori* upper bound for the rate of decrease for the energy E when the data are radially symmetric and smooth. More recently, these results have been extended in [5] to the case of weak solutions with data in $L^1 \cap L^\infty(\mathbb{R}^N)$ and we state the result here. For this class of weak data, we must consider a weaker form of (2.4).

Definition 2.6. *A function u is a weak solution of (2.4) on [0,T] for nonnegative initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$, if it satisfies*

1. *$u \in L^\infty([0, T]; L^1(\mathbb{R}^N))$ and ess sup $_{0 \leqslant s \leqslant T} \|u(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} < \infty$.*
2. *(Weak differentiability in time.) Solutions must satisfy $u_t \in L^\infty([0, T]; V^*)$ and*

$$\int_0^T \langle u_t, \phi \rangle \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} (u - u_0)\phi_t \, \mathrm{d}x \, \mathrm{d}t = 0, \qquad (2.9)$$

for every test function $\phi \in W^{1,1}([0, T]; V)$ such that $\phi(T) = 0$. Here, $V = \{f \in L^\infty(\mathbb{R}^N) : \nabla f \in L^2(\mathbb{R}^N)\}$ endowed with the norm $\|f\|_V = \|f\|_{L^\infty(\mathbb{R}^N)} + \|\nabla f\|_{L^2(\mathbb{R}^N)}$.

(Weak solution of evolution equation.) Solutions satisfy a weak form of (2.4):

$$\int_0^T \langle u_t, \phi \rangle \, \mathrm{d}t - \int_0^T \int_{\mathbb{R}^N} \langle (\nabla K * u)u, \nabla \phi \rangle \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{2.10}$$

for all $\phi \in L^1([0, T]; V)$.

With mild decay conditions at infinity and the same conditions on the kernel K as above, we have local-in-time well-posedness of the problem and continuation of solutions. For simplicity we state the result for data with compact support.

Theorem 2.7 (Maximal time interval of existence for $L^1 \cap L^\infty$ data [5]). *Let u be a weak solution of (2.4) in R^N, $N \geqslant 2$, with K acceptable, and with compactly supported nonnegative initial data in L^∞. Then there exists a maximal time T^* and a unique weak solution u to the problem (2.4). Moreover if $T^* < \infty$ then*

$$\lim_{t \to T^*} \|u(\cdot, t)\|_{L^q(\mathbb{R}^N)} = \infty \qquad \text{for } q \in [2, \infty] \text{ if } N > 2 \quad \text{and} \quad q \in (2, \infty] \text{ if } N = 2.$$

Existence of solutions for L^∞ data is proved by constructing first the characteristics for the weak problem. This approach requires unique solutions to the characteristic equation, which requires a certain degree of regularity of the velocity field v. Provided u is bounded, it is shown in [5] that v is Lipschitz continuous and moreover $\nabla \cdot v$ is log-Lipschitz continuous (Lipschitz continuous) in dimension two (three and higher).

Since the mass of the solution is conserved on its interval of existence, another way to prove finite-time blow-up is to derive an estimate for the size of the support of the solution. If an upper bound for the size of the support shrinks to zero in finite time, this also guarantees that the time interval of existence of the L^∞ solution is less than infinity. In this section, we show how to extend the analysis from the ODE case to the continuum problem.

Lemma 2.8 (Center of mass conservation). *For compactly supported weak solutions corresponding to nonnegative bounded compactly supported initial data, the first moment is conserved.*

Proof. It suffices to substitute $\phi(x) = x_i \Psi(x)$ in definition 2.6 where Ψ is a cutoff function outside the support of u. The rest follows by symmetrizing the convolution integral and noting that ∇K is an odd function. $\square$

Proposition 2.9 (Frozen-in-time velocity estimate). *Assume $k'(r)/r$ is a monotone decreasing function of r. Consider $u(y)$ is a nonnegative function with total mass M, first moment zero and compact support. Consider any $B_R(0)$ containing the support of u. Then, for any $x \in \partial B_R(0)$ we have*

$$v(x) \cdot x \leqslant -\frac{k'(2R)R}{2} M \leqslant 0.$$

Proof. The argument is along the lines of the discrete particle estimates in the previous section. We have the velocity field given by

$$v(x) = -\int_{\mathbb{R}^N} \nabla K(x - y)u(y) \, \mathrm{d}y$$

and thus,

$$
\begin{aligned}
v(x) \cdot x &= -\int_{\mathbb{R}^N} x \cdot \nabla K(x - y) u(y) \, \mathrm{d}y \\
&= -\int_{\mathbb{R}^N} \frac{x \cdot (x - y)}{|x - y|} k'(|x - y|) u(y) \, \mathrm{d}y \\
&\leqslant -\frac{k'(2R)}{2R} \int_{\mathbb{R}^N} x \cdot (x - y) u(y) \, \mathrm{d}y \\
&= -\frac{k'(2R)}{2R} (M R^2) = -\frac{k'(2R)}{2} M R.
\end{aligned}
$$

Here we use the fact that $x \cdot (x - y) \geqslant 0$ and $|x - y| \leqslant 2R$ for any y in the support of $u \geqslant 0$ and $x \in \partial B_R(0)$, along with the monotonicity constraints on k' and the moments of u. $\qquad\square$

The above proposition is now used to prove the following theorem. This is a generalization of [6, theorem 6] and [5, theorem 6.2] to the case of less singular kernels satisfying (1.4) and the monotonicity conditions in proposition 2.9. Also, significantly, the radial symmetry of the initial data, required in the proofs from [5, 6], is no longer necessary.

Theorem 2.10 (Finite time blow-up for compactly supported solution in L^∞). *Let u be a weak solution of (2.4) with nonnegative compactly supported initial data in $L^\infty(\mathbb{R}^N)$. Let K satisfy conditions (1.4) and $k'(r)/r$ monotone decreasing, $k'(r) > 0$. Then there exists a maximal time $T^* < \infty$ and a unique weak solution u to the problem (2.4) on the interval $[0, T^*]$. Moreover*

$$
\lim_{t \to T^*} \|u(\cdot, t)\|_{L^q(\mathbb{R}^N)} = \infty \qquad \text{for } q \in [2, \infty] \quad \text{if } N > 2 \text{ and } q \in (2, \infty] \quad \text{if } N = 2.
$$

Proof. Given the existing continuation theorem, it suffices to prove that the solution ceases to exist in finite time. To do that, we prove a comparison principle for the support of the solution:

Proposition 2.11 (Comparison principle). *Let $u(x, t)$ be the weak solution in theorem 2.10. Let $B_{R_0}(c_M)$ contain the support of the solution at time zero. Let $\tilde{R}(t)$ be the unique solution of the ordinary differential equation $\mathrm{d}R/\mathrm{d}t = -Mk'(2R)/2$. On any time interval of existence of the $L^1 \cap L^\infty(\mathbb{R}^N)$ solution $u(x, t)$, the support of u must lie inside $B_{\tilde{R}(t)}(c_M)$.*

Proof. Without loss of generality, we move the coordinate system to have centre of mass at $c_M = 0$ (the translation invariance of the problem allows for this). We note that at time zero, the characteristics $X(t, x)$ for any x on the boundary of the ball $B_{R_0}(0)$ satisfy $\mathrm{d}R/\mathrm{d}t \leqslant -Mk'(2R)/2$ where $R(t) = |X(t, x)|$. Thus the characteristics are shrinking towards the origin faster than the boundary of the ball. This estimate only holds on the boundary of the ball, not in the interior. However, on any time interval of existence of the bounded solution, we have unique characteristic paths that are Lipschitz continuous in space and time. Moreover they cannot cross, by the classical Picard theory for ODEs. This guarantees that none of the characteristics associated with the support of u will exit the ball. Since the ball continues to shrink at a rate less than the characteristic speed on the boundary, the characteristics associated with the support of u remain inside the shrinking ball of radius $\tilde{R}(t)$ on any time interval of existence of the weak solution. Another way to look at this comparison would be to recast the problem in a moving frame with coordinates shrinking at the uniform rate associated with the ODE for $\tilde{R}(t)$. Then the ball stays fixed and becomes an absorbing ball

for the ODE associated with the characteristics. This concludes the proof of the proposition and of theorem 2.10.   □

