Lévy searches based on *a priori* information: The Biased Lévy Walk.

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Searching for objects with unknown locations based on random walks can be optimized when the walkers obey Lévy distributions with a critical exponent. We consider the problem of optimizing statistical searches when a priori information, such as location densities, are known. We consider both spatially dependent exponents and biased search directions. For spatially localized target distributions and non-destructive searches, the search is most improved by biasing the search direction.

The best statistical strategy for efficient searching for randomly located objects is a subject of ongoing investigation [1–4]. Lévy random walks are known to outperform Brownian (normal) random walks [3] when the precise location of the targets is not known *a priori* but their spatial distribution is uniform. Lévy flights are characterized by a distribution function

$$\Pi(l_j) \sim l_j^{-\alpha}, \tag{1}$$

where l_j is the flight length and $1 < \alpha \leq 3$ [2]. The lower bound of 1 is required in order to have a normalized probability distribution (1). Such statistical searching patterns are also robust as they result in space filling paths. For exponent values exceeding 3, the central limit theorem implies that the distributions are Gaussian. The optimally efficient searching exponent can be computed analytically for a nondestructive search for a spatially uniform target distribution [2]. In the limit of large mean free path between targets one obtains $\alpha = 2$; for destructive searches $\alpha \downarrow 1$ is optimal [3].

A uniform target distribution applies to problems in biology such a foraging for food [5]. However, for human applications, in which the searchers are software, robots, humans, etc., there is often presumptive knowledge and there are many approaches to best use this information. For example, a recent study of U-boat search and destroy missions [6] proposes a human-in-the-loop integrated with a heuristic optimization method. In this paper, we propose an autonomous search method based on the classical Lévy method but applied to a problem in which the distribution of targets is spatially non-uniform. We consider both spatially dependent exponents α and biasing the choice of direction for each successive flight. For simplicity we consider a Gaussian target distribution. Future work would generalize to more complex target distributions.

We build our model on an *unbiased* Lévy search with flight lengths satisfying Eq. (1), in which the searcher "walks" the path as opposed to making an instantaneous jump [2]. In this model the searcher behaves as follows: (a) If there is a target located within a 'direct vision' distance r_v, then the searcher detects the target. (b) If there are no targets within a distance r_v, then the searcher chooses a direction at random and a distance l_j from a Lévy distribution. The searcher then incrementally moves to this new point, constantly searching for a target site within a distance r_v. At the end of the jump, the searcher picks a new jump distance and direction and repeats the process. Multiple targets could, in principle, be detected during any individual jump path.

The model on which we build has only two parameters, the vision distance r_v and the exponent α of the Lévy distribution. The vision distance is a fixed parameter for the model determined by the physical limitations of the searcher. Although we develop theory and algorithms for an infinite domain, for practical purposes and for numerical simulation, we restrict to a finite system size. In this paper we contrast two methods of incorporating *a priori* information into the standard Lévy model: the first allows the distribution exponent α to depend on space. The second biases the direction chosen for each successive flight path (the standard model has equal probability to jump in any direction).

We first review the results in [2] on the optimal exponent α for both nondestructive and destructive searches. Let λ denote the mean free path of the searcher between successive target. The mean flight distance is

$$\langle l \rangle \approx \frac{\int_{r_v}^{\lambda} x^{1-\alpha}dx + \lambda \int_{\lambda}^{\infty} x^{-\alpha}dx}{\int_{r_v}^{\infty} x^{-\alpha}dx} = \left(\frac{\alpha-1}{2-\alpha}\right)\frac{\lambda^{2-\alpha} - r_v^{2-\alpha}}{r_v^{1-\alpha}} + \frac{\lambda^{2-\alpha}}{r_v^{1-\alpha}}. \tag{2}$$

Here we assume that the mean distance between successive targets is a fixed λ which thus serves as a maximum necessary jump length. Likewise, no flights are less than the "vision distance" r_v. The search efficiency function $\eta(\alpha)$ is the ratio of the number of target visited to the total distance traveled by the searcher,

$$\eta = \frac{1}{\aleph \langle l \rangle}, \tag{3}$$

where $\aleph$ is the mean number of flights needed to travel between two successive targets. $\aleph$ can be computed as

$$\aleph \sim (\lambda/r_v)^{\alpha-1} \text{for destructive searching,} \tag{4}$$

$$\aleph \sim (\lambda/r_v)^{\frac{\alpha-1}{2}} \text{for nondestructive searching,} \tag{5}$$

for $1 < \alpha \leq 3$. In the case where targets are sparsely distributed, $\lambda \gg r_v$, then substituting Eq. (4) and Eq.

