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POSITIVITY-PRESERVING NUMERICAL SCHEMES FOR LUBRICATION-TYPE EQUATIONS*

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Abstract. Lubrication equations are fourth order degenerate diffusion equations of the form $h_t + \nabla \cdot (f(h)\nabla\Delta h) = 0$, describing thin films or liquid layers driven by surface tension. Recent studies of singularities in which $h \to 0$ at a point, describing rupture of the fluid layer, show that such equations exhibit complex dynamics which can be difficult to simulate accurately. In particular, one must ensure that the numerical approximation of the interface does not show a false premature rupture. Generic finite difference schemes have the potential to manifest such instabilities especially when underresolved. We present new numerical methods, in one and two space dimensions, that preserve positivity of the solution, regardless of the spatial resolution, whenever the PDE has such a property. We also show that the schemes can preserve positivity even when the PDE itself is only known to be nonnegativity preserving. We prove that positivity-preserving finite difference schemes have unique positive solutions at all times. We prove stability and convergence of both positivity-preserving and generic methods, in one and two space dimensions, to positive solutions of the PDE, showing that the generic methods also preserve positivity and have global solutions for sufficiently fine meshes. We generalize the positivity-preserving property to a finite element framework and show, via concrete examples, how this leads to the design of other positivity-preserving schemes.

Key words. lubrication equations, entropy dissipation, positivity-preserving, finite differences, finite elements, fourth order degenerate diffusion

AMS subject classifications. 65M06, 65M12, 65M60, 76D08, 35K65

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1. Introduction.

Lubrication-type equations arise in the study of thin liquid films and fluid interfaces driven by surface tension. The general problem we consider is of the form[1]

$$h_t + \nabla \cdot (f(h)\nabla\Delta h) = 0, \tag{1.1}$$

where

$$f(h) \sim h^n \text{ as } h \to 0. \tag{1.2}$$

This equation was first derived in [1] as a model for the surface tension dominated motion of thin viscous films and spreading droplets. For such problems, the power n depends on the boundary condition on the liquid solid interface: no-slip gives $n = 3$ while various Navier slip conditions can yield $n < 3$. The same equation in one space dimension with $f(h) = h$ was also shown to model a thin neck in the Hele–Shaw cell [2]. Other applications include Cahn–Hilliard models with degenerate mobility [3], population dynamics [4], and problems in plasticity [5]. In all examples, in order to have a physical solution, h must be nonnegative.

Equation (1.1) describes fourth order degenerate diffusion. Like the second order porous media equation,

$$u_t = \Delta(u^m), \quad m > 1, \tag{1.3}$$

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[1] Here we use Δ to denote the Laplacian and later on we use it to denote space or time step. The meaning of each should be clear from the context.

its solutions are smooth whenever they are positive. This result is known in one space dimension but only conjectured in more than one space dimension. And where the solution vanishes, there is typically a loss of regularity. Unlike the (nondegenerate) heat equation (1.3 with $m = 1$), periodic solutions of degenerate diffusion equations can have finite speed of propagation for their support [6, 7]. Second order degenerate diffusion equations satisfy a maximum principle; the structure of the differential operator guarantees that solutions are bounded from above and below by their initial data.[2] The fourth order analogues do not possess such a property. The linear fourth order heat equation ((1.1) with $f(h) = 1$) has no maximum principle. In particular, positive initial data can easily lead to solutions that change sign. This always occurs for the Cauchy problem with $L^2(\mathbb{R}^d)$ initial data. Other degenerate fourth order diffusion equations also do not preserve sign of the solutions (see [8]).

What is unusual about (1.1) is that for sufficiently large values of n, the equation *does* preserve positivity of the solution. This was proved in one space dimension for $n \geq 4$ [9] and later extended to $n \geq 3.5$ [10]. For the two-dimensional (2D) problem, numerical computations of thin film flows suggest the same may be true. However, for smaller values of $n > 0$, numerical simulations [10, 11, 12] show that solutions can develop singularities of the form $h \to 0$, which physically describe the rupture of the liquid film. If the singularity forms in finite time, then the solution past this time can be defined through a regularization method [9, 13, 14] as a limit of strictly positive smooth solutions of regularized problems.

We briefly review some properties of positive solutions of (1.1) on a periodic domain $\Omega = (S^1)^d$ or in $\Omega = \mathbb{R}^d$. First note that solutions of (1.1) satisfy conservation of mass

$$\int_\Omega h(x,t)dx = \int_\Omega h(x,0)dx \tag{1.4}$$

and surface energy dissipation

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla h|^2 dx = -\int_\Omega f(h)|\nabla\Delta h|^2 dx. \tag{1.5}$$

The two results combine to yield an a priori H^1 bound for solutions of the PDE (1.1). Furthermore, positive solutions also satisfy a nonlinear entropy dissipation. Let $G(y)$ satisfy $G''(y) = 1/f(y)$. Then

$$\frac{d}{dt}\int_\Omega G(h)dx = -\int_\Omega |\Delta h|^2 dx. \tag{1.6}$$

Bernis and Friedman used these three properties in [9] to prove positivity of solutions of (1.1) in one dimension with Neumann-type boundary conditions whenever $n \geq 4$.

The nonlinear structure of the PDE presents a challenge in the design of efficient and accurate numerical methods. Even when the analytical solution is strictly positive, the solution of a generic scheme (see 2.2 and Definition 2.1) may become negative, especially when the grid is underresolved. Since the PDE becomes degenerate as $h \to 0$ this may lead to numerical instabilities. When a positive approximation of the solution is desired, it may be necessary to do computationally expensive local

[2]The precise version of this statement of course depends on the boundary conditions for (1.3). In this paper we consider simple periodic boundary conditions, and in this case the statement holds as is.

mesh refinement near the minimum of the solution in order to avoid such premature or "false" singularities [10]. Physically important examples of situations when such singularities may arise include flow down an inclined plane [15], where resolution was required at the apparent contact line [16].

For the computation of nonnegative weak solutions, a nonnegativity-preserving finite element method is proposed in [17]. Nonnegativity of the solution is imposed as a constraint, so one has to solve a variational problem involving a Lagrange multiplier at every time step to advance the nonnegative solution. In this work the authors showed the convergence of their finite element method to the weak nonnegative solution of the PDE. The advantage of using this method lies in the fact that computation of nonnegative solutions requires no regularization of the PDE or initial condition which seems particularly useful for tracking the moving contact lines. However, this method allows for solutions with positive initial data to lose positivity, which makes it less capable of capturing singularity formation comparable to the positivity-preserving method described here.

There are three main points of this paper. First, we show that it is possible to design a finite difference (in space) scheme to satisfy discrete analogues of properties (1.4)–(1.6) above. Such a scheme preserves positivity of the solution (whenever $n \geq 2$) and has solutions that exist at all times, regardless of the size of the grid. This method improves upon previous methods (e.g., in [10]) that required mesh refinement in order to avoid a premature "numerical" singularity. Second, we study the convergence properties of a larger class of finite difference schemes, satisfying discrete analogues of properties (1.4) and (1.5) but not necessarily (1.6). Although such schemes in general do not have global solutions for any grid size, we show that for a sufficiently fine mesh, the solution exists at all times and converges to a positive solution of the PDE. To our knowledge, this is the first paper to address the order of convergence of numerical schemes to smooth positive solutions of equations of the type (1.1) in one and two dimensions. Third, we show that it is possible to generalize the positivity-preserving property to finite element methods on arbitrary element spaces (including those involving nonuniform grids). Even though we do not prove convergence of these finite element methods in this paper, we anticipate that such methods will be useful for computations involving nonuniform and locally refined meshes.

This paper is organized as follows. We first consider, in section 2, a general finite difference (in space) scheme and prove local existence and uniqueness of its solution. The solutions of the difference schemes exist globally in time, provided that they remain positive. We then show, in section 3, that the key to preserving positivity is to construct a scheme that dissipates a discrete form of a nonlinear entropy, used in [9, 14, 13, 18] to prove positivity and existence results of nonnegative solutions for the continuous PDE. The scheme constructed in section 3 preserves positivity whenever the exponent in (1.2) satisfies $n \geq 2$. Sections 4.1 and 4.2 prove consistency and stability of general difference schemes. In section 5 we prove that both the generic and entropy dissipating schemes converge to a positive strong solution, with second order accuracy, under mesh refinement. Section 6 shows how to modify the entropy dissipating scheme to preserve positivity whenever $0 < n < 2$. Section 7 extends the results obtained in earlier sections to finite difference schemes in a 2D case. Section 8 generalizes the finite difference schemes to a general finite element framework, yielding positivity-preserving schemes on more complicated grids with higher order elements. Section 9 presents a computational example illustrating the advantage of using a positivity-preserving numerical scheme over a generic one.