We briefly discuss the aggregation equation in one space dimension. This case is somewhat special. First of all, k'' plays an important role in the blow-up dynamics because the amplification factor for u along characteristics is $k'' * u$. In the case of a kernel with a Lipschitz point at the origin, one has finite-time blow-up because k'' is a delta to leading order, and thus the blow-up is driven by a quadratic function of u. This argument was presented in [21] and made rigorous in [10]. For power-law potentials smoother than the Lipschitz case, one can read the paper by Li and Toscani [27]. In both [10, 27] the transformation $w = \int_{-\infty}^{x} u$ is used resulting in a non-local scalar conservation law for w:

$$w_t - k'' * w w_x = 0. \tag{2.11}$$

We now see that when k'' is a delta, the problem reduces to Burgers equation and the blow-up is simply shock formation in w. For more regular kernels than Lipschitz, one needs an existence theory. The work of [23] proves local existence in one dimension for sufficiently smooth initial data and kernels satisfying $k'' = C\delta + P$, where P is L^1. To the best of our knowledge, the full existence and continuation theory for general K and bounded initial data, in 1D, has not been derived in the most general setting—however, the *a priori* bounds presented in this section still hold and apply to this problem. For completeness, we remind the reader that in dimensions two and higher, if the kernel K has a Lipschitz point at the origin, then $\Delta K *$ as a convolution operator provides additional smoothness (typically a gain of $N - 1$ derivatives in dimension N), that is lacking in one dimension.

2.3. Global existence of solutions for Osgood potentials

In this section we prove global existence of solutions for the case of Osgood potentials satisfying monotonicity conditions. To do this, we obtain refined estimates on the L^∞-norm of $\nabla \cdot v$. Note first that the Osgood condition is more general than $K \in C^2$. For example, $K(x) = |x|^2 |\ln|x||$ satisfies this condition. Moreover, one does not, in general, have boundedness of $\nabla \cdot v$ whenever the density is given by a general nonnegative measure μ, so that one has to rely on the nonlinearity in the evolution equation to provide an *a priori* bound for $\|u\|_{L^\infty}$. For example, if v is log-Lipschitz then the modulus of continuity ρ only guarantees particle paths that are Hölder continuous, which is insufficient to guarantee that they would map L^∞ densities to L^∞. Instead, we have to examine the evolution equation and use the fact that a smoother density yields a more regular velocity field.

Remark 2.12 (Density argument). In this section we derive some differential inequalities for $\|u\|_{L^\infty}$. Rather than writing the integral form, which would be more appropriate for weak solutions, for simplicity of notation we compute the differential form for strong solutions. However, the results give *a priori* bounds that are uniform for smooth approximations of the weak solutions. By the construction of the weak solutions in [5], namely by smoothing the initial data but not the kernel, the same bounds are inherited by the weak solutions.

We begin by reviewing the C^2 case, which has already been studied in the literature. Along the characteristics $X(t, \alpha)$ associated with $v(t, x)$, we have $u_t + v \cdot \nabla u = -u \nabla \cdot v$, and this holds in the integral form [5], for the case of L^∞-weak solutions. Thus, by taking the L^∞-norm along all characteristics, we have a bound on the time evolution of $\|u\|_{L^\infty}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^\infty} \leqslant \|\Delta K * u\|_{L^\infty} \|u\|_{L^\infty}. \tag{2.12}$$

In the case where K is C^2, we immediately get that

$$\|\Delta K * u\|_{L^\infty} \leqslant \|\Delta K\|_{L^\infty} \|u\|_{L^1},$$

which is *a priori* bounded and thus, by Grönwall's lemma, gives a global bound for $\|u\|_{L^\infty}$. Combining this with theorem 2.7 provides the following result (the *a priori* bound has been proved in [35]):

Theorem 2.13 (Global-in-time solutions for C^2 potentials). *Let K be an admissible C^2 kernel. Then the weak solution of theorem 2.7 exists for all time and we have a global-in-time bound*

$$\|u(\cdot, t)\|_{L^\infty} \leqslant e^{Ct} \|u(\cdot, 0)\|_{L^\infty},$$

where C depends on $\|\Delta K\|_{L^\infty}$ and the mass of u.

We also obtain the following corollary of the previous section:

Corollary 2.14 (Infinite time blow-up for C^2 potentials). *Let K be an admissible C^2 kernel satisfying the conditions of proposition 2.9. If the global-in-time weak solution of theorem 2.13 has compact support, then it converges to a Dirac mass at the centre of mass c_M as $t \to \infty$.*

Proof. The proof follows by applying proposition 2.11 to the global solution and noting that the solution $\tilde{R}$ of the ODE goes to zero as $t \to \infty$. □

We now show that the same result holds for other potentials satisfying the Osgood condition

$$\int_0^1 \frac{1}{k'(r)} \, dr = \infty$$

that are not C^2. Consider, for example, the aggregation equation with v log-Lipschitz. This is slightly more singular than C^2, yet it satisfies the Osgood condition. To construct a global potential that also satisfies the conditions of proposition 2.9 we consider

$$k''(r) = -(\ln r)\phi(r) + 1 - \phi(r), \tag{2.13}$$

$$k'(r) = \int_0^r k''(r), \tag{2.14}$$

where ϕ is a C_0^∞ cutoff function supported inside a ball of radius one at the origin. This potential satisfies the conditions of proposition 2.9, namely $k'(r)/r$ is monotone decreasing and has a logarithmic singularity at the origin, similar to k''. Moreover, this potential is slightly more singular than C^2, yet it satisfies the Osgood condition.

Theorem 2.15 (Log-quadratic potential). *Consider the aggregation equation with K as defined in (2.13), (2.14). Then the weak solution of theorem 2.7 exists for all time with an a priori bound*

$$\|u(\cdot, t)\|_{L^\infty} \leqslant C e^{C e^{Ct}},$$

where the constant C depends on the dimension of space N, the mass M and the initial L^∞-norm.

Proof. To prove this result, we need a refined estimate on $\Delta K * u$, since ΔK is no longer bounded. The idea is to derive a potential theory estimate that is log-linear in $\|u\|_{L^\infty}$ where we split the convolution up into two pieces, one near the singularity in ΔK and one far from that:

$$|\Delta K * u(x)| \leqslant \int_{|x-y|<\delta} |\Delta K(x-y)| u(y) \, dy + \int_{|x-y|\geqslant \delta} |\Delta K(x-y)| u(y) \, dy = T_1 + T_2.$$

We estimate

$$T_1 \leqslant \|\Delta K\|_{L^1(B_\delta(0))} \|u\|_{L^\infty}$$

and

$$T_2 \leqslant \Delta K(\delta) \|u\|_{L^1}.$$

Using the fact that $|\Delta K(x)| \leqslant k''(r) + (N-1)\frac{k'(r)}{r} \leqslant C(N)(1 + |\ln r|)$, we estimate

$$T_1 \leqslant C(N)\delta^N(1 + |\ln(\delta)|)\|u\|_{L^\infty}$$

and

$$T_2 \leqslant C(N)(1 + |\ln(\delta)|)\|u\|_{L^1}.$$

Choosing $\delta = (M/\|u\|_{L^\infty})^{1/N}$, which has dimensions of length, we have

$$T_1 \leqslant C(N, M)(1 + |\ln \|u\|_{L^\infty}|) \tag{2.15}$$

and

$$T_2 \leqslant C(N, M)(1 + |\ln \|u\|_{L^\infty}|). \tag{2.16}$$

Combining (2.15) and (2.16) with (2.12) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^\infty} \leqslant C(N, M)\|u\|_{L^\infty}(1 + |\ln \|u\|_{L^\infty}|),$$

where $C(N, M)$ depends only on the dimension of space N and the total mass of the solution M. A standard Grönwall argument results in an *a priori* bound for $\|u(\cdot, t)\|_{L^\infty} \leqslant C\mathrm{e}^{C\mathrm{e}^{ct}}$. Here the constants depend only on M and the initial L^∞-norm. Thus, we have an *a priori* bound for the L^∞ norm of u on any time interval of existence. The continuation theorem guarantees a global-in-time weak L^∞ solution. $\qquad\square$

Moreover, as in the case of C^2 kernel, we have convergence to a Dirac delta mass at the centre of mass as $t \to \infty$, as in corollary 2.14.

