Image Inpainting Using a Fourth-Order Total Variation Flow

Carola-Bibiane Schönlieb* Andrea Bertozzi† Martin Burger‡ Lin He§

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Abstract

We introduce a fourth-order total variation flow for image inpainting proposed in [5]. The well-posedness of this new inpainting model is discussed and its efficient numerical realization via an unconditionally stable solver developed in [15] is presented.

1 Introduction

An important task in image processing is the process of filling in missing parts of damaged images based on the information obtained from the surrounding areas. It is essentially a type of interpolation and is referred to as inpainting. Given an image f in a suitable Banach space of functions defined on $\Omega \subset \mathbb{R}^2$, an open and bounded domain, the problem is to reconstruct the original image u in the damaged domain $D \subset \Omega$, called inpainting domain. In the following we are especially interested in so called non-texture inpainting, i.e., the inpainting of structures, like edges and uniformly colored areas in the image, rather than texture.
In the pioneering works of Caselles et al. [6] (with the term disocclusion instead of inpainting) and Bertalmio et al. [2] partial differential equations have been first proposed for digital non-texture inpainting. In subsequent works variational models, originally derived for the tasks of image denoising, deblurring and segmentation, have been adopted to inpainting. The most famous variational inpainting model is the total variation (TV) model, cf. [8, 10, 13, 14]. Here, the inpainted image u is computed as a minimizer of the functional

$$\mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2}\|\lambda(f-u)\|^2_{L^2(\Omega)},$$

where $|Du|(\Omega)$ is the total variation of u (cf. [1]), and λ is the indicator function of $\Omega \setminus D$ multiplied by a (large) constant, i.e., $\lambda(x) = \lambda_0 >> 1$ in $\Omega \setminus D$ and 0 in D. The corresponding steepest descent for the total variation inpainting model reads

$$u_t = -p + \lambda(f-u), \quad p \in \partial|Du|(\Omega), \tag{1}$$

where p is an element in the subdifferential of the total variation $\partial|Du|(\Omega)$. The steepest-descent approach is used to numerically compute a minimizer of $\mathcal{J}$, whereby it is iteratively solved until one is close enough to a minimizer of $\mathcal{J}$. For the numerical computation an element p of the subdifferential is approximated by the anisotropic diffusion $\nabla \cdot (\nabla u/|\nabla u|_\epsilon)$, where $|\nabla u|_\epsilon = \sqrt{|\nabla u|^2 + \epsilon}$.
Now, TV inpainting, while preserving edge information in the image, fails in propagating level lines (sets of image points with constant grayvalue) smoothly into the damaged domain, and in connecting edges over large gaps in particular. In an attempt to solve these issues from second order image diffusions, a number of third and fourth order diffusions have been suggested for image inpainting, e.g., [7, 9].

In this paper we present a fourth-order variant of total variation inpainting, called TV-H^{-1} inpainting. The inpainted image u of $f \in L^2(\Omega)$, shall evolve via

$$u_t = \Delta p + \lambda(f-u), \quad p \in \partial TV(u), \tag{2}$$

with

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

This inpainting approach has been proposed by Burger, He, and Schönlieb in [5] as a generalization of the sharp interface limit of Cahn-Hilliard inpainting [3, 4] to grayvalue images. The L^∞ bound in the definition (3) of the total variation functional $TV(u)$ is motivated by this sharp interface limit and is part of the technical setup, which made it easier to derive rigorous results for this scheme. A similar form of this higher-order TV approach already appeared in the context of decomposition and restoration of grayvalue images, see for example [12]. In the following we shall recall the main rigorous results obtained in [5], present an unconditionally stable solver for (2), and show a numerical example emphasizing the superiority of the fourth-order TV flow over the second-order one.

2 Well-Posedness of the Scheme

In contrast to its second-order analogue, the well-posedness of (2) strongly depends on the L^∞ bound introduced in (3).

*Department of Applied Mathematics and Theoretical Physics (DAMTP), Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom. Email: c.b.s.schonlieb@damtp.cam.ac.uk

†Department of Mathematics,UCLA (University of California Los Angeles), 405 Hilgard Avenue, Los Angeles, CA 90095-1555, USA. Email: bertozzi@math.ucla.edu

‡Institut für Numerische und Angewandte Mathematik, Fachbereich Mathematik und Informatik, Westfälische Wilhelms Universität (WWU) Münster, Einsteinstrasse 62, D 48149 Münster, Germany. Email: martin.burger@wwu.de

§Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, Austria. Email: lin.he@oeaw.ac.at

This is because of the lack of maximum principles which, in the second-order case, guarantee the well-posedness for all smooth monotone regularizations of p.

The existence of a steady state for (2) is given by the following theorem.

Theorem 1 *[5, Theorem 1.4] Let $f \in L^2(\Omega)$. The stationary equation*

$$\Delta p + \lambda(f-u) = 0, \quad p \in \partial TV(u) \tag{4}$$

admits a solution $u \in BV(\Omega)$.

Results for the evolution equation (2) are a matter of future research. In particular it is highly desirable to achieve asymptotic properties of the evolution. Note that additionally to the fourth differential order, a difficulty in the convergence analysis of (2) is that it does not follow a variational principle.

3 Unconditionally Stable Solver

Motivated by the idea of convexity splitting schemes, e.g., [11], Bertozzi and Schönlieb propose in [15] the following time-stepping scheme for the numerical solution of (2):

$$\begin{array}{l} \frac{U_{k+1}-U_k}{\Delta t} + C_1\Delta\Delta U_{k+1} + C_2 U_{k+1} = C_1\Delta\Delta U_k \\ -\Delta(\nabla\cdot(\frac{\nabla U_k}{|\nabla U_k|_\epsilon})) + C_2 U_k + \lambda(f-U_k), \end{array} \tag{5}$$

with $C_1 > 1/\epsilon$, $C_2 > \lambda_0$. Here, U_k is the kth iterate of the time-discrete scheme, which approximates a solution u of the continuous equation at time $k\Delta t$, $\Delta t > 0$. It can be shown that (5) defines a numerical scheme that is unconditionally stable, and of order 2 in time, cf. [15].

4 Numerical Results

In Figure 1 a result of the TV-H^{-1} inpainting model computed via (5) and its comparison with the result obtained by the second order TV-L^2 inpainting model for a crop of the image is presented. The superiority of the fourth-order TV-H^{-1} inpainting model to the second order model with respect to the desired continuation of edges into the missing domain is clearly visible.

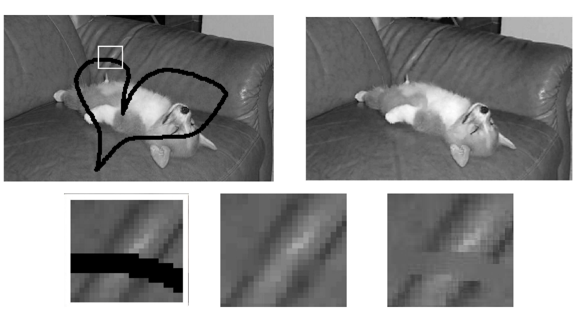

Figure 1: First row: TV-H^{-1} inpainting (2): u(1000) with $\lambda_0 = 10^3$. Second row: (l.) u(1000) with TV-H^{-1} inpainting, (r.) u(5000) with TV-L^2 inpainting (1)

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