2. Finite difference schemes for lubrication-type equations in one space dimension: Existence, uniqueness, and continuation of solutions. We consider a family of continuous-time, discrete-space finite difference schemes for the one-dimensional (1D) lubrication equation

$$\partial_t h + \partial_x (f(h)\partial_x^3 h) = 0, \quad x \in S^1, \quad f(h) \sim h^n \text{ as } h \to 0, \tag{2.1}$$

with periodic boundary conditions and strictly positive initial data $h_0(x) \in H^1(S^1)$. We prove, for positive initial data, local existence and uniqueness of solutions of the schemes. We also derive a global upper bound for positive solutions and show that the solution to the scheme can be continued in time, provided it stays positive. In the next section, we show that a particular choice of scheme guarantees a positive global solution.

Notation. We divide our periodic domain S^1 into N equally spaced regions of length Δx and introduce the following discrete analogues of the space derivatives:[3]

$$y_{x,i} = \frac{y_{i+1} - y_i}{\Delta x}, \qquad y_{\bar{x},i} = \frac{y_i - y_{i-1}}{\Delta x},$$
$$y_{\bar{x}x,i} = \frac{y_{\bar{x},i+1} - y_{\bar{x},i}}{\Delta x}, \quad y_{\bar{x}x\bar{x},i} = \frac{y_{\bar{x}x,i} - y_{\bar{x}x,i-1}}{\Delta x}.$$

The discrete $H_{1,\Delta x}$ norm is

$$\|y\|_{1,\Delta x} = \left(\sum_i (y_{\bar{x},i}^2 + y_i^2)\Delta x \right)^{1/2}.$$

This is equivalent to the norm

$$\left(\left(\sum_i y_{\bar{x},i}^2 \Delta x \right) + \bar{y}^2 \right)^{1/2},$$

where $\bar{y}$ denotes the mean $\sum_i y_i \Delta x$.

The numerical scheme

$$\begin{aligned} &y_{i,t} + (a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i})_x = 0, \\ &i = 0, 1, \dots, N-1, \\ &y_i(0) = h_0(x_i), \end{aligned} \tag{2.2}$$

which we can view as a coupled system of ODEs for the y_i, is a continuous-time discrete-space approximation of the PDE (2.1) provided that the nonlinear coefficient $a(s_1, s_2)$ satisfies the following conditions.

DEFINITION 2.1 (generic finite difference scheme).

(a) $a(s,s) = f(s)$,

(b) $a(s_1, s_2) = a(s_2, s_1)$,

(c) $a(s_1, s_2) \in C^4((0,\infty) \times (0,\infty)) \cap C([0,\infty) \times [0,\infty))$,

(d) $\forall \delta > 0$, *there exists* $\gamma > 0$ *such that* $s_1, s_2 > \delta \implies a(s_1, s_2) \geq \gamma > 0$.

[3]We append the i onto the x and $\bar{x}$ subscripts to make it clear that we are computing a finite difference as opposed to a spatial derivative. Throughout the rest of this paper, we use ∂_x, ∇, etc., to denote space derivatives and subscripts to denote finite differences in space.

In section 4.1 we use conditions (a)–(c) to show that the scheme (2.2) is second order consistent with a positive smooth solution of (2.1). Note that condition (c) does not require $a(s_1, s_2)$ to be differentiable at the origin. Condition (d) says that $a(s_1, s_2)$ is positive whenever both its arguments are positive but may become degenerate if at least one of its arguments approaches zero. This matches the property of the analytical solution (2.1) since $f(h) \sim h^n$ is positive whenever h is positive but $f(h) \to 0$ as $h \to 0$. We will need this last property to prove convergence of a solution of the scheme (2.2) to a smooth positive solution of the PDE (2.1). Both $a(s_1, s_2) = f(0.5(s_1 + s_2))$ and $a(s_1, s_2) = 0.5(f(s_1) + f(s_2))$ are examples of allowed discretizations. The former discretization was used, for example, in [10].

First we prove existence and uniqueness of solutions to the system (2.2). In order to do this we use the following important properties of the numerical scheme, which are analogues of the continuous case:

• Discrete conservation of mass.

$$\sum_{i=0}^{N-1} y_{i,t}\Delta x + \sum_{i=0}^{N-1} (a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i})_x \Delta x = 0$$

or

$$\sum_{i=0}^{N-1} y_i(t)\Delta x = \sum_{i=0}^{N-1} y_i(0)\Delta x. \tag{2.3}$$

• Discrete energy dissipation.

$$\frac{1}{2}\sum_i (y_{\bar{x},i}(t))^2\Delta x + \int_0^t \sum_i a(y_{i-1}(s), y_i(s))(y_{\bar{x}x\bar{x},i}(s))^2 \Delta x ds = \frac{1}{2}\sum_i (y_{\bar{x},i}(0))^2 \Delta x. \tag{2.4}$$

The discrete conservation of mass can be obtained by multiplying (2.2) by Δx, summation over $i = 0, 1, \ldots, N-1$, and time integration while the discrete energy dissipation can be obtained by multiplying (2.2) by $y_{\bar{x}x,i}\Delta x$, summation over $i = 0, 1, \ldots, N-1$, summation by parts, and time integration.

Lemma 2.2 (existence, uniqueness, and continuation of solutions of the numerical scheme). *Given the coupled ODE system* (2.2) *with strictly positive initial data* $y_i(0)$ *and diffusion coefficient* $a(s_1, s_2)$ *satisfying conditions* (a)–(d) *from Definition* 2.1, *there is a time interval* $[0, \sigma]$ *for which there exists a unique positive solution of the coupled ODE system* (2.2). *Moreover, there exists a constant* C*, depending only on the discrete* $H_{1,\Delta x}$ *norm of the initial data* $y_i(0)$*, such that* $y_i(t) \le C$ $\forall i$ *and* $\forall t \le \sigma$. *The solution can be continued for arbitrarily long times provided it remains positive.*

Proof. The system (2.2) can be rewritten in the form

$$\frac{dY}{dt} = F(Y),$$

where $Y = (y_0, y_1, \ldots, y_{N-1})$ and the ith component of the vector $F(Y)$ is given by

$$F_i(y_0, \ldots, y_{N-1}) = -\frac{1}{\Delta x}\left(a(y_i, y_{i+1}) y_{\bar{x}x\bar{x},i+1} - a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i}\right).$$

The standard theory of existence and uniqueness for systems of ODEs (see, for example, [19]) guarantees the local existence of a unique solution if the function

$F(Y)$ is uniformly Lipschitz continuous in a neighborhood of the initial data $Y(0)$. For this function to be Lipschitz continuous in all of its arguments, it suffices if the functions $a(s_1, s_2)$, $\frac{\partial a}{\partial s_1}(s_1, s_2)$, and $\frac{\partial a}{\partial s_2}(s_1, s_2)$ are bounded. But this is guaranteed by condition (c) of Definition 2.1 as long as the components of vector Y stay away from zero and infinity. This proves the local existence and uniqueness of system (2.2). Now, assuming that the solution exists and is unique, we will prove the existence of the uniform upper bound.

The discrete energy dissipation property implies that the discrete $H_{1,\Delta x}$ norm is bounded, which gives an upper bound on the discrete $C^{1/2}_{\Delta x}$ norm:

$$|y_i - y_j| \le C\,|(i-j)\Delta x|^{1/2}. \tag{2.5}$$

This fact can be proved by applying the Sobolev lemma to piecewise linear functions.

Let $i_*(t)$ denote a position where the maximum of $y_i(t)$ occurs. Then (2.5) implies

$$y_{i_*} - y_j \le C$$

$\forall j$. Multiplication by Δx and summation over j give

$$y_{i_*} - \sum_j y_j(t)\Delta x \le C$$

or

$$\max_i y_i(t) = y_{i_*} \le \sum_j y_j(0)\Delta x + C.$$

Classical continuation theory for ODEs guarantees that we can continue the solution until it leaves the region in which $F(Y)$ is uniformly Lipschitz. Thus we have global existence whenever we can guarantee that each component y_i of the solution stays bounded away from zero and infinity. We have proved a uniform upper bound depending only on the $H_{1,\Delta x}$ norm of the initial data. Thus, either $\min_i y_i(t)$ stays positive at all times, in which case we have a global in time solution or there is a maximal time of existence $\tilde{T} < \infty$ which is defined to be the earliest time when one of the y_i's goes to zero. ∎

In the next section we show that if $n \ge 2$ in (2.1), then there exists a way to choose the function $a(s_1, s_2)$ in (2.2) so that, for positive initial data, the solution of (2.2) stays positive at all times. This property is a discrete analogue of the weak maximum principle for the PDE (2.1).

3. A positivity-preserving finite difference scheme.

In the previous section we introduced a general finite difference scheme

$$\begin{aligned} &y_{i,t} + (a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i})_x = 0,\\ &i = 0, 1, \dots, N-1,\\ &y_i(0) = h_0(x_i) \end{aligned} \tag{3.1}$$

for the 1D lubrication equation

$$\partial_t h + \partial_x (f(h)\partial_x^3 h) = 0, \quad x \in S^1, \quad f(h) \sim h^n \text{ as } h \to 0. \tag{3.2}$$

We showed that such a scheme satisfies the discrete versions (2.3) and (2.4) of the conservation of mass (1.4) and energy dissipation (1.5) of the continuous PDE. In addition, solutions of the PDE dissipate a nonlinear entropy (1.6) which can be used to prove positivity of the solution for sufficiently large n. Our goal is to choose $a(s_1, s_2)$ so that the numerical scheme (3.1) satisfies a discrete form of (1.6) and show that this is sufficient to guarantee that its solutions also preserve positivity.