Corollary 2.16 (Infinite time blow-up for log-quadratic potentials). *For the kernel K as defined in (2.13), (2.14), any nonnegative L^∞-weak solution with compact support converges in infinite time to a Dirac delta with mass M located at the centre of mass c_M of the solution.*

Proof. The proof follows the arguments above, noting that this particular kernel yields global-in-time solutions, but since k satisfies the conditions of proposition 2.9, the support of the solution stays inside a ball of radius R satisfying the ODE $\dot{\tilde{R}} = -k'(2\tilde{R})M/2$. Thus we obtain an upper bound on the radius of the support of u, which vanishes as $t \to \infty$. Thus u must converge to a Dirac delta in infinite time. $\qquad\square$

We now show that this special case of the log-quadratic potential can be extended to any Osgood potential satisfying the conditions of proposition 2.9 that is smooth away from the origin.

Theorem 2.17. *(Global-in time L^∞ and infinite time blow-up for Osgood potentials) Assume $k''(r) > 0$ and that $k'(r)/r$ monotone decreasing in r. Then on the interval of existence $(0, T^*)$*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^\infty}^{-1/N} \geqslant -C(N, M)\, k'(M^{1/N}\|u\|_{L^\infty}^{-1/N}) \tag{2.17}$$

holds. As a consequence, if K satisfies the Osgood condition (1.4) then for any compactly supported nonnegative L^∞ solution of the aggregation equation stays bounded for all time and converges as $t \to \infty$ to a Dirac mass of size M located at its centre of mass c_M.

Proof. The proof follows the argument of the log-quadratic case in theorem 2.15. The fact that $k'(r)/r$ is monotone decreasing means that $k''(r)/r - k'(r)/r^2 \leqslant 0$ away from the origin and thus $0 < k''(r) \leqslant k'(r)/r$ away from the origin. More specifically, we have

$$T_1 \leqslant \|\Delta K\|_{L^1(B_\delta(0))} \|u\|_{L^\infty} \leqslant C \int_0^\delta |k'(r)| r^{N-2} \, \mathrm{d}r \, \|u\|_{L^\infty}.$$

Note that $k''(r) > 0$ implies $k'(r)$ is increasing. Therefore $\sup_{[0,\delta]}(k'(r)) = k'(\delta)$. Thus, in dimension $N \geqslant 2$

$$T_1 \leqslant C k'(\delta) \int_0^\delta r^{N-2} \, \mathrm{d}r \|u\|_{L^\infty} \leqslant C \frac{k'(\delta)}{\delta} \delta^N \|u\|_{L^\infty}.$$

In one space dimension, $1/r$ is not integrable, however the original integral can be written directly in terms of k'' as $\int_0^\delta |k''| \, \mathrm{d}r \leqslant k'(\delta)$. Also

$$T_2 \leqslant C \frac{k'(\delta)}{\delta} \|u\|_{L^1}.$$

As in the proof of theorem 2.15, we choose $\delta = (M/\|u\|_{L^\infty})^{1/N}$ so that

$$T_1 + T_2 \leqslant C(N, M) \, k'[(M/\|u\|_{L^\infty})^{1/N}] \, (\|u\|_{L^\infty}/M)^{1/N}.$$

We have the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^\infty} \leqslant C(N, M) \, k'[(M/\|u\|_{L^\infty})^{1/N}] \, (\|u\|_{L^\infty}/M)^{1/N} \|u\|_{L^\infty}.$$

Dividing by $\|u\|_{L^\infty}^{1+1/N}$ gives the estimate in (2.17). The long-time behaviour of the solution follows from the same arguments used to prove corollaries 2.14 and 2.16. $\qquad\square$

Remark 2.18 (General Osgood potentials). The global existence result in theorem 2.17 also holds for more general potentials satisfying the Osgood condition. For example, we can take k'' bounded from below by a negative constant, thereby allowing kernels that change curvature outside some neighbourhood of the origin. Very general kernels can be considered that are C^2 outside some neighbourhood and satisfy the more strict assumptions inside a neighbourhood. The monotonicity of $k'(r)/r$ is used in the global existence argument in order to bound k'' by $k'(r)/r$ in the estimates. Since this is a pointwise estimate for the growth rate of u, we need to control the second derivative of k since it explicitly appears in the formula for the rate of growth of u along characteristics. Of course, to prove convergence to the delta in infinite time, the monotonicity condition is required everywhere on the support of u since it plays a key role in the estimate of the size of the support.

3. Radially symmetric solutions

In this section we consider the special case of radially symmetric solutions. The first part of the section refines the blow-up arguments of section 2.2 to include weaker monotonicity conditions on the kernel. This can be accomplished by writing the problem in polar coordinates. In the second part of this section we consider the special case of similarity solutions for the most well-known model of $K(x) = |x|$. In this case we are able to derive some exact solutions. We show that the problem in higher dimensions is very different from the one-dimensional problem. Similarity solutions are often used to understand the structure of the finite-time blow-up. In this section we prove that in odd space dimension higher than one, there are no radially symmetric similarity solutions with compact support on an open set of positive Lebesgue measure. Likewise there are no L^p symmetric similarity solutions with support

containing an open set ($1 < p < \infty$). The example solutions that we are able to construct all involve collapsing delta-rings in dimension higher than one. This raises an interesting question of what the blow-up profile for finite-time singularities in L^∞ solutions looks like near the blow-up time, in higher dimensions.

3.1. Finite time blow-up in the radial case

In section 2, in order to obtain 'Osgood-like' estimates, it was necessary that the potential $K(x) = k(|x|)$ satisfies the monotonicity condition:

$$\frac{k'(r)}{r} \text{ is decreasing.}$$

In this section we obtain 'Osgood-like' estimates under the milder monotonicity condition:

$$k'(r) \text{ is monotone in a neighbourhood of zero.}$$

The price to pay in order to consider this more general class of potential is that we now have to work with a smaller class of solutions, i.e. radially symmetric solutions. Interestingly, the techniques used in this section, despite the fact that they lead to similar 'Osgood-like' estimates, are of a different nature from the one used in section 2. Let us summarize the assumptions on the potential $K(x)$ in this section:

(i) K is radially symmetric $K(x) = k(|x|)$.
(ii) $k : [0, +\infty) \to [0, +\infty)$ is smooth away from 0 and $k(0) = 0$.
(iii) $k'(r) \geqslant 0$ and the inequality is strict when $r > 0$.
(iv) There exists a neighbourhood $[0, \delta)$ of 0 on which k' is monotonic.
(v) K is acceptable in the sense of definition 2.4.

Our main result in this section is the following theorem:

Theorem 3.1 (Blow-up: radial case). *Consider equation (2.4) where the potential K satisfies (i)–(v) together with*

$$\int_0^1 \frac{1}{k'(r)} \, dr < +\infty.$$

Suppose the initial data are radially symmetric, compactly supported and bounded. Then there exists a finite time T^* such that the unique weak solution $u(x, t)$ of (2.4) satisfies

$$\lim_{t \to T^*} \sup_{0 \leqslant \tau < t} \|u(\cdot, \tau)\|_{L^q} = +\infty$$

for all $q \geqslant 2$ ($q > 2$ for $N = 2$).

The proof of theorem 3.1 relies on two key ingredients. The first one is the continuation theorem 2.7. The second one is a refined estimate of the speed at which the support of radially symmetric solutions shrinks (theorem 3.5). This estimate shows that the radius of the support becomes zero in finite time, and thus, using the continuation theorem, the L^q norm must blow up.