(2) into (3), one finds that for destructive searching, the mean efficiency has no maximum, with lower values of α leading to more efficient searching. Thus for destructive searching the optimal choice is α as close to 1 as possible. For nondestructive searches, replacing Eq. (4) with (5) in Eq. (3) and differentiating with respect to α, the optimal efficiency is achieved at

$$\alpha_*(\lambda, r_v) \equiv \text{argmax}_{\alpha>1}\eta(\alpha, \lambda, r_v) = 2 - \delta(\lambda, r_v), \quad (6)$$

where $\delta \sim 1/[\ln(\lambda/r_v)]^2$. In summary, for a nondestructive search, an optimal searching exponent α_* is predicted; where α_* is a function of the mean target distance λ and the vision radius r_v. These results are independent of the dimension of the search domain.

Searching in one dimension. We propose two methods for constructing a spatially dependent exponent α. The first method directly applies the above arguments for uniform target distribution to a nonuniform distribution. The second method is a heuristic model motivated by observational data from simulation. Both models require an upper bound on the distance between targets as a function of position in the spatial domain. Assume a probability distribution of target locations $P(x)$ ($P \geq 0$, $\int P = 1$). Then the expected length between a fixed point x in the search domain and a point z chosen from the target distribution is given by

$$\lambda(x) = \int_{-\infty}^{\infty} |x - z| P(z) dz. \quad (7)$$

This gives an effective upper bound on the distance to the nearest target. One strategy for improving a searching method for a spatially variable target distribution is to take a spatially dependent α based on a local value of λ. We compare two contrasting options using $\lambda(x)$ above.

Option 1A. Use α_* in formula (6), plugging in for λ the spatially dependent bound computed in (7). Note that for a fixed vision radius r_v, α_* increases (to the value 2) as λ increases. A larger value of α makes the search closer to Brownian motion, thus keeping the searcher more contained in a region. This is because the larger the exponent α, the smaller the average jump length, with the jumps approximating Brownian motion as $\alpha \to 3$.

Option 1B. For a spatially localized probability distribution, we might heuristically expect that a better search strategy uses a smaller α where targets are sparse (larger λ) and a larger α where targets are closer together. This way a searcher would take smaller hop lengths in a dense target region and larger hop lengths in a less dense region.

Using the mean distance to a target computed in (7), we propose to choose α according to $\alpha_2(x) =$

$$H(\lambda(x) - r_v)\left(1 + \frac{\max(P(x)) + 1}{\lambda(x) + 1}\right) + 3H(r_v - \lambda(x)). \quad (8)$$

Here H denotes the Heaviside function which is one for positive arguments and otherwise zero. Note that Option 1B has α decreasing as λ increases. Figure 1 contrasts α_* and α_2 for the case of a Gaussian target P with zero mean and unit variance and a vision radius $r_v = 0.5$. For this particular choice of P, λ can be computed explicitly as $\lambda(x) = x\text{erf}\left(\frac{x}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}e^{-x^2/2}$. Note that α_* increases for larger x while α_2 decreases. The significance of this is that, for the Gaussian distribution, Option 1A results in shorter jump lengths in regions of lower target density while Option 1B results in longer jump lengths in these regions.

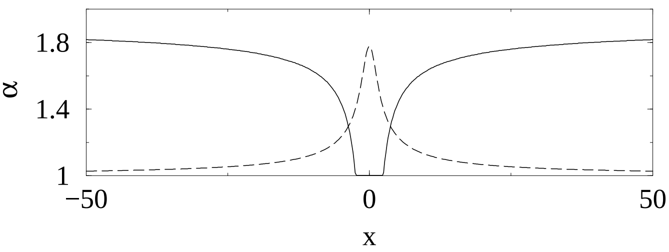

FIG. 1: The figure contrasts α_* (solid line) with α_2 (dashed line) for a Gaussian distribution $P(x)$ with variance 1, zero mean, and a vision radius $r_v = 0.5$.

Option 2. We propose a third method that biases the jump direction based on the *a priori* information. After choosing a jump length from the Lévy distribution, a choice of direction is made using information about the derivative P_x of the *a priori* target distribution. Since we wish to retain the space filling nature of the search, we choose a nonzero probability of moving in either direction. We propose the following algorithm in which β denotes a number between 0 and $\frac{1}{2}$ and p_r and p_l denote the probabilities of jumping to the right and to the left.

- $P_x > 0$, (the *a priori* distribution is increasing), then choose $p_r = 1 - \beta$, $p_l = \beta$.
- $P_x < 0$, (the *a priori* distribution is decreasing), then $p_r = \beta$ and $p_l = 1 - \beta$.
- $P_x = 0$, then perform unbiased Lévy search.

In the computations below, we choose $\beta = 0.25$. In one dimension, this method is very similar to the classical biased random walk [7]. Here we use local spatial information to determine the bias. In two dimensions (discussed below) the method is more complicated.