LEMMA 3.1. *Let $G''(v) = \frac{1}{f(v)}$ be a nonlinear entropy function. If we choose $a(s_1, s_2)$ in* (3.1) *to satisfy*

$$a(s_1, s_2) = \begin{cases} \frac{s_1 - s_2}{G'(s_1) - G'(s_2)} & \text{if } s_1 \neq s_2, \\ f(s_1) & \text{if } s_1 = s_2, \end{cases} \tag{3.3}$$

then the solution of (3.1), *with positive initial data, satisfies the following discrete entropy dissipation property:*

$$\sum_i G(y_i(t))\Delta x + \int_0^t \left\{ \sum_i (y_{\bar{x}x,i}(s))^2 \Delta x \right\} ds = \sum_i G(y_i(0))\Delta x. \tag{3.4}$$

Proof. First we would like to remark that $a(s_1, s_2)$ defined in (3.3) satisfies all of the properties in Definition 2.1. In particular, (c) can be checked using the Taylor series expansions near $s_2 = s_1$ and the definition of $G'(h)$.

Direct computation shows that

$$\begin{aligned} \frac{d}{dt} \sum_i G(y_i)\Delta x &= \sum_i G'(y_i) y_{i,t} \Delta x \\ &= -\sum_i G'(y_i)(a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i})_x \Delta x \\ &= \sum_i (G'(y_i))_{\bar{x}} a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i} \Delta x \\ &= -\sum_i \left\{ (G'(y_i))_{\bar{x}} a(y_{i-1}, y_i) \right\}_x y_{\bar{x}x,i} \Delta x. \end{aligned}$$

Now we would like the resulting expression to be $-\sum_i (y_{\bar{x}x,i})^2 \Delta x$, which is an analogue of $-\int_0^1 h_{xx}^2 dx$ for the continuous case, thus forcing us to choose $a(s_1, s_2)$ as in (3.3).

Integration in time gives

$$\sum_i G(y_i(t))\Delta x + \int_0^t \left\{ \sum_i (y_{\bar{x}x,i}(s))^2 \Delta x \right\} ds = \sum_i G(y_i(0))\Delta x. \qquad \square$$

Note that (3.4) gives an a priori upper bound on $\sum_i G(y_i(t))\Delta x$. This allows us to get a positive lower bound on $\min_t \min_i y_i(t)$.

PROPOSITION 3.2. *Let $f(h) \sim h^n$, as $h \to 0$, $n \geq 2$, and*

$$a(s_1, s_2) = \begin{cases} \frac{s_1 - s_2}{G'(s_1) - G'(s_2)} & \text{if } s_1 \neq s_2, \\ f(s_1) & \text{if } s_1 = s_2. \end{cases} \tag{3.5}$$

Then given positive initial data $y_i(0) > 0$, for every fixed Δx there exists $\bar{\delta} > 0$ such that $\min_t \min_i y_i(t) \geq \bar{\delta}$. Moreover, if $n \geq 4$, then $\bar{\delta}$ depends only on $\|y_i\|_{1,\Delta x}$ and $\min_i y_i$. In particular, it does not depend directly on Δx.

Proof. For simplicity, we take $f(h) = h^n$. The more general case is proved in an analogous fashion. First we prove the result for $n \geq 2$. Choose $G(v) = \frac{v^{-n+2}}{(-n+1)(-n+2)}$ for $n > 2$ and $G(v) = -\ln(v)$ for $n = 2$.

Using the discrete entropy dissipation property (3.4), we get

$$\sum_j \frac{1}{y_j^{n-2}} \Delta x \le \tilde{C}, \quad \text{if } n > 2,$$

$$\sum_j \ln\left(\frac{1}{y_j}\right) \Delta x \le \tilde{C}, \quad \text{if } n = 2,$$

where $\tilde{C}$ depends on the uniform positive lower and (for $n = 2$) upper bounds of the initial condition $y_i(0)$. This implies that

$$\min_t \min_i y_i(t) \ge \left(\frac{\Delta x}{\tilde{C}}\right)^{\frac{1}{n-2}} \equiv \bar{\delta}, \quad \text{if } n > 2,$$

$$\min_t \min_i y_i(t) \ge \exp\left(-\frac{\tilde{C}}{\Delta x}\right) \equiv \bar{\delta}, \quad \text{if } n = 2.$$

This estimate is valid for any $n \ge 2$. However, it depends strongly on Δx; that is, as Δx gets smaller the lower bound gets smaller as well. Since one can derive a positive lower bound on the solution of the PDE (3.2) for n large enough [9] (see also [10] for a sharper result), one would like to obtain a lower bound independent of Δx for the discrete case as well. The following argument, similar to that for the continuous case [9], shows that this indeed is possible for $n \ge 4$.

Recall that the discrete energy dissipation property and the Sobolev lemma imply the existence of an upper bound on the discrete $C^{1/2}_{\Delta x}$ norm (2.5) of $y_i(t)$. Let $\delta(t) = \min_{0 \le i \le N-1} y_i(t)$, which occurs at $i(t)$. Then we get

$$\begin{aligned}
\tilde{C} &\ge \sum_j \frac{1}{y_j^{n-2}} \Delta x \ge \sum_j \frac{\Delta x}{\left(\delta(t) + C\,|(j - i(t))\Delta x|^{1/2}\right)^{n-2}} \\
&\ge 2 \sum_j \frac{\Delta x}{\left(\delta(t) + C\,(j\Delta x)^{1/2}\right)^{n-2}} \ge 2 \sum_j \frac{\Delta x/\delta^2}{\delta^{n-4}\left(1 + C\,(j\Delta x/\delta^2)^{1/2}\right)^{n-2}} \\
&\ge 2 \frac{1}{\delta^{n-4}} \int_0^{1/\delta^2} \frac{ds}{\left(1 + s^{1/2}\right)^{n-2}}.
\end{aligned}$$

Now to obtain a bound independent of Δx, we consider two cases: $\delta > 1$ and $\delta \le 1$. In the former case we already have the bound we wanted. In the latter case we get

$$\tilde{C} \ge 2 \frac{1}{\delta^{n-4}} \int_0^{1/\delta^2} \frac{ds}{\left(1 + s^{1/2}\right)^{n-2}} \ge 2 \frac{1}{\delta^{n-4}} \int_0^{1} \frac{ds}{\left(1 + s^{1/2}\right)^{n-2}} = \frac{C'}{\delta^{n-4}}, \quad \text{if } n > 4,$$

$$\tilde{C} \ge 2 \int_0^{1/\delta^2} \frac{ds}{\left(1 + s^{1/2}\right)^{2}} \ge C' \ln\left(\frac{1}{\delta^2}\right), \quad \text{if } n = 4,$$

which again implies the existence of a lower bound independent of Δx. $\square$

Thus, with the choice of spatial discretization (3.5) the finite difference scheme (3.1) is positivity-preserving $\forall n \ge 2$, and the lower bound is independent of Δx, provided $n \ge 4$. It is interesting to note that for $2 \le n < 3.5$, we have proved positivity of the numerical scheme when we do not know that such a result is true for the continuous PDE. Proposition 3.2 can be sharpened slightly following the convergence argument in section 5. We can show that the solution has a positive lower bound independent of the grid size $\forall n \geq 3.5$ (see Remark 1 at the end of section 5).

Note that the above results, coupled with Lemma 2.2 in section 2, prove global existence of positive solutions of the scheme (3.1) with the special choice of $a(s_1, s_2)$ in (3.5).

COROLLARY 3.3. *Let $f(h) \sim h^n$ with $n \geq 2$. The finite difference scheme* (3.1), *with the special choice of $a(s_1, s_2)$ satisfying* (3.5), *and positive initial data $y_i(0) = h_0(x_i) > 0$ has a global in time positive solution $y_i(t)$ regardless of the size of the mesh Δx.*

4. Consistency and stability of difference schemes in one space dimension.

In section 2 we proved the existence and uniqueness of solution of a class of finite difference schemes (2.2). In section 3 we showed that a special choice of scheme from this class dissipates a discrete form of an entropy, creating a scheme that preserves positivity of solutions. In this section we prove that the general scheme (2.2) is second order consistent with a smooth positive solution of the PDE. In section 4.2 we show that the scheme is stable whenever its solution remains bounded away from zero.