The rest of the section is devoted to the proof of this theorem. For $r, \rho \in (0, +\infty)$, define $\psi(r, \rho)$ as follows (figure 1):

$$\psi(r, \rho) := \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0,\rho)} \nabla K(re_1 - y) \cdot e_1 \, d\sigma(y)$$

$$= \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0,\rho)} k'(|re_1 - y|) \frac{re_1 - y}{|re_1 - y|} \cdot e_1 \, d\sigma(y), \qquad (3.1)$$

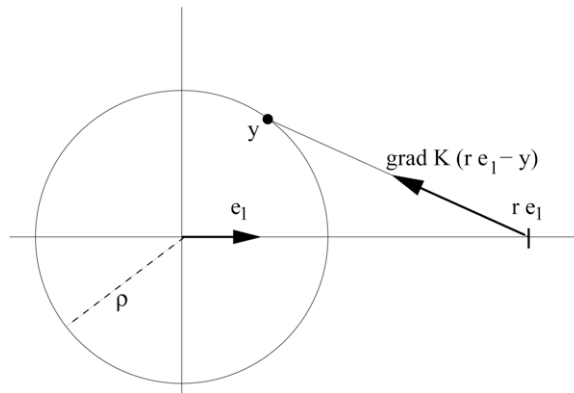where e_1 is the first vector of the canonical basis of $\mathbb{R}^N$.

Figure 1. Geometric interpretation of the function ψ.

Lemma 3.2 (Convolution in polar coordinates). *Suppose u is radially symmetric: $u(x) = \tilde{u}(|x|)$. Define $\hat{u}$ as follows:*

$$\hat{u}(r) = \omega_N r^{N-1} \tilde{u}(r). \tag{3.2}$$

Then

$$(u * \nabla K)(x) = \left(\int_0^{+\infty} \psi(|x|, \rho)\hat{u}(\rho) \, d\rho \right) \frac{x}{|x|}. \tag{3.3}$$

Remark 3.3 (Polar coordinates). *One can easily check that in polar coordinates, the aggregation equation becomes*

$$\begin{cases} \dfrac{d}{dt}\hat{u} + \dfrac{d}{dr}(\hat{u}\hat{v}) = 0, \\[2mm] \hat{v}(r,t) = -\displaystyle\int_0^{+\infty} \psi(r, \rho)\hat{u}(\rho, t) \, d\rho. \end{cases} \tag{3.4}$$

Note that $\hat{u}(r, t)$ has been defined such that

$$\int_{r_1}^{r_2} \hat{u}(r) \, dr = \int_{r_1 \leqslant |x| \leqslant r_2} u(x) \, dx.$$

Proof of lemma 3.2.

$$(u * \nabla K)(x) = \int_{\mathbb{R}^N} u(y)\nabla K(x - y) \, dy$$

$$= \int_0^{+\infty} \tilde{u}(\rho) \left(\int_{\partial B(0,\rho)} \nabla K(x - y) \, d\sigma(y) \right) d\rho$$

$$= \int_0^{+\infty} \hat{u}(\rho) \left(\frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0,\rho)} \nabla K(x - y) \, d\sigma(y) \right) d\rho. \tag{3.5}$$

Since K is radially symmetric, it is not difficult to see that the vector

$$a(x) = \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0,\rho)} \nabla K(x - y) \, d\sigma(y)$$

is parallel to x. Moreover, since by definition $a(x)$ is rotationally invariant, there exists a function $b(r)$ such that
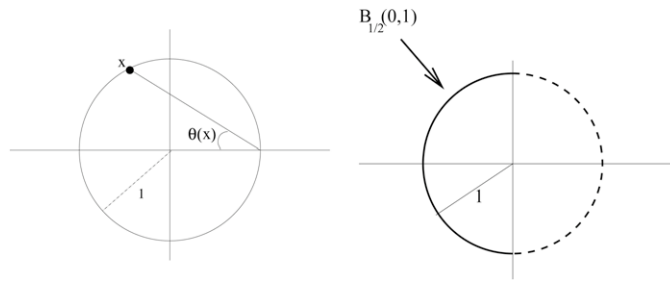
$$a(x) = b(|x|)\frac{x}{|x|}. \tag{3.6}$$

Figure 2. Geometric interpretation of α_N and β_N.

Plugging $x = r\,e_1$ in (3.6) and taking the scalar product with e_1, one easily finds that

$$b(r) = a(re_1) \cdot e_1 = \psi(r, \rho). \tag{3.7}$$

Collecting (3.5) and (3.7), we obtain (3.3). □

Our next goal is to derive estimates for $\psi(r, \rho)$. Let us define the following constants:

$$\alpha_N = \frac{1}{\omega_N} \int_{\partial B(0,1)} \frac{e_1 - x}{|e_1 - x|} \cdot e_1 \, \mathrm{d}\sigma(x) = \frac{1}{\omega_N} \int_{\partial B(0,1)} \cos(\theta(x)) \, \mathrm{d}\sigma(x)$$

$$\beta_N = \frac{1}{\omega_N} \int_{\partial B_{1/2}(0,1)} \frac{e_1 - x}{|e_1 - x|} \cdot e_1 \, \mathrm{d}\sigma(x) = \frac{1}{\omega_N} \int_{\partial B_{1/2}(0,1)} \cos(\theta(x)) \, \mathrm{d}\sigma(x),$$

where $\partial B_{1/2}(0, \rho)$ is the half sphere of radius ρ, i.e. $\partial B_{1/2}(0, \rho) = \{x \in \partial B(0, \rho) : x_1 \leqslant 0\}$ and $\theta(x)$ is defined in figure 2. It is easy to check that $0 < \beta_N < \alpha_N < 1$.

Lemma 3.4 (Estimate for $\psi(r, \rho)$). *Suppose the potential K satisfies (i)–(iv)*

(a) If $k'(r)$ is decreasing in the interval $[0, \delta)$, then

$$\psi(r, \rho) \geqslant \alpha_N \, k'(2r) \qquad \text{for all } \rho < r < \frac{\delta}{2} \tag{3.8}$$

(b) If $k'(r)$ is increasing in the interval $[0, \delta)$, then

$$k'(2r) \geqslant \psi(r, \rho) \geqslant \beta_N \, k'(r) \qquad \text{for all } \rho < r < \frac{\delta}{2} \tag{3.9}$$

(c) For any $R_1 > 0$, given $\frac{\delta}{2} \leqslant r \leqslant R_1$, then

$$\psi(r, \rho) \geqslant \beta_N \inf_{r \in [\frac{\delta}{2}, 2R_1]} k'(r). \tag{3.10}$$

Proof. Let us write the function ψ as

$$\psi(r, \rho) = \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0,\rho)} k'(|re_1 - y|) \, F(r, y) \, \mathrm{d}\sigma(y), \tag{3.11}$$

where the function F is defined by

$$F(r, y) = \frac{re_1 - y}{|re_1 - y|} \cdot e_1$$

whenever $|re_1 - y| \neq 0$. If y is parallel to e_1, then the function F is defined by

$$F(r, y) = \frac{r - y_1}{|r - y_1|} = \begin{cases} 1 & \text{if } r > y_1 \\ 0 & \text{if } r = y_1 \\ -1 & \text{if } r < y_1. \end{cases}$$

Before proving the estimates on ψ, let us verify some simple properties of F:

(Fa) $-1 \leqslant F(r, y) \leqslant 1 \ \forall (r, y) \in \mathbb{R} \times \mathbb{R}^N$,

(Fb) $F(r, y) \geqslant 0$ if $r \geqslant y_1$,

(Fc) $F(\rho, y) \leqslant F(r, y)$ if $\rho \leqslant r$.

If y is parallel to e_1 then (Fa), (Fb) and (Fc) are obvious. Assume y is not parallel to e_1. Then $F(r, y)$ can be written

$$F(r, y) = \frac{r - y_1}{\sqrt{(r - y_1)^2 + c^2}} \qquad \text{where } c = \left(\sum_{i=2}^{d} y_i^2 \right)^{1/2} > 0. \qquad (3.12)$$

From this (Fa) and (Fb) are obvious. Then write

$$\frac{1}{|F(r, y)|^2} = 1 + \frac{c^2}{(r - y_1)^2}.$$

From this it is clear that, as a function of r, $F_y(r) := F(r, y)$ decreases on $(-\infty, y_1)$ and increases on $(y_1, +\infty)$. On the other hand, $F_y(r)$ is negative on $(-\infty, y_1)$ and positive on $(y_1, +\infty)$. Combining the two, we obtain that $F_y(r)$ is increasing on R. This proves (Fc).