We present simulations of the three methods above for nondestructive searching on a fixed box of length 200. The vision radius $r_v = 1.5$ and the searcher is started randomly in the domain. Figure 2 shows the result of each method as a function of N, the number of targets. Each datapoint is an average over 1000 runs of 100 jumps each. Figure 2 shows that the biased Lévy search outperforms all unbiased searches. The variable α_2 (Option 1B) method is the next efficient followed by the standard unbiased with $\alpha = 2$. Finally the simulations that used $\alpha = \alpha_*$ (Option 1A) are least efficient. At the surface this

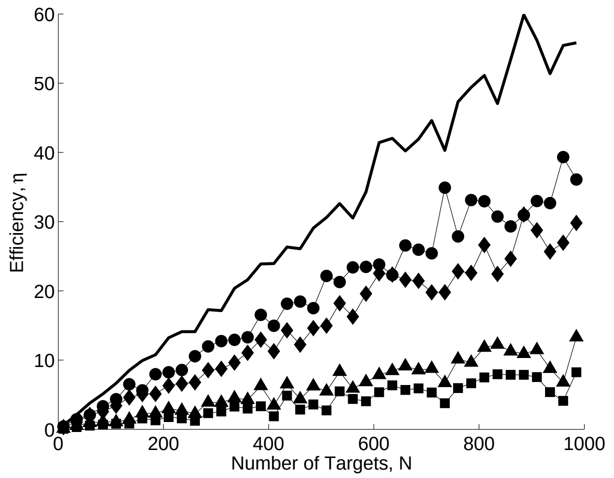

FIG. 2: One dimensional Lévy walks, variable α compared with biased searching for a Gaussian target distribution with variance 1. Efficiencies as a function of the number of targets N averaged over 1000 runs of 100 jumps each. The vision radius $r_v = 1.5$. The symbols denote ■ $= \alpha_*$, ▲ $= \alpha_*$ with bias, ♦ = unbiased with $\alpha = 2$, • $= \alpha_2$, solid line = biased with $\alpha = 2$.

is surprising as α_* is originally derived to maximize efficiency. However with a concentrated target distribution, this choice of α tends to have a larger jump length which gives a higher probability of "jumping over" the concentration of targets at the origin. Figure 2 also shows that bias search outperforms the variable α approach. We expect that higher dimensionality will enhance this effect.

Two dimensions. The proposed 1D search methods can be extended to higher dimensions. Given a 2D probability distribution P of target sites, the expected length to a target can be computed as

$$\lambda(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{(x-u)^2 + (y-v)^2} P(u, v) du dv. \tag{9}$$

The proposed methods in Options 1AB above can be directly extended two dimensions with this new choice of λ.

Extending the biased Lévy search requires more discussion. Given an *a priori* two dimensional target distribution $P(x, y)$, we choose the jump direction from a continuum as opposed to left or right as in the one dimensional case. We choose an angular direction from a distribution with mean

$$\mu = \tan^{-1}\left(\frac{\partial_y P(x, y)}{\partial_x P(x, y)}\right). \tag{10}$$

In the case that $\nabla P(x, y)$ is undefined or zero, the angle is chosen uniformly in $[0, 2\pi)$ (i.e. it reduces to an unbiased Lévy step), otherwise we choose the angle from a *von Mises* distribution [8] with mean μ. The von Mises distribution for points on a circle is analogous to the normal distribution of points on a line. Its properties are well documented in [9]. An angular random variable θ has a von Mises distribution $VM(\mu, \kappa)$ if its probability density function has the form

$$\Phi(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)}, \quad -\pi \leq \theta < \pi, \quad -\pi \leq \mu < \pi,$$

where $I_0(\kappa)$ is the zeroth order modified Bessel function of the first kind. We use the wrapped Cauchy method [10] to generate variates from this type of density. The variables μ and κ are analogous to the mean and variance, respectively, of a normal distribution on the line. We choose the angle of direction from a von Mises distribution with parameters μ given by (10) and κ constant. If the gradient of P is zero or undefined, then θ is chosen uniformly in $[0, 2\pi)$.

Again, we simulate and compare the different search methods. As before we choose P to be Gaussian with zero mean and unit variance. The vision radius, $r_v = 1.5$, and we average over 1000 runs, each truncated after 100 jumps. A smaller domain $[-10, 10] \times [-10, 10]$ is used; on the domain $[-100, 100] \times [-100, 100]$, the majority of searching is done in sparse regions preventing meaningful statistics with the 100 jump cutoff. On the smaller domain $[-10, 10] \times [-10, 10]$ we can achieve a similar hit rate for targets in the Gaussian distribution as we did on the $[-100, 100]$ domain in one dimension. We compute $\lambda(x, y)$ numerically on the lattice with spacing $\frac{1}{25}$, and interpolate as needed at each jump point in the simulation. Figure 3 shows the efficiencies as a function of target number for the different methods. In two dimensions the increase in efficiency by using a biased search (Option 2) is more pronounced than in one dimension. All three computations without bias (fixed $\alpha = 2$, $\alpha = \alpha_2$ and $\alpha = \alpha_2$) performed similarly. The biased simulation with $\alpha = 2$ outperforms all the others, including the biased simulation with $\alpha = \alpha_*$.