4.1. Consistency.

As before, let $h(x, t)$ denote a positive solution of the original PDE (2.1) and $y_i(t)$ denote the solution of the numerical scheme (2.2). To measure consistency, we introduce the local truncation error $\tau_i(t)$, defined as the result of substituting the solution of the PDE (2.1) into (2.2):

$$h_{i,t} + (a(h_{i-1}, h_i) h_{\bar{x}x\bar{x},i})_x = \tau_i(t). \tag{4.1}$$

Since a positive solution of (2.1) is infinitely differentiable [9], we can use Taylor series expansions to examine the consistency of the numerical scheme.

LEMMA 4.1. *Let $a(s_1, s_2)$ satisfy the conditions* (a)–(d) *of Definition* 2.1 *and let $h(x, t)$ be a smooth positive solution of* (2.1). *Then the local truncation error $\tau_i(t)$ in* (4.1) *is $O((\Delta x)^2)$ uniformly in t.*

Proof. By using a Taylor series expansion we obtain[4]

$$h_{\bar{x}x\bar{x},i} = \frac{h_{i+1} - 3h_i + 3h_{i-1} - h_{i-2}}{(\Delta x)^3} = h'''(x_{i-\frac{1}{2}}, t) + \alpha(x_{i-\frac{1}{2}}, t)(\Delta x)^2 + O((\Delta x)^4),$$

$$\begin{aligned} a(s_1, s_2) = a(s + \Delta s, s - \Delta s) &= a(s, s) + \frac{\partial a}{\partial s_1}(s, s)\Delta s - \frac{\partial a}{\partial s_2}(s, s)\Delta s \\ &+ \tilde{\beta}(s, s)(\Delta s)^2 + O((\Delta s)^3) = f(s) + \beta(s)(\Delta s)^2 + O((\Delta s)^3), \end{aligned}$$

where $s = \frac{s_1+s_2}{2}$, $\Delta s = \frac{s_1-s_2}{2}$, $\alpha(x, t) = \frac{1}{8} h^{(5)}(x, t)$, and

$$\tilde{\beta}(s, s) = \beta(s) = \frac{1}{2}\left(\frac{\partial^2 a(s, s)}{\partial s_1^2} - 2\frac{\partial^2 a(s, s)}{\partial s_1 \partial s_2} + \frac{\partial^2 a(s, s)}{\partial s_2^2}\right).$$

We used the symmetry of $a(s_1, s_2)$ to cancel the first order in Δs terms. Note that $\alpha(x, t)$ is infinitely differentiable and $\beta(s)$ is twice continuously differentiable for $s > 0$ by property (c) of Definition 2.1.

[4] Here $'$ denotes space derivative.

Therefore,

$$
\begin{aligned}
(a(h_{i-1},h_i)h_{\bar{x}x\bar{x},i})_x &= \frac{1}{\Delta x}\left(a(h_i,h_{i+1})h_{\bar{x}x\bar{x},i+1} - a(h_{i-1},h_i)h_{\bar{x}x\bar{x},i}\right)\\
&= \frac{1}{\Delta x}\Bigg[\left\{f\left(\frac{h_i+h_{i+1}}{2}\right)+\beta\left(\frac{h_i+h_{i+1}}{2}\right)\left(\frac{h_{i+1}-h_i}{2}\right)^2+O((\Delta x)^3)\right\}\\
&\qquad \left(h'''(x_{i+\frac{1}{2}})+\alpha(x_{i+\frac{1}{2}})(\Delta x)^2+O((\Delta x)^4)\right)\\
&\qquad -\left\{f\left(\frac{h_{i-1}+h_i}{2}\right)+\beta\left(\frac{h_{i-1}+h_i}{2}\right)\left(\frac{h_i-h_{i-1}}{2}\right)^2+O((\Delta x)^3)\right\}\\
&\qquad \left(h'''(x_{i-\frac{1}{2}})+\alpha(x_{i-\frac{1}{2}})(\Delta x)^2+O((\Delta x)^4)\right)\Bigg].
\end{aligned} \tag{4.2}
$$

Note that for any twice continuously differentiable function $g(s)$

$$
g\left(\frac{h_i+h_{i+1}}{2}\right) = g(h_{i+\frac{1}{2}}) + g'(h_{i+\frac{1}{2}})\frac{h''_{i+\frac{1}{2}}}{2}\left(\frac{\Delta x}{2}\right)^2 + O((\Delta x)^4), \tag{4.3}
$$

$$
\left(\frac{h_{i+1}-h_i}{2}\right)^2 = (h'_{i+\frac{1}{2}})^2\left(\frac{\Delta x}{2}\right)^2 + O((\Delta x)^4). \tag{4.4}
$$

By using property (c) of Definition 2.1 we obtain that function $\beta(s)$ is $C^2(0,\infty)$, so we are allowed to use an asymptotic expansion (4.3) for $\beta(s)$. Taking function $g(s)$ in (4.3) to be $f(s)$ and $\beta(s)$ and plugging the results into (4.2) gives

$$
(a(h_{i-1},h_i)h_{\bar{x}x\bar{x},i})_x = (f(h_i)h'''(x_i,t))' + O((\Delta x)^2). \qquad \square
$$

Note that the error depends on the size of the higher derivatives of h, which are known empirically [10, 11] to become unbounded when a positive solution approaches zero.

4.2. Stability near flat steady state. We now show that the flat steady state solutions of the finite difference schemes introduced in the previous sections are stable. The results of this subsection are not needed to prove convergence in the next section.

Consider the inhomogeneous numerical scheme

$$
y_{i,t} + (a(y_{i-1},y_i)y_{\bar{x}x\bar{x},i})_x = f_i(t) \tag{4.5}
$$

with periodic boundary conditions, positive initial data $y_i(0)$, and small right-hand side $f_i(t)$.

From the results of section 2 we know that the following is true for the homogeneous problem:

• For any constant $c > 0$, $\{y_i \equiv c\}$ is a solution.

• If the initial condition is positive, then the solution is bounded and can be continued in time, provided it stays positive.

In this subsection we prove that if the initial data is close to the constant solution, then the solution of the forced (inhomogeneous) equation will stay close to the solution of the unforced equation provided that $f_i(t)$ are small. Moreover, for positive initial data and bounded $f_i(t)$ the solution of the forced equation is bounded and can be continued in time, provided it stays positive. While the results of this section are independent of the results of the next section on convergence, the energy methods used here are similar to some of the arguments used there for convergence.

THEOREM 4.2. *Let $y_i(t)$ be a solution of* (4.5) *with positive initial data $y_i(0)$.*

(1) *Suppose $f_i(t)$ is small in $C([0,T];H_{1,\Delta x})$ and $y_i(0) = \bar{y}(0) + \epsilon_i(0)$, where $\bar{y} = \sum_i y_i \Delta x$ is the mean of y and $\sum_i \epsilon_{\bar{x},i}(0)^2 \Delta x$ is small in $H_{1,\Delta x}$. Then $y_i(t) = \bar{y}(0) + \epsilon_i(t)$ with $\epsilon_i(t)$ small in $C([0,T];H_{1,\Delta x})$.*

(2) *Suppose $f_i(t)$ is bounded in $C([0,T];H_{1,\Delta x})$. Then there exists a constant $C > 0$ such that $y_i(t) \le C$ $\forall i$ and t provided that $y_i(t) > 0$.*

Proof. Multiply (4.5) by $y_{\bar{x}x,i}\Delta x$, sum in i over a period, and apply summation by parts to the left-hand side. We get

$$\frac{d}{dt}\left\{\frac{1}{2}\sum_i y_{\bar{x},i}^2 \Delta x\right\} + \sum_i a(y_{i-1}, y_i) y_{\bar{x}x\bar{x},i}^2 \Delta x = -\sum_i f_i y_{\bar{x}x,i}\Delta x. \tag{4.6}$$

Using the fact that the second term on the left-hand side of (4.6) has a fixed sign, we apply summation by parts to the right-hand side of (4.6) and the Schwarz inequality to obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\sum_i y_{\bar{x},i}^2 \Delta x\right\} \le \left(\sum_i f_{\bar{x},i}^2 \Delta x\right)^{1/2} \left(\sum_i y_{\bar{x},i}^2 \Delta x\right)^{1/2}.$$

Dividing by $\left(\sum_i y_{\bar{x},i}^2 \Delta x\right)^{1/2}$ and integrating in time give

$$\left(\sum_i y_{\bar{x},i}^2(t) \Delta x\right)^{1/2} \le \left(\sum_i y_{\bar{x},i}^2(0) \Delta x\right)^{1/2} + \int_0^t \left(\sum_i f_{\bar{x},i}^2(s) \Delta x\right)^{1/2} ds. \tag{4.7}$$

To prove part (1) we use the fact that $\sum_i y_{\bar{x},i}^2(t)\Delta x = \sum_i \epsilon_{\bar{x},i}^2(t)\Delta x$ to obtain

$$\left(\sum_i \epsilon_{\bar{x},i}^2(t) \Delta x\right)^{1/2} \le \left(\sum_i \epsilon_{\bar{x},i}^2(0) \Delta x\right)^{1/2} + \int_0^t \left(\sum_i f_{\bar{x},i}^2(s) \Delta x\right)^{1/2} ds. \tag{4.8}$$

The mean of the numerical solution satisfies

$$\frac{d}{dt}\sum_i y_i(t)\Delta x = \sum_i f_i(t)\Delta x,$$

which implies that

$$\sum_i y_i(t)\Delta x = \sum_i y_i(0)\Delta x + \int_0^t \left(\sum_i f_i(s)\Delta x\right) ds \tag{4.9}$$

or

$$\bar{e}(t) = \int_0^t \left(\sum_i f_i(s)\Delta x\right) ds. \tag{4.10}$$

Combining (4.8) and (4.10) proves that the discrete $H_{1,\Delta x}$ norm of $\epsilon_i(t)$ is controlled by the discrete $H_{1,\Delta x}$ norm of the initial data and the $C([0,T];H_{1,\Delta x})$ norm of f_i, concluding the proof of part (1).