Let us prove statement (a). Using (Fb) we see that, since $\rho < r$, $F(r, y) \geqslant 0$ for all $y \in \partial B(0, \rho)$. Moreover $|re_1 - y| \leqslant 2r$. Combining them together with the fact that k' is decreasing on $[0, \delta]$, we obtain that if $r < \frac{\delta}{2}$ then

$$\psi(r, \rho) \geqslant k'(2r) \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0, \rho)} F(r, y) \, d\sigma(y).$$

Using (Fc) we then get that

$$\frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0, \rho)} F(r, y) \, d\sigma(y) \geqslant \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0, \rho)} F(\rho, y) \, d\sigma(y) = \alpha_N,$$

where the last identity has been obtained by doing the change of variable $z = \rho y$.

Let us prove statements (b) and (c). For the same reason as in the proof of (a), $F(r, y) \geqslant 0$ for all $y \in \partial B(0, \rho)$. Moreover $k' \geqslant 0$ so the integrand in (3.11) is positive and we have

$$\psi(r, \rho) \geqslant \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B_{1/2}(0, \rho)} k'(|re_1 - y|) F(r, y) \, d\sigma(y). \qquad (3.13)$$

Note that $|re_1 - y| \geqslant |r - y_1|$. Therefore, since $y_1 \leqslant 0$ for all $y \in \partial B_{1/2}(0, \rho)$ we have $|re_1 - y| \geqslant r$. Moreover, $|re_1 - y| \leqslant 2r$ for all $y \in \partial B_{1/2}(0, \rho)$. So combining the two inequalities, we obtain

$$r \leqslant |re_1 - \rho y| \leqslant 2r \qquad \text{for all } y \in \partial B_{1/2}(0, \rho) \text{ and for all } \rho < r.$$

Combining this with (3.13) we obtain

$$\psi(r, \rho) \geqslant \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B_{1/2}(0, \rho)} F(r, y) \, d\sigma(y) \min_{s \in [r, 2r]} k'(s)$$

$$\geqslant \beta_N \min_{s \in [r, 2r]} k'(s), \qquad (3.14)$$

where once again we have used the fact that $F(r, y) \geqslant F(\rho, y)$ and then the change of variable $z = \rho y$. Inequality (3.10) and the right side of inequality (3.9) are a direct consequence of (3.14). We are then left to prove that $\psi(r, \rho) \leqslant k'(2r)$ for all $\rho < r < \frac{\delta}{2}$. Since k' is increasing on $[0, \delta)$, we have

$$k'(|re_1 - y|) \leqslant k'(2r) \qquad \text{for all } y \in \partial B(0, \rho) \text{ and for all } \rho < r$$

so

$$\psi(r, \rho) \leqslant k'(2r) \frac{1}{\omega_N \rho^{N-1}} \int_{\partial B(0, \rho)} F(r, y) \, d\sigma(y).$$

To conclude, one just needs to use the bound $F(r, y) \leqslant 1$ from (Fa). $\qquad \square$

Theorem 3.5 (Estimate of the size of the support). *Consider equation (2.4) where the potential K satisfies (i)–(v) and where the initial data are radially symmetric, compactly supported and bounded. Let $u(x, t)$ be the unique weak solution defined on $[0, T^*)$, where T^* is the maximal time of existence. Let $R(t)$ be the radius of the support of $u(\cdot, t)$ with R_0 its initial value. Then*

(a) If k' is increasing in $[0, \delta)$, there exists a finite time $T_1 \in [0, T^*)$ depending on δ, k and R_0 such that

$$- Mk'(2R(t)) \leqslant R'(t) \leqslant -M\beta_N k'(R(t)) \qquad \forall t \in [T_1, T^*) \qquad (3.15)$$

(b) If k' is decreasing in $[0, \delta)$, there exists a finite time $T_1 \in [0, T^*)$ depending on δ, k and R_0 such that

$$R'(t) \leqslant -M\alpha_N k'(2R(t)) \qquad \forall t \in [T_1, T^*) \qquad (3.16)$$

Proof. From lemma 3.2 it is clear that

$$R'(t) = - \int_0^{+\infty} \psi(R(t), \rho)\hat{u}(\rho, t)\, \mathrm{d}\rho$$

$$= - \int_0^{R(t)} \psi(R(t), \rho)\hat{u}(\rho, t)\, \mathrm{d}\rho. \qquad (3.17)$$

Note that $\rho \leqslant R(t)$, this allows us to use lemma 3.4. We also observe that $R'(t) \leqslant 0$ resulting from $\psi(r, \rho) \geqslant 0$ for all $\rho \leqslant r$ due to (Fb) together with $k' \geqslant 0$.

Assume first that $R_0 \in [0, \frac{\delta}{2})$. Since $R(t)$ is decreasing, $R(t)$ will stay in $[0, \frac{\delta}{2})$ and we easily get (3.15) and (3.16) from (3.17) together with parts (a) and (b) of lemma 3.4.

Assume next that $R_0 > \frac{\delta}{2}$. Then, since R is decreasing, $R(t)$ belongs to $\left[\frac{\delta}{2}, R_0\right]$ for some time interval $[0, T_1]$ where T_1 might be finite or infinite. Combining (3.17) with part (c) of lemma 3.4, we get

$$R'(t) \leqslant -M\beta_N \inf_{r \in \left[\frac{\delta}{2}, 2R_0\right]} k'(r) \qquad \forall t \in [0, T_1).$$

Since $k'(r) > 0$ for $r > 0$, this tells us that $R(t)$ decreases at least linearly when $t \in [0, T_1)$, therefore $R(t)$ will be smaller than $\frac{\delta}{2}$ in finite time (i.e. $T_1 < \infty$). In this way, we are reduced to the previous case since $R(T_1) \in [0, \frac{\delta}{2})$. $\qquad \square$

We are now ready to prove the blow-up theorem.

Proof of theorem 3.1. Suppose first that $k'(r)$ increases on $[0, \delta)$ and let T_1 and $R(t)$ be as in theorem 3.5. Consider the ODE

$$\begin{cases} y'(t) = -M\beta_N k'(y(t)) \\ y(T_1) = R(T_1) \end{cases}$$

Since the Osgood condition is violated, i.e.

$$\int_0^1 1/k'(r)\, \mathrm{d}r < \infty,$$

we know that there exists $T^{**} < \infty$ at which y reaches 0. Using part (a) of the previous theorem we see that the maximal time of existence T^* must be less than T^{**}. If it were not the case, then at $t = T^{**}$, the support of $u(\cdot, t)$ would be reduced to a point, therefore $u(\cdot, t)$ would not be in $L^1 \cap L^\infty$, which contradicts the fact that u is a weak solution of (2.4) on $[0, T^*)$. Thus, we have $T^* \leqslant T^{**} < \infty$. We can then use the continuation theorem 2.7 to get the blow-up of the L^q-norm. The proof for the case where $k'(r)$ decreases in $[0, \delta)$ is exactly similar. $\qquad \square$

3.2. Self-similar solutions for $K = |x|$

In this final section, we explore the possibility of describing more in detail the finite-time blow-up proved in previous sections. With this purpose, we focus on finding certain blow-up self-similar solutions of (1.1) with homogeneous potentials. Suppose $K(x)$ is a potential with a Lipschitz point at the origin:

$$K(x) \sim C|x| \qquad \text{as } x \to 0 \tag{3.18}$$

and suppose u is a solution of (1.1) which blows up at $t = T^*$. We choose this special kernel for the following reasons: (1) kernels with Lipschitz points are one of the most common examples in the aggregation literature, (2) the special homogeneity of this kernel simplifies some of the analysis and (3) when the blow-up occurs at a point, it is only the local structure of the kernel at the origin that is important and, moreover, to do similarity analysis we take it to be homogeneous. Close to the blow-up time, one would expect u to have small support (or at least to be highly concentrated). Therefore the velocity can be well approximated by

$$v = -Cu * \nabla|x|.$$

From this remark, one would expect that the blow-up profile of (1.1) with a potential $K(x)$ satisfying (3.18) can be well approximated by the blow-up profile of (1.1) with $K(x) = C|x|$.