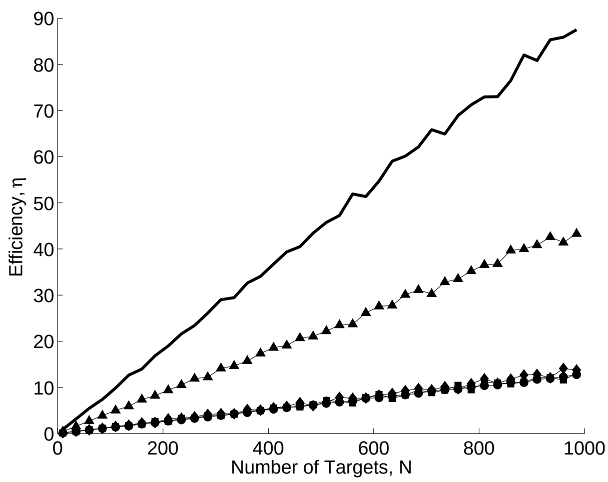

FIG. 3: Two dimensional Lévy walks, efficiencies vs. target number. The symbols are as in Figure 2, ■ $= \alpha_*$, ▲ $= \alpha_*$ with bias, ♦ = unbiased with $\alpha = 2$, • $= \alpha_2$, solid line = biased with $\alpha = 2$.

In two dimensions, Option 1B, defined by Eq. (8) performs less well than in one dimension. This is related to the fact that direction is more important than in the one dimension. This effect is illustrated in the trajectories seen in Fig. 4. The left panel denotes the trajectory from

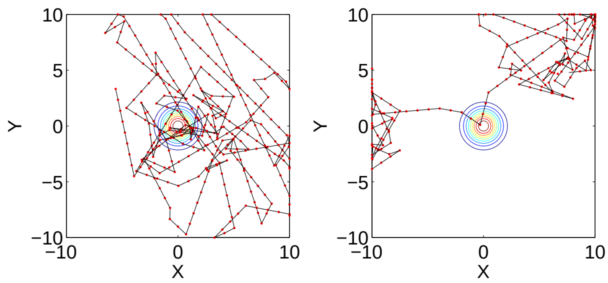

FIG. 4: The left panel shows a biased ($\kappa = 1$) two dimensional trajectory. The right panel shows a dynamic α trajectory (using α_* (Option 1A)). $r_v = 0.5$, and 100 jumps are allowed before cutoff. The step size is 1 and every 5 steps are plotted.

a biased ($\kappa = 1$ in the von Mises distribution) Lévy flight, and the right panel the trajectory from a Lévy flight with variable α. The circles are the level sets of the *a priori* target distribution. When the trajectory intersects with the boundary of the search domain $[-10, 10] \times [-10, 10]$, the jump is truncated and the searcher jumps again. The maximal jump length is restricted to be the minimum of $\lambda(x)$ and 20. With these constraints, we see that the majority of the variable α search is spatially near the boundary of the search domain, while the biased search repeatedly passes through the peak of the *a priori* target distribution. The variable α search has the same probability of jumping in any direction, therefore, when the searcher is near the boundary, there is a 50% chance that the searcher will jump towards the edge of the search domain. In the biased case, when the searcher is near the boundary, there is a large probability that the searcher will jump towards the peak of the Gaussian. Increasing the computational domain size will only serve to worsen the effect shown in Figure 4 (right); the searcher can wander very far from the center of the target distribution before returning to the concentrated area of targets.

We conclude that when searching with *apriori* information, biasing the searching direction has the most positive effect of the methods considered. Our results concern nondestructive search, which is the predominant type seen in nature [11]. As a point for further investigation, we propose a simple way to extend our algorithms to destructive searching. Start by performing a nondestructive search with *a priori* target information in the form of a probability density P, as above. As targets are located and destroyed, dynamically update P to include information about areas that have been searched and thus are now known to contain no targets. In this way a biased search would continue to look toward new areas that have not been searched. The conclusions about bias being more effective than the dynamic α search are largely based on our computational results for single point Gaussian probability distribution. As another point of further study, it would be interesting to see how the different methods perform for bimodal and more complex distributions of target sites. Moreover, a recent study [12] considers the dynamics of Lévy flights in the presence of external deterministic potentials. It would be interesting to consider optimal search strategies given an external field or flow. This is particularly relevant to underwater searching which is affected by ocean currents and to aerial searching affected by wind and weather patterns.

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