To prove the second part of the statement we note that (4.7) and (4.9) imply that the discrete $H_{1,\Delta x}$ norm of $y_i(t)$ is bounded. By applying the discrete Sobolev lemma we conclude that the discrete $C_{\Delta x}^{1/2}$ norm is bounded, which gives an a priori upper bound on the solution $y_i(t)$ by the same argument as we used at the end of section 2. ∎

5. Convergence of finite difference schemes in one space dimension. In this section, we prove convergence of finite difference schemes to positive solutions of the PDE

$$h_t + \partial_x(f(h)\partial_x^3 h) = 0, \quad x \in S^1. \tag{5.1}$$

Convergence of finite difference methods in two space dimensions is considered in section 7.

Recall that the continuous time-discrete space numerical method

$$\begin{aligned} &y_{i,t} + (a(y_{i-1}, y_i)y_{\bar{x}x\bar{x},i})_x = 0,\\ &i = 0, 1, \dots , N-1,\\ &y_i(0) = h_0(x_i) \end{aligned} \tag{5.2}$$

with $a(s_1, s_2)$ satisfying Definition 2.1 is a generic scheme (GS) for solving (5.1). The special finite difference scheme (5.2) with $a(s_1, s_2)$ defined by

$$a(s_1, s_2) = \begin{cases} \frac{s_1 - s_2}{G'(s_1) - G'(s_2)} & \text{if } s_1 \neq s_2,\\ f(s_1) & \text{if } s_1 = s_2 \end{cases} \tag{5.3}$$

is an entropy dissipating scheme (EDS) that preserves positivity of the solution for sufficiently degenerate f. This scheme is a special case of GS.

In section 2 we proved the local existence and uniqueness of solutions to GS. At the end of section 3, in Corollary 3.3, we proved that the EDS is guaranteed to have global positive solutions whenever $f(h) \sim h^n$ as $h \to 0$, $n \geq 2$. In this section we investigate the question of global existence of solutions of GS as well as convergence of solutions of the numerical schemes to the solution of PDE (5.1) as $\Delta x \to 0$. We prove the following result.

THEOREM 5.1. *Let $h(x, t)$ be a smooth solution of* (5.1) *such that $h(x, t) \geq \delta > 0$ for some $\delta > 0$ and $t \in [0, T]$. Then for Δx sufficiently small there exists a unique solution $y_i(t)$ of* GS *$\forall t \in [0, T]$. Moreover, there exists a constant $C > 0$ such that* $\sup_{t\in[0,T]} \| y_i(t) - h(x_i, t) \|_{1,\Delta x} \leq C\Delta x^2$.

Proof. By Lemma 2.2 of section 2 a solution of the numerical scheme can be continued as long as it stays positive. In Corollary 3.3 we proved that the EDS has globally positive solutions whenever $n \geq 2$. In the case of GS, or EDS with $n < 2$, the numerical solution may not exist globally $\forall \Delta x$. However, it exists globally ($\forall t \in [0, T]$) if Δx is small enough. To prove this we use the fact that the analytical solution $h(x, t) \geq \delta > 0$.

At time $t = 0$ we have $y_i(0) = h_i(0) \equiv h(x_i, 0) \geq \delta/2$ $\forall i$. According to Lemma 2.2 of section 2, for every Δx there exists a time interval $[0, \sigma]$, $\sigma < T$ such that $\forall t \in [0, \sigma]$

$$y_i(t) \geq \delta/2 \quad \forall i. \tag{5.4}$$

We now estimate, on this time interval, the difference between the exact solution $h_i(t) \equiv h(x_i, t)$ and the approximate solution $y_i(t)$ on $[0, \sigma]$. From section 4.1, we know that the solution of the PDE (5.1) satisfies the finite difference equation

$$h_{i,t} + (a(h_{i-1}, h_i)h_{\bar{x}x\bar{x},i})_x = \tau_i(t) \tag{5.5}$$

with a local truncation error $\tau_i(t) = O((\Delta x)^2)$ uniformly in t. Define an error $e_i(t) = y_i(t) - h_i(t)$. Subtracting (5.5) from (5.2) gives an equation for the error

$$e_{i,t} + [h_{\bar{x}x\bar{x},i}(a(y_{i-1}, y_i) - a(h_{i-1}, h_i))]_x + [a(y_{i-1}, y_i)e_{\bar{x}x\bar{x},i}]_x = -\tau_i(t). \tag{5.6}$$

Multiplication by $e_{\bar{x}x,i}\Delta x$, summation over i, and summation by parts give

$$\frac{d}{dt}\left[\frac{1}{2}\sum_i e_{\bar{x},i}^2\Delta x\right] + \sum_i h_{\bar{x}x\bar{x},i}(a(y_{i-1},y_i) - a(h_{i-1},h_i))e_{\bar{x}x\bar{x},i}\Delta x + \sum_i a(y_{i-1},y_i)e_{\bar{x}x\bar{x},i}^2\Delta x = \sum_i \tau_i e_{\bar{x}x,i}\Delta x. \tag{5.7}$$

To estimate the second term on the left we first use the continuity of h_{xxx} and the Schwarz inequality:

$$\left|\sum_i h_{\bar{x}x\bar{x},i}(a(y_{i-1},y_i) - a(h_{i-1},h_i))e_{\bar{x}x\bar{x},i}\Delta x\right| \le C\left[\sum_i (a(y_{i-1},y_i) - a(h_{i-1},h_i))^2\Delta x\right]^{1/2}\left[\sum_i e_{\bar{x}x\bar{x},i}^2\Delta x\right]^{1/2}. \tag{5.8}$$

Using property (c) of Definition 2.1, the fact that both the analytical and numerical solutions are bounded away from zero and infinity, and the mean value theorem give

$$|a(y_{i-1},y_i) - a(h_{i-1},h_i)| \le \|\nabla a\|_{L^\infty([\delta/2,M]^2)}\,(|e_{i-1}| + |e_i|).$$

Therefore,

$$\sum_i (a(y_{i-1},y_i) - a(h_{i-1},h_i))^2\Delta x \le C\sum_i e_i^2\Delta x \le C\left(\bar{e}^2 + \sum_i e_{\bar{x},i}^2\Delta x\right) \tag{5.9}$$

by the discrete Poincaré inequality, where the constant $C = 4\,\|\nabla a\|^2_{L^\infty([\delta/2,M]^2)}$. Here $\bar{e} = \sum_i e_i\Delta x$ is the mean of the error, which satisfies an ODE

$$\frac{d\bar{e}}{dt} = -\sum_i \tau_i(t)\Delta x.$$

This implies that

$$\bar{e}(t) = -\int_0^t \left(\sum_i \tau_i(s)\Delta x\right) ds. \tag{5.10}$$

By Definition 2.1 since $y_i(t) \ge \delta/2\ \forall i$ there exists $\gamma > 0$ such that $a(y_{i-1},y_i) \ge \gamma$. Therefore, the third term on the left-hand side of (5.7) can be moved to the right with a negative sign

$$-\sum_i a(y_{i-1},y_i)e_{\bar{x}x\bar{x},i}^2\Delta x \le -\gamma\sum_i\sum_i e_{\bar{x}x\bar{x},i}^2\Delta x.$$

Now by using the discrete Poincaré inequality, the Schwarz inequality, and the ϵ-inequality $2ab \le (a\epsilon)^2 + (b/\epsilon)^2$ we can get rid of the $e_{\bar{x}x}$ and $e_{\bar{x}x\bar{x}}$ dependence of the second and the fourth terms in (5.7) to get the differential inequality

$$\frac{d}{dt}\left[\frac{1}{2}\sum_i e_{\bar{x},i}^2\Delta x\right] \le C\sum_i \tau_i^2\Delta x + C\sum_i e_{\bar{x},i}^2\Delta x + C\left(\int_0^t \left(\sum_i \tau_i(s)\Delta x\right) ds\right)^2,$$

where C depends on γ. Since $\tau_i = O((\Delta x)^2)$ uniformly in $t \in [0, \sigma]$,

$$\frac{d}{dt}\left[\frac{1}{2}\sum_i e_{\bar{x},i}^2 \Delta x\right] \le C \sum_i e_{\bar{x},i}^2 \Delta x + C(\Delta x)^4.$$

Also, $|\bar{e}| \le C(\Delta x)^2$ by (5.10).