3.2.1. One dimension.
Let $U(x)$ be the uniform distribution on $[-1, 1]$ and define the rectangle-like similarity solution

$$u_S(x, t) := \frac{1}{R(t)} U\left(\frac{x}{R(t)}\right),$$

where $R(t) = T^* - t$. The next result states that $u_S(x, t)$ is a weak solution of (1.1) with $K(x) = |x|$.

Proposition 3.6. *Let $T^* > 0$ and $R(t) = T^* - t$. Then, for all $\phi \in C_0^\infty(\mathbb{R} \times [0, T^*))$, we have*

$$\int_0^{T^*} \int_{\mathbb{R}} (u_S \phi_t + u_S v_S \phi_x) \, dx \, dt + \int_{\mathbb{R}} u_S(x, 0) \phi(x, 0) \, dx = 0, \tag{3.19}$$

where $v_S := -u_S * \nabla|x|$.

Remark 3.7. Let us first notice that $\phi \in C_0^\infty(\mathbb{R} \times [0, T^*))$ means that there exists a $\tilde{\phi} \in C_0^\infty(\mathbb{R} \times (-T^*, T^*))$ such that $\tilde{\phi} = \phi$ on $R \times [0, T^*)$. Moreover, note that $u \in L^1(\mathbb{R} \times (0, T^*))$ with $\|u_S\|_{L^1} = T^*$ and $v_S \in L^\infty(R \times (0, T^*))$ with $\|v_S\|_{L^\infty} = 1$. Therefore, (3.19) is well defined. Finally, let us point out that this solution blows up at time $T^* = R_0$ and we have

$$u_S(x, t) \rightharpoonup \delta_0 \qquad \text{as } t \to T^*.$$

Proof. Given the domain $D = \{(x, t) \in \mathbb{R} \times (0, T^*) : -R(t) < x < R(t)\}$, we have

$$u_S(x, t) = \begin{cases} \dfrac{1}{2R(t)} & \text{if } (x, t) \in D \\ 0 & \text{otherwise.} \end{cases}$$

Since $K'(x) = \text{sgn}(x)$, we can explicitly compute $v_S = -u_S * K'$:

$$v_S(x, t) = \begin{cases} -\dfrac{x}{R(t)} & \text{if } (x, t) \in D \\ -\text{sgn}(x) & \text{otherwise} \end{cases}$$

From this it is clear that u_S and v_S are C^∞ on D. Moreover, since $R'(t) = -1$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2R(t)}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{x}{2R(t)^2}\right] = 0$$

and therefore u_S and v_S satisfy $u_t + (uv)_x = 0$ on D. Since $\mathrm{supp}\{u\} = D$, (3.19) is equivalent to

$$\iint_D u_S\,\phi_t + u_S v_S\,\phi_x\,\mathrm{d}x\,\mathrm{d}t + \int_\mathbb{R} u_S(x,0)\,\phi(x,0)\,\mathrm{d}x = 0. \tag{3.20}$$

We then integrate by parts the first term of (3.20). Since u_S and v_S satisfy $u_t + (uv)_x = 0$ inside D, we are just left with the boundary terms:

$$\int_{\partial D}(u_S\,\phi v_t + u_S v_S\,\phi v_x)\,\mathrm{d}\sigma + \int_\mathbb{R} u_S(x,0)\,\phi(x,0)\,\mathrm{d}x = 0. \tag{3.21}$$

Here, $v_t(x,t)$ and $v_x(x,t)$ are the t- and the x-coordinates of the outward unit normal vector to ∂D at the point (x,t). We then break the boundary into 3 parts $\partial D = \gamma_0 \cup \gamma_+ \cup \gamma_-$, where γ_0, γ_+ and γ_- are the three segments defined by $\gamma_0 = \{(x,t) \in \partial D : t = 0\}$ and $\gamma_\pm = \{(x,t) \in \partial D : t > 0 \text{ and } \pm x > 0\}$. Clearly $v_t = -1$ on γ_0 and $v_x = 0$ on γ_0, so we are reduced to showing

$$\int_{\gamma_+\cup\gamma_-} u_S\phi(v_t + v_S v_x)\,\mathrm{d}\sigma = 0. \tag{3.22}$$

It is straightforward to check that $v_t + v_S v_x = 0$ on $\gamma_+ \cup \gamma_-$, which proves (3.22). □

3.2.2. Delta ring examples in multiple dimensions. We will show that in any dimension we have certain self-similar solutions of (1.1). This family of solutions gives different similarity solutions even in one dimension to those found in the previous subsection. Recall that in radial coordinates, the aggregation equation becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u} + \dfrac{\mathrm{d}}{\mathrm{d}r}(\hat{u}\hat{v}) = 0 \\[2mm] \hat{v}(r,t) = -\displaystyle\int_0^{+\infty}\psi(r,\rho)\hat{u}(\rho,t)\,\mathrm{d}\rho \end{cases}$$

with the function ψ given by (3.1). Suppose that $R_0(t), \ldots, R_n(t)$ satisfy the ODE

$$R_i'(t) = -\sum_{j=0}^n m_j\psi(R_i(t), R_j(t)) \tag{3.23}$$

then one can easily check that

$$u(r,t) = \sum_{i=0}^n m_i\,\delta(r - R_i(t)) \tag{3.24}$$

is formally a weak solution of (1.1). Going back to the normal coordinates, we can see that the solution described by (3.24) is a linear combination of measures which are uniformly distributed on $\partial B(0, R_i)$.

Remark 3.8 (Shrinking self-similar delta). In particular, in the case of a single delta on a sphere, we have a self-similar shrinking delta Dirac solution where the radius $R_0(t)$ follows (3.23), i.e. $R_0'(t) = -M\psi(R_0(t), R_0(t)) = -M\psi(R_0(t), R_0(t)) = -M\phi(\frac{R_0(t)}{R_0(t)}) = -M\phi(1)$,

with ϕ defined below, and therefore, it shrinks linearly in time.

$$\psi(r, p) = \phi\left(\frac{r}{\rho}\right) \qquad \text{with} \qquad \phi(r) = \frac{1}{\omega_N} \int_{\partial B(0,1)} \frac{re_1 - y}{|re_1 - y|} \cdot e_1 \, d\sigma(y). \qquad (3.25)$$

Now, let us work with general linear combination of deltas; we say that such a solution is a similarity solution if there exist positive constants β_{ij} such that

$$\frac{R_i(t)}{R_j(t)} = \beta_{ij} \qquad i, j = 0, 1, \ldots, n$$

or, equivalently, if there exist positive constants a_i, $i = 1, \ldots, n$ such that

$$R_i(t) = a_i R_0(t) \qquad i = 1, \ldots, n. \qquad (3.26)$$

Note that if $K(x) = |x|$, then $k'(r) = 1$ and $\psi(r, \rho)$ can be written as in (3.25). In this way (3.23) becomes

$$R_i'(t) = -\sum_{j=0}^{n} m_j \phi\left(\frac{R_i(t)}{R_j(t)}\right) \qquad i = 0, \ldots, n.$$

Thus, if $R_i(t)$ result from a self-similar solution, then they must satisfy

$$R_0'(t) = -m_0 \phi(1) - \sum_{j=1}^{n} m_j \phi\left(\frac{1}{a_j}\right) \qquad (3.27)$$

$$R_i'(t) = -m_0 \phi(a_i) - \sum_{j=1}^{n} m_j \phi\left(\frac{a_i}{a_j}\right) \qquad i = 1, \ldots, n \qquad (3.28)$$

but from (3.26) it is clear that

$$R_i'(t) = a_i R_0'(t). \qquad (3.29)$$

Combining (3.27), (3.28) and (3.29), we see that

$$a_i = \frac{m_0 \phi(a_i) + \sum_{j=1}^{n} m_j \phi\left(\frac{a_i}{a_j}\right)}{m_0 \phi(1) + \sum_{j=1}^{n} m_j \phi\left(\frac{1}{a_j}\right)}.$$

Having noticed this, let us define:

$$\mathcal{V} := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : 1 < a_1 < \cdots < a_n\}$$

and also the function $G : \mathcal{V} \to \mathbb{R}^n$ given by

$$G_i(a_1, \ldots, a_n) = \frac{m_0 \phi(a_i) + \sum_{j=1}^{n} m_j \rho\left(\frac{a_i}{a_j}\right)}{m_0 \phi(1) + \sum_{j=1}^{n} m_j \phi\left(\frac{1}{a_j}\right)}.$$

Note that, since ϕ is strictly increasing, $G(\mathcal{V}) \subset \mathcal{V}$. We can summarize the previous discussion as

Theorem 3.9. *Suppose* $(a_1, \ldots, a_n) \in \mathcal{V}$ *is a fixed point of G and define*

$$c = m_0 \phi(1) + \sum_{j=1}^{n} m_j \phi\left(\frac{1}{a_j}\right),$$

$$R_0(t) = 1 - ct \qquad \forall t \in \left[0, \frac{1}{c}\right),$$

$$R_i(t) = a_i R_0(t) \qquad \forall t \in \left[0, \frac{1}{c}\right), \quad i = 1, \ldots, n$$

Then $R_0(t), R_1(t), \ldots, R_n(t)$ *satisfy*

$$R_i'(t) = -\sum_{j=0}^{n} m_j \phi\left(\frac{R_i(t)}{R_j(t)}\right) \qquad i = 0, \ldots, n \tag{3.30}$$

with $R_0(t) < R_1(t) < \cdots < R_n(t)$ *on the time interval* $[0, \frac{1}{c})$.

Thus, if G has a fixed point in $\mathcal{V}$, then the solution

$$u(t) = \sum_{i=0}^{n} m_i \delta(r - R_i(t)),$$

where the $R_i(t)$ are given by (3.30), is formally a self-similar weak solution of the aggregation equation with $K(x) = |x|$. This solution blows up at time $T^* = \frac{1}{c}$ and we have

$$u(t) \rightharpoonup \delta_0 \qquad \text{as } t \to T^*.$$

3.2.3. Existence of two-delta-ring solution in $\mathbb{R}^2$. We prove the existence of a two-delta-ring similarity solution in the special case of two space dimensions. In this case, the function G becomes

$$G(x) = \frac{m_0 \phi(x) + m_1 \phi(1)}{m_0 \phi(1) + m_1 \phi(1/x)}.$$

We want to prove that G has a fixed point in $\mathcal{V} = (1, \infty)$. Note first that, since $\lim_{x \to \infty} \phi(x) = 1$ and $\phi(0) = 0$,

$$\lim_{x \to \infty} G(x) = \frac{m_0 + m_1 \phi(1)}{m_0 \phi(1)} < \infty.$$

Therefore, for large enough x, $G(x) < x$. Furthermore,

$$G'(1) = \frac{\phi'(1)}{\phi(1)}. \tag{3.31}$$

So, if we can show that $\phi'(1) > \phi(1)$, then $G'(1) > 1$ and therefore $G(x) > x$ on some interval $(1, 1 + \delta)$, which is enough to conclude that G has a fixed point on $(1, \infty)$ and therefore there exists a self-similar solution comprising two delta rings.

We conclude this subsection with a proof of $\phi'(1) > \phi(1)$. Recall that

$$\phi(r) = \frac{1}{\omega_N} \int_{\partial B(0,1)} F(r, y) \, d\sigma(y),$$

where as before

$$F(r, y) = \frac{re_1 - y}{|re_1 - y|} \cdot e_1.$$

Write $y = (y_1, y')$, where $y_1 \in \mathbb{R}$ and $y' \in \mathbb{R}^{N-1}$. One can check that

$$\frac{\partial}{\partial r} F(r, y) = \frac{|y'|^2}{|re_1 - y|^3} = \frac{|y'|^2}{\{(r - y_1)^2 + |y'|^2\}^{3/2}}.$$

Note that, if $y \in \partial B(0, 1)$ and if $y \to e_1$, then

$$\frac{\partial}{\partial r} F(1, y) = \frac{|y'|^2}{\{(1 - y_1)^2 + |y'|^2\}^{3/2}} \sim \frac{1}{|y'|}$$

and therefore, when $N = 2$,

$$\phi'(1) = \frac{1}{\omega_N} \int_{\partial B(0,1)} \frac{\partial}{\partial r} F(1, y) \, \mathrm{d}\sigma(y) = \infty$$

and obviously $\phi'(1) > \phi(1)$.

3.2.4. On the support of similarity solutions in odd dimensions.   In this section we prove that similarity solutions cannot exist with support on open sets in any odd dimension larger than one.

A function $u(t, x)$ is said to be a similarity solution to (1.1) if it has the form

$$u(t, x) = u(t)(x) = \frac{1}{R(t)^N} u_0 \left(\frac{x}{R(t)} \right) \tag{3.32}$$

with $R(0) = 1$. This is equivalent to saying that $u(t) = T_t \# u_0$ with $T_t : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ the dilation map $T_t(x) = R(t)x$. This formulation of self-similarity has perfect sense for measure-valued solutions, and thus

Definition 3.10. *A time dependent measure-valued solution on $\mathbb{R}^N$ is a mass-conserving similarity solution if it is a weak solution of (1.1) having the following form $u(t) = T_t \# \mu$ with μ a nonnegative Radon measure with total finite mass.*

Plugging the general form into (1.1) for the special case $K(x) = |x|$ gives, via separation of variables, that $R(t)$ must vary linearly in time. Moreover, by the similarity ansatz, the velocity field generated by the convolution $v = \nabla K * \mu$, is well defined and satisfies

$$v = -\lambda x \tag{3.33}$$

for some constant λ, on the support of μ. Another way to see this is just to realize that the solutions of the continuity equation (1.1) must satisfy $u(t) = X(t) \# u_0$ as discussed in section 2. Therefore, $u(t) = T_t \# \mu$ is a weak solution of (1.1) with initial data μ if and only if $X(t) = T_t$ on supp(μ), and thus, for all $x \in$ supp(μ) we get $X(t, x) = T_t(x) = R(t)x$. As a consequence, for all $x \in$ supp(μ) we deduce

$$R'(t)x = v(t, R(t)x) = -(\nabla|x| * u(t))(R(t)x) = -(\nabla|x| * u_0)(x)$$

by the homogeneity of $|x|$. From this, $R'(t) = -\lambda$ with $\lambda \in \mathbb{R}^+$ and thus (3.33).

Using these facts, we arrive at the following nonexistence theorems for radially symmetric similarity solutions in odd dimensions N higher than one.

Theorem 3.11 (Non-existence of similarity solutions). *Let N be an odd space dimension larger than one and $K(x) = |x|$. Then there does not exist a nonnegative similarity solution in $L^p(\mathbb{R}^N)$ for $p > 1$ whose support contains an open set.*

Proof. We start with the 3D case. Note that if μ has a density ρ in L^p then $\nabla \cdot v$ is a constant times the Newtonian potential convolved with ρ. Therefore, the distributional Laplacian of $\nabla \cdot v$ is a constant multiple of the L^p function ρ. However, by taking distributional derivatives of equation (3.33), i.e. by multiplying by suitable test functions and integrating by parts for the right-hand side, we have that the distributional Laplacian of $\nabla \cdot v$ is zero on any open set on the support of ρ giving that ρ is zero a.e. in any open set inside its support.

To extend this result from 3D to odd higher dimensions, we note that there is always some power of the Laplacian that can be applied to $\nabla \cdot v$ to obtain a constant times the identity functional, since the Newtonian potential is a constant multiple of $1/|x|^{2-N}$ in N dimensions. The rest of the argument follows in higher dimensions as well. $\square$

Theorem 3.12 (Non-existence of similarity solutions). *Let N be an odd space dimension larger than one and $K(x) = |x|$. Then there do not exist any nonnegative nontrivial measure-valued similarity solutions, compactly supported on $\mathbb{R}^N$, whose support contains an open set.*

Proof. The proof is similar to that of theorem 3.11 except that we now consider μ to be a compactly supported measure. This means that $\nabla K * \mu$ and $1/|x| * \mu$ can be understood in the sense of distributions (a distribution convolved with a distribution of compact support is a distribution). The distributional Laplacian of $\nabla \cdot v$ equals a constant times μ in the sense of distributions and due to (3.33) as above, the distributional Laplacian is zero on any open set contained in the support of μ. Thus μ has no support on open sets. $\square$

Remark 3.13 (Even dimensions). Note that the argument fails in even dimensions because there is no local differential operator that inverts the convolution operator $1/|x|$. Rather the appropriate operator is a non-local pseudo-differential operator.