This gives the a priori bound on the discrete $H_{1,\Delta x}$ norm of the error

$$\| e(t) \|_{1,\Delta x}^2 \le C_1 (\Delta x)^4 \left(e^{CT} - 1\right), \tag{5.11}$$

which proves that $\| e(t) \|_{1,\Delta x}$ is $O((\Delta x)^2)$ uniformly in $t \in [0, \sigma]$. Estimate (5.11) holds any time interval $[0, \sigma]$ for which (5.4) is true. Now let $\tilde{T} \le T$ be the maximal time interval for which the solution satisfies (5.4). Continuity in time of the solution y guarantees the existence of such a time. Then the estimate (5.11) holds with $\sigma = \tilde{T}$.

We now show that the pointwise lower bound (5.4) persists and we can continue the solution on $[0, T]$, provided Δx is sufficiently small. Note that from the above,

$$\sum_i e_i^2 \Delta x \le C(\Delta x)^4,$$

which implies a pointwise estimate

$$|e_i| = |y_i - h_i| \le C(\Delta x)^{3/2} \to 0 \text{ as } \Delta x \to 0,$$

and hence

$$y_i(t) \ge \delta - C(\Delta x)^{3/2}. \tag{5.12}$$

Suppose Δx satisfies

$$C(\Delta x)^{3/2} < \delta/2 \tag{5.13}$$

and $\tilde{T}$ is strictly less than T. Then (5.12) gives a better bound than (5.4), contradicting the assumption that $\tilde{T}$ is a maximal time for which (5.4) holds. Thus for sufficiently small Δx, we can continue the solution $y_i(t)$ $\forall t \in [0, T]$ so that (5.11) still holds. $\square$

REMARK 1. *Note that the results of this theorem allow us to sharpen the last sentence in Proposition* 3.2. *That is, the solution of EDS of section* 3 *is bounded away from zero uniformly in* Δx *whenever* $n \ge 3.5$. *This is because* (1) *from Proposition* 3.2 *we have a lower bound* $\delta(\Delta x)$ *whenever* $n \ge 2$, *where the function* δ *is monotone decreasing as* $\Delta x \to 0$; (2) *the numerical solution converges to the analytical solution, which is guaranteed to remain positive whenever* $n \ge 3.5$. *Combining these results gives a uniform lower bound for the solution of EDS whenever* $n \ge 3.5$.

6. Modified entropy dissipating scheme. In Corollary 3.3 of section 3 we showed that if $n \ge 2$, then $\forall \Delta x$ there exists a unique positive solution of the EDS $\forall t$. However, for $n < 2$ the EDS (5.2)–(5.3) may not be positivity-preserving. In examples such as Hele–Shaw ($n = 1$), we may wish to use a positivity-preserving scheme to approximate a positive solution. We now present a modification of the EDS that yields a positivity-preserving scheme $\forall n > 0$. To accomplish this, we introduce the regularization of the PDE, which was introduced in [9] and later used by [13],

$$h_{\epsilon t} + \partial_x (f_\epsilon(h_\epsilon) \partial_x^3 h_\epsilon) = 0, \; f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 f(h_\epsilon)}{\epsilon f(h_\epsilon) + h_\epsilon^4},$$

and the corresponding numerical scheme

$$\begin{aligned}&y_{\epsilon,i,t} + (a_\epsilon(y_{\epsilon,i-1}, y_{\epsilon,i})y_{\epsilon\bar{x}x\bar{x},i})_x = 0,\\ &i = 0, 1, \dots, N-1,\\ &y_{\epsilon,i}(0) = h_0(x_i),\end{aligned} \tag{6.1}$$

$$a_\epsilon(s_1, s_2) = \begin{cases} \frac{s_1 - s_2}{G'_\epsilon(s_1) - G'_\epsilon(s_2)} & \text{if } s_1 \neq s_2,\\ f_\epsilon(s_1) & \text{if } s_1 = s_2.\end{cases} \tag{6.2}$$

Note that numerical method (6.1)–(6.2) depends on two independent parameters, namely, Δx and ϵ. However, we could choose $\epsilon = \epsilon(\Delta x)$ to be a function of the grid size.

Since $f_\epsilon(h_\epsilon) \sim \frac{h_\epsilon^4}{\epsilon}$ as $h_\epsilon \to 0$ we know that $\forall \epsilon > 0$ and $\Delta x > 0$ the solution of (6.1)–(6.2) is positive. Moreover, the following theorem shows that the solution of the modified EDS converges to the solution of the original (nonregularized) PDE as $\epsilon \to 0$ and $\Delta x \to 0$ as long as the solution of the latter stays positive. The proof relies on the fact that $\forall \bar{\delta} > 0$, $m, n \in \mathbb{N}$, there exists a constant $C(m, n, \bar{\delta})$ so that

$$\left\| \frac{\partial^{n+m}}{\partial s_1^n \partial s_2^m} a_\epsilon(s_1, s_2) - \frac{\partial^{n+m}}{\partial s_1^n \partial s_2^m} a(s_1, s_2) \right\|_{L^\infty([\bar{\delta}, M]^2)} \leq C(m, n, \bar{\delta})\epsilon. \tag{6.3}$$

THEOREM 6.1. *Let $h(x,t)$ be a smooth solution of* (5.1) *such that $h(x,t) \geq \delta > 0$ for some $\delta > 0$ and $t \in [0,T]$. Then $\forall \Delta x$ and ϵ there exists a unique solution $y_{\epsilon,i}(t)$ of the modified EDS* (6.1)–(6.2) *$\forall t \in [0,T]$. Moreover, there exists a constant $C > 0$ independent of Δx and ϵ such that*

$$\sup_{t\in[0,T]} \| y_{\epsilon,i}(t) - h(x_i, t) \|_{1,\Delta x} \leq C(\Delta x^2 + \epsilon).$$

Proof. As in section 4.1, to analyze consistency we introduce the local truncation error

$$\tau_{\epsilon,i}(t) = h_{i,t} + (a_\epsilon(h_{i-1}, h_i)h_{\bar{x}x\bar{x},i})_x. \tag{6.4}$$

Following the arguments from section 4.1 and using (6.3) gives us $\tau_{\epsilon,i}(t) = O((\Delta x)^2 + \epsilon)$.

Next note that from Proposition 3.2, the solution y_i has an a priori pointwise lower bound independent of Δx but possibly depending strongly on ϵ. Thus the solution y_i exists and remains positive on $[0, T]$.

Since the positive lower bound of $y_{\epsilon,i}(t)$ may depend strongly on ϵ, to prove convergence we need to use the inequality $h(x,t) \geq \delta > 0$. Applying the argument from the proof of Theorem 5.1 of the previous section, for each Δx and ϵ there exists a time $\sigma = \sigma(\epsilon, \Delta x)$ such that

$$y_{\epsilon,i} \geq \delta/2 \ \forall i \text{ and } \ \forall t \in [0, \sigma]. \tag{6.5}$$

Now we can follow the proof of Theorem 5.1 to obtain

$$\| e_\epsilon(t) \|_{1,\Delta x} \leq C(\Delta x^2 + \epsilon) \ \forall i \text{ and } t \in [0, \sigma], \tag{6.6}$$

where $e_\epsilon(t) = y_{\epsilon,i}(t) - h_i(t)$.

In order to obtain (6.6) we needed $a_\epsilon(y_{\epsilon,i}, y_{\epsilon,i-1}) > \tilde{\delta} > 0$ and $\|\nabla a_\epsilon\|_{L^\infty} < \tilde{C}$ for some $\tilde{\delta}$ and $\tilde{C}$ independent of Δx and ϵ. The existence of these constants is guaranteed by the inequalities $y_{\epsilon,i}(t) \geq \delta/2$ and the uniform convergence of $a_\epsilon(s_1, s_2)$ and its derivatives to $a(s_1, s_2)$ on $[\bar{\delta}, M]^2$ (6.3). The inequality (6.6) implies that for sufficiently small Δx and ϵ, the estimate (6.5) can be improved. Thus we can extend the estimate (6.5) to the whole time interval $[0, T]$ for sufficiently small Δx and ϵ, following an argument similar to the one at end of the proof of Theorem 5.1. ∎

REMARK 2. *We can choose* $\epsilon \sim (\Delta x)^2$ *to make the scheme* (6.1)–(6.2) *second order in* Δx.

We discuss the relevance of this scheme for weak nonnegative solutions of (5.1) in the conclusions.

7. Finite difference schemes in two dimensions. In two space dimensions the lubrication equation with periodic boundary conditions is

$$\begin{aligned} &h_t + \partial_x(f(h)\partial_x(\partial_x^2 h + \partial_y^2 h)) \\ &\qquad + \partial_y(f(h)\partial_y(\partial_x^2 h + \partial_y^2 h)) = 0,\ x \in S^1,\ y \in S^1. \end{aligned} \tag{7.1}$$

We now show that the ideas used to construct and analyze finite difference schemes in one space dimension can be extended to two space dimensions. The main difference, as we show below, is the fact that the Sobolev embedding $H^1 \subset C^\alpha$ for some positive α fails in two space dimensions. As a consequence, we need to assume below that we have an analytical solution that is smooth and bounded from above and below. Let $z_{ij}(t)$ be a solution of the following finite difference scheme:

$$\begin{aligned} &z_{i,j,t} + (a(z_{i-1,j}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{x}})_x \\ &\qquad + (a(z_{i,j-1}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{y}})_y = 0, \end{aligned} \tag{7.2}$$

where $a(s_1, s_2)$ is an approximation of $f(h)$ satisfying Definition 2.1.

As in the 1D case, consider the finite difference scheme (7.2) as a coupled system of ODEs. Given positive initial data there exists, locally in time, a unique solution of (7.2); the proof follows from the same arguments as in Lemma 2.2. Moreover, following the argument in Lemma 2.2, we can uniquely continue the solution in time, provided it stays bounded away from zero and infinity. Therefore, a unique solution exists globally whenever one can prove the existence of a priori upper and lower bounds. We now show that a generic finite difference scheme of the form (7.2) always possesses such an upper bound. We then show that a special entropy dissipating form of (7.2) has an additional lower bound and hence global solutions.

A genetic finite difference scheme (7.2) has the following properties:

- Discrete conservation of mass

$$\begin{aligned} &\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} z_{ij,t}\Delta x\Delta y + \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}(a(z_{i-1,j}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{x}})_x\Delta x\Delta y \\ &\qquad + \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}(a(z_{i,j-1}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{y}})_y\Delta x\Delta y = 0, \end{aligned}$$

or

$$\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} z_{ij}(t)\Delta x\Delta y = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} z_{ij}(0)\Delta x\Delta y. \tag{7.3}$$

• Discrete energy dissipation.

Multiplication by $(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})\Delta x \Delta y$, summation over i, j, and integration in time give

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j}\left\{(z_{\bar{x},ij}(t))^2 + (z_{\bar{y},ij}(t))^2\right\}\Delta x\Delta y \\
&\quad + \int_0^t \left\{\sum_{i,j} a(z_{i-1,j}(s), z_{ij}(s))(z_{\bar{x}x,ij}(s) + z_{\bar{y}y,ij}(s))_{\bar{x}}^2 \Delta x\Delta y\right\} ds \\
&\quad + \int_0^t \left\{\sum_{i,j} a(z_{i,j-1}(s), z_{ij}(s))(z_{\bar{x}x,ij}(s) + z_{\bar{y}y,ij}(s))_{\bar{y}}^2 \Delta x\Delta y\right\} ds \\
&\quad = \frac{1}{2}\sum_{i,j}\left\{(z_{\bar{x},ij}(0))^2 + (z_{\bar{y},ij}(0))^2\right\}\Delta x\Delta y.
\end{aligned}
\tag{7.4}
$$

The above mass conservation and energy dissipation combine to give the following estimate.

LEMMA 7.1. *There exists a constant C such that for any z_{ij} a positive solution of* (7.2), *on the time interval* $0 \le t \le T$, *with initial data* $z_{ij}(0) = h_0(x_i, y_j) > 0$, *we have the a priori upper bound*

$$
z_{ij}(t) \le \frac{C\|z_{ij}(0)\|_{1,\Delta x,\Delta y}}{\sqrt{\Delta x \Delta y}},
\tag{7.5}
$$

where $\|\cdot\|_{1,\Delta x,\Delta y}$ *denotes the discrete* H^1 *norm in two dimensions.*

Proof. Mass conservation (7.3) and energy dissipation (7.4) imply that the discrete H^1 norm of the solution is bounded by the initial data. The discrete Poincaré inequality then implies a discrete L^2 bound, which in turn yields the inequality (7.5). □

We now construct a special entropy dissipating scheme that gives an a priori pointwise lower bound. The following lemma shows that the same discretization of the diffusion coefficient that was used to make a scheme dissipate nonlinear entropy in one dimension works in two dimensions as well.

• Nonlinear entropy dissipation.

LEMMA 7.2. *Let* $G''(v) = \frac{1}{f(v)}$ *be a nonlinear entropy function. Let* $f(h) \sim h^n$ *as* $h \to 0$. *Then if we choose*

$$
a(s_1, s_2) = \begin{cases} \frac{s_1 - s_2}{G'(s_1) - G'(s_2)} & \text{if } s_1 \ne s_2, \\ f(s_1) & \text{if } s_1 = s_2, \end{cases}
\tag{7.6}
$$

then any solution of (7.2) *with positive initial data* $z_{ij}(0) = h_0(x_i, y_j) > 0$ *satisfies the following discrete entropy dissipation property:*

$$
\begin{aligned}
&\sum_{i,j} G(z_{ij}(t))\Delta x\Delta y + \int_0^t \left\{\sum_{i,j}(z_{\bar{x}x,ij}(s) + z_{\bar{y}y,ij}(s))^2 \Delta x\Delta y\right\} ds \\
&= \sum_{i,j} G(z_{ij}(0))\Delta x\Delta y.
\end{aligned}
\tag{7.7}
$$

This property implies that the solution $z_{ij}(t)$ *has the following pointwise lower bound:*

$$\min_t \min_{ij} z_{ij}(t) \geq \left(\frac{\Delta x \Delta y}{\tilde{C}}\right)^{\frac{1}{n-2}} \equiv \bar{\delta} \quad \textit{if } n > 2, \tag{7.8}$$

$$\min_t \min_{ij} z_{ij}(t) \geq \exp\left(-\frac{\tilde{C}}{\Delta x \Delta y}\right) \equiv \bar{\delta} \quad \textit{if } n = 2. \tag{7.9}$$

Proof. We first prove (7.7) following the same arguments from section 3 for the 1D case:

$$\begin{aligned}
&\frac{d}{dt}\sum_{i,j} G(z_{ij})\Delta x \Delta y = \sum_{i,j} G'(z_{ij}) z_{ij,t} \Delta x \Delta y \\
&= -\sum_{i,j} G'(z_{ij})(a(z_{i-1,j}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{x}})_x \Delta x \Delta y \\
&\quad - \sum_{i,j} G'(z_{ij})(a(z_{i,j-1}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{y}})_y \Delta x \Delta y \\
&= -\sum_{i,j} \left\{(G'(z_{ij}))_{\bar{x}} a(z_{i-1,j}, z_{i,j})\right\}_x (z_{\bar{x}x,ij} + z_{\bar{y}y,ij})\Delta x \Delta y \\
&\quad - \sum_{i,j} \left\{(G'(z_{ij}))_{\bar{y}} a(z_{i,j-1}, z_{i,j})\right\}_y (z_{\bar{x}x,ij} + z_{\bar{y}y,ij})\Delta x \Delta y \\
&= -\sum_{i,j} (z_{\bar{x}x,ij} + z_{\bar{y}y,ij})^2 \Delta x \Delta y.
\end{aligned}$$

To prove the pointwise lower bound, we again follow the arguments from the 1D case in section 3, which use the scaling behavior of G near $z = 0$. ☐

Note that in two dimensions we do not have a uniform lower bound independent of the grid size for large values of n as we did in the 1D case. This is due to the fact that the 1D argument used the a priori $C^{1/2}$ bound, from Sobolev embedding, which we do not know for the 2D case. The fact that we can derive an a priori pointwise lower bound implies the following corollary.

COROLLARY 7.3 (global existence of solutions of the 2D entropy dissipating scheme). *Consider* $f(h) \sim h^n$ *as* $h \to 0$*, for* $n \geq 2$*. Then* $\forall \Delta x, \Delta y$*, the entropy dissipating scheme* (7.2) *with* $a(s_1, s_2)$ *satisfying* (7.6) *and positive initial data* $z_{ij}(0) = h_0(x_i, y_j) > 0$ *has a unique positive global in time solution.*

In general the generic scheme (7.2) may have only local solutions in time. The issue is that the solution may lose positivity at some finite time and can not be continued after that time. We now show that both the generic and entropy dissipating schemes converge to positive, bounded smooth solutions of the PDE, which has a consequence that a generic scheme has global existence for sufficiently small grid size, as in the 1D case.

The proof of convergence follows the same argument as the one for the 1D case. We establish first consistency, then stability, and finally convergence.

LEMMA 7.4 (consistency). *Let* $h(x, y, t)$ *be a smooth positive solution of the PDE* (7.1) *and let* $a(s_1, s_2)$ *satisfy the conditions of Definition* 2.1*. Define the local truncation error* $\tau_{ij}(t)$ *to be the error made when plugging the smooth solution of the*

PDE into the scheme. That is, $\tau_{ij}(t)$ *satisfies*

$$\begin{aligned} h_{i,j,t} &+ (a(h_{i-1,j}, h_{i,j})(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{x}})_x \\ &+ (a(h_{i,j-1}, h_{i,j})(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{y}})_y = \tau_{ij}(t), \end{aligned} \tag{7.10}$$

where $h_{i,j} = h(x_i, y_j)$*. Then there exists a constant* $C(T)$*, depending on bounds for the smooth solutions* h *and its derivatives, such that the local truncation error* $\tau_{ij}(t)$ *satisfies*

$$\sup_{i,j} \sup_{0<t<T} |\tau_{ij}(t)| \leq C(T)((\Delta x)^2 + (\Delta y)^2).$$

Proof. As in the 1D proof of Lemma 4.1, Taylor series expansions give the desired result. □

In one dimension, we proved H^1 stability which implied pointwise stability. Again, because of the Sobolev lemma, we are not guaranteed pointwise stability from H^1 stability in two dimensions. Nevertheless we can still prove H^1 stability of the numerical scheme. As in the 1D case, we do not directly use the stability lemma to prove convergence.

LEMMA 7.5 (H^1-stability). *Let* $z_{ij}(t)$ *be a solution of the inhomogeneous problem*

$$z_{i,j,t} + (a(z_{i-1,j}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{x}})_x + (a(z_{i,j-1}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{y}})_y = f_{ij}(t) \tag{7.11}$$

with positive initial data $z_{ij}(0)$*.*

(1) *Suppose* $f_{ij}(t)$ *is small in the* $C([0,T]; H_{1,\Delta x,\Delta y})$ *norm and* $z_{ij}(0) = \bar{z}(0) + \epsilon_{ij}(0)$*, where* $\bar{z} = \sum_{i,j} z_{ij}\Delta x\Delta y$ *is the mean of* z *and* $\sum_{i,j}(\epsilon^2_{\bar{x},i}(0) + \epsilon^2_{\bar{y},i}(0))\Delta x\Delta y$ *is small. Then* $z_{ij}(t) = \bar{z}(0) + \epsilon_{ij}(t)$ *with* $\epsilon_{ij}(t)$ *small in the* $C([0,T]; H_{1,\Delta x,\Delta y})$ *norm.*

(2) *Suppose* $f_{ij}(t)$ *is bounded in* $C([0,T]; H_{1,\Delta x,\Delta y})$ *norm. Then provided the solution* $z_{ij}(t)$ *stays positive, it is bounded in the discrete* $H_{1,\Delta x,\Delta y}$ *norm.*

Proof. Multiplication of (7.11) by $(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})\Delta x\Delta y$, summation over i, j, and summation by parts give

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\sum_{i,j}(z^2_{\bar{x},ij} + z^2_{\bar{y},ij})\Delta x\Delta y\right\} + \sum_{i,j} a(z_{i-1,j}, z_{ij})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})^2_{\bar{x}}\Delta x\Delta y \\ &\quad + \sum_{i,j} a(z_{i,j-1}, z_{ij})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})^2_{\bar{y}}\Delta x\Delta y = -\sum_{i,j} f_{ij}(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})\Delta x\Delta y. \end{aligned}$$

Estimating this expression in the same way as we did in Theorem 4.2, we get

$$\begin{aligned} \left(\sum_{ij}(\epsilon^2_{\bar{x},ij}(t) + \epsilon^2_{\bar{y},ij}(t))\Delta x\Delta y\right)^{1/2} &\leq \left(\sum_{ij}(\epsilon^2_{\bar{x},ij}(0) + \epsilon^2_{\bar{y},ij}(0))\Delta x\Delta y\right)^{1/2} \\ &\quad + \int_0^t \left(\sum_{ij}(f^2_{\bar{x},ij}(s) + f^2_{\bar{y},ij}(s))\Delta x\Delta y\right)^{1/2} ds. \end{aligned}$$

This estimate, together with the identity

$$\sum_{i,j} z_{ij}(t)\Delta x\Delta y = \sum_{i,j} z_{ij}(0)\Delta x\Delta y + \int_0^t \left(\sum_{i,j} f_{ij}(s)\Delta x\Delta y\right) ds,$$

implies both statements of the lemma. □

We now use the above results on existence and consistency to prove convergence of the scheme to a positive solution of the PDE.

THEOREM 7.6 (convergence). *Let $h(x,y,t)$ be a smooth solution of* (7.1) *such that $C \geq h(x,y,t) \geq \delta > 0$ for some $C > \delta > 0$ and $t \in [0,T]$. Let $z_{ij}(t)$ be a solution of the numerical scheme* (7.2) *with $(\Delta x)^3 \ll \Delta y \ll (\Delta x)^{1/3}$. Then for $\Delta x, \Delta y$ sufficiently small the solution $z_{ij}(t)$ can be continued $\forall t \in [0,T]$ and there exists a constant $C > 0$ such that we have the H^1 convergence*

$$\sup_{t\in[0,T]} \| z_{ij}(t) - h(x_i, y_j, t) \|_{1,\Delta x,\Delta y} \leq C(\Delta x^2 + \Delta y^2)$$

and the pointwise convergence

$$\sup_{t\in[0,T]} \sup_{i,j} |z_{ij}(t) - h(x_i, y_j, t)|^2 \leq C \frac{\left((\Delta x)^2 + (\Delta y)^2\right)^2}{\Delta x \Delta y}.$$

Proof. As in the proof of Theorem 5.1 we first find a time interval $[0,\sigma]$ such that $2C \geq z_{ij}(t) \geq \delta/2$ hold $\forall i,j$ and $t \in [0,\sigma]$, then estimate the error between the analytical and numerical solutions and finally continue numerical solution to the whole interval $[0,T]$ for sufficiently small Δx and Δy.

The analytical solution satisfies (7.10). By Lemma 7.4 the local truncation error $\tau_{ij}(t) = O((\Delta x)^2 + (\Delta y)^2)$. Introducing $e_{ij}(t) = z_{ij}(t) - h_{ij}(t)$ and subtracting (7.10) from (7.2) we obtain

$$\begin{aligned} e_{ij}(t) &+ [(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{x}}(a(z_{i-i,j}, z_{i,j}) - a(h_{i-i,j}, h_{i,j})]_x \\ &+ [a(z_{i-i,j}, z_{i,j})(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})_{\bar{x}}]_x \\ &+ [(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{y}}(a(z_{i,j-1}, z_{i,j}) - a(h_{i,j-1}, h_{i,j})]_y \\ &+ [a(z_{i,j-1}, z_{i,j})(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})_{\bar{y}}]_y = -\tau_{ij}(t). \end{aligned}$$

Multiplication by $(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})\Delta x \Delta y$, summation over i,j, and summation by parts gives

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\sum_{i,j}(e^2_{\bar{x},ij} + e^2_{\bar{y},ij})\Delta x \Delta y\right] \\ &+ \sum_{i,j}(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{x}}(a(z_{i-1,j}, z_{i,j}) - a(h_{i-1,j}, h_{i,j}))(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})_{\bar{x}}\Delta x \Delta y \\ &+ \sum_{i,j} a(z_{i-1,j}, z_{i,j})(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})^2_{\bar{x}}\Delta x \Delta y \\ &+ \sum_{i,j}(h_{\bar{x}x,ij} + h_{\bar{y}y,ij})_{\bar{y}}(a(z_{i,j-1}, z_{i,j}) - a(h_{i,j-1}, h_{i,j}))(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})_{\bar{y}}\Delta x \Delta y \\ &+ \sum_{i,j} a(z_{i,j-1}, z_{i,j})(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})^2_{\bar{y}}\Delta x \Delta y = -\sum_{i,j}\tau_{ij}(e_{\bar{x}x,ij} + e_{\bar{y}y,ij})\Delta x \Delta y. \end{aligned} \tag{7.12}$$

Using the boundedness of the numerical solution and property (d) in Definition 2.1, and estimating the second and fourth terms of (7.12) in a way similar to (5.8) in section 5, we obtain

$$\| e(t) \|^2_{1,\Delta x,\Delta y} \leq C\left((\Delta x)^2 + (\Delta y)^2\right)^2 \ \forall i,j \text{ and } t \in [0,\sigma]. \tag{7.13}$$

The constant C is independent of $\sigma < T$. Therefore,

$$\sum_{i,j} e_{ij}^2 \Delta x \Delta y \leq C \left((\Delta x)^2 + (\Delta y)^2 \right)^2,$$

which implies

$$e_{ij}^2(t) \leq C \frac{\left((\Delta x)^2 + (\Delta y)^2 \right)^2}{\Delta x \Delta y}. \tag{7.14}$$

The expression on the right-hand side of (7.14) tends to zero as $\Delta x, \Delta y$ tend to zero if $(\Delta x)^3 \ll \Delta y \ll (\Delta x)^{1/3}$. Making this assumption we note that for $\Delta x, \Delta y$ small enough we can improve the upper and lower bounds, $2C \geq z_{ij}(t) \geq \delta/2$, from which we started. Therefore, for $\Delta x, \Delta y$ sufficiently