Remark 3.14 (Blow-up profiles). All known examples of similarity solutions in dimensions two and higher are collapsing delta rings. It would be interesting to know whether these are the only similarity solutions or whether, for some reason the odd dimensions are special. Moreover, it raises the question of what is the blow-up profile in higher dimensions for blow-up from bounded data. It cannot be exactly a delta ring because that solution is more singular than the bounded-blowing up solution. The existence and continuation theory only guarantees a blow-up in L^p for $p \geqslant 2$ so we do not know if the initial singularity for bounded data involves a mass concentration or instead something a bit less singular for the initial blow-up time.

4. Conclusions and discussion

We summarize the main results of this work as follows. Given radial, Lipschitz continuous, attractive potentials with an additional monotonicity condition on k' and given bounded nonnegative compactly supported initial data, the Osgood condition (1.4) is a necessary and sufficient condition to have global-in-time bounded solutions in multiple space dimensions. Moreover, the monotonicity of the kernel guarantees that the global in time solutions converge to a Dirac delta singularity, as $t \to \infty$, at the centre of mass (which is conserved). Particle paths are well-posed globally in time and the solutions constructed are unique. In the case the Osgood condition does not hold (1.3), then all initially bounded solutions blow-up in finite time. Moreover, the blow-up time is uniformly bounded by the blow-up time of an ODE of the form $\mathrm{d}R/\mathrm{d}t = -M \, k'(2R)/2$. Therefore, the Osgood condition is sharp for

deciding the finite or infinite time blow-up among a certain class of potentials. However, we do not know the typical blow-up profile of bounded solutions, since solutions for certain initial data will form singularities before the time in which all the mass might concentrate in a single Dirac delta. It is an open problem to clarify the different blow-up scenarios that may happen. For non-Osgood potentials, particle paths are unique up to the initial blow-up time.

In a single space dimension, the results hold provided there is an existence theory for the PDE with bounded data. This problem has been studied in some particular contexts; however to our knowledge the general existence problem has not been proved for bounded data and all kernels that are Lipschitz continuous or smoother. Three recent papers that consider the one-dimensional inviscid problem are [10, 23, 27] and discuss a number of results for various kernels. By making a transformation $\int_{-\infty}^{x} u = w$, one finds that the 1D problem is equivalent to a scalar conservation law with a non-local flux function. The Lipschitz point kernel gives Burgers equation to leading order.

Part of this paper focuses on blow-up from radially symmetric data. We are able to prove the same blow-up results as in the case of general initial data; however, we can slightly weaken the monotonicity constraints on the kernel. A special class of radially symmetric solutions are similarity solutions that converge to a Dirac mass in finite time. Motivated by the particular problem of blow-up for potentials with a Lipschitz point, we consider the question of existence of similarity solutions for the special kernel $|x|$ in different space dimensions. We note that in one dimension there is a special solution described by a self-similar rectangle that converges to a delta. This is the analogue of the classical shock solution example for Burgers equation. In multiple dimensions, we describe families of solutions of nested delta rings, in which the support is concentrated on the surface of spheres with shrinking radii. In two dimensions we prove the existence of a 2-ring solution. In any dimension there is a single-ring solution. This brings up the question of whether there are any similarity solutions in higher space dimensions in which the profile of the solution is bounded. In odd space dimensions higher than one, we prove that no such similarity solutions exist in which the support of the solution contains an open set. This strongly suggests, at least in odd dimensions, that the only radially symmetric similarity solutions for this potential are the delta-ring solutions. It also raises the question of what is the blow-up profile in dimension two and higher. The result for odd dimensions tells us that the blow-up profile from bounded initial data must be unusual.

Another interesting problem is the case of measure-valued initial data. We consider a subproblem, namely data concentrated in a finite number of deltas, which results formally in an ODE for the motion of particles. For Osgood potentials, we derive, via lemma A.15 in the appendix, that the full ODE particle system satisfies the well-known Osgood uniqueness condition. This, combined with the *a priori* bound on the support of the particles, guarantees global existence and uniqueness of a solution. We note that a related result was recently proved in [1] for the linear transport equation with signed measure-valued initial data and a characteristic vector field satisfying the classical ODE Osgood condition. In the case of non-Osgood potentials, we prove uniqueness of a special solution forward in time in which particles are defined to merge when they collide. Uniqueness is clearly not satisfied going backward in time. Our results for ODEs suggest that there may be a way to give sense to global existence and uniqueness for measure-valued solutions, in the case of non-Osgood potentials, based on a kind of uniqueness of forward-in-time characteristics. The optimal transport point of view is definitely an approach that can deal with measure-valued solutions for smooth potentials, see [2]. We will pursue this different approach for non-smooth non-Osgood potentials elsewhere.

Acknowledgments

Appendix

Here we prove the following lemma:

Lemma A.15 (Osgood modulus of continuity). *Let $k(r)$ be a function defined on $(0, \infty)$ satisfying $k'' \geqslant 0$ on $(0, \infty)$ and $k'(r)/r$ nonincreasing in r. Then the vector field in $\mathbb{R}^N$ defined by $v(x) = \nabla k(|x|)$ has a modulus of continuity that satisfies*

$$|v(x_1) - v(x_2)| \leqslant 2k'(2|x_1 - x_2|),$$

for all $x_1, x_2 \in \mathbb{R}^N$.

Proof. First note that the monotonicity conditions imply that $[k'(r)/r]' = k''/r - k'/r^2 \leqslant 0$ and hence $0 \leqslant k''(r) \leqslant k'(r)/r$. Let x_1 and x_2 be two points in $\mathbb{R}^N$ and without loss of generality take $0 \leqslant |x_1| \leqslant |x_2|$. Now we consider two cases.

Case 1: $|x_1 - x_2| > |x_2|/2$. In this case we can estimate

$$|\nabla K(x_1) - \nabla K(x_2)| \leqslant k'(|x_1|) + k'(|x_2|) \leqslant 2k'(|x_2|) \leqslant 2k'(2|x_1 - x_2|).$$

Case 2: $|x_1 - x_2| \leqslant |x_2|/2$. In this case, we can reduce to $0 < \frac{|x_2|}{2} \leqslant |x_1| \leqslant |x_2|$ and then

$$|\nabla K(x_1) - \nabla K(x_2)| \leqslant k'(|x_1|) \left| \frac{x_1}{|x_1|} - \frac{x_2}{|x_2|} \right| + [k'(|x_2|) - k'(|x_1|)] = T_1 + T_2$$

$$T_1 \leqslant \frac{\sqrt{2}k'(|x_2|)}{|x_2|}|x_1 - x_2| \leqslant \frac{\sqrt{2}k'(2|x_1 - x_2|)}{2|x_1 - x_2|}|x_1 - x_2| \leqslant \frac{k'(2|x_1 - x_2|)}{\sqrt{2}}.$$

$$T_2 \leqslant \sup_{|x_1| \leqslant r \leqslant |x_2|} k''(r)|r_2 - r_1| \leqslant \frac{k'(r_2)}{r_2}(r_2 - r_1) \leqslant \frac{k'(2(r_2 - r_1))}{2(r_2 - r_1)}|r_2 - r_1|$$

$$= \frac{k'(2(r_2 - r_1))}{2} \leqslant \frac{k'(2|x_1 - x_2|)}{2}.$$

Above we use the notation $|x_i| = r_i$ and the inequality

$$\left| \frac{x_1}{|x_1|} - \frac{x_2}{|x_2|} \right| \leqslant \left(\frac{1}{|x_1||x_2|} \right)^{1/2} |x_1 - x_2|$$

for all non-zero $x_1, x_2 \in \mathbb{R}^N$. $\square$

As mentioned earlier in the paper, the sign constraint on k'' is used in the proof of the lemma, combined with the monotonicity assumption, to bound it by the first derivative divided by r. The constraint can be weakened as long as one can still obtain such a bound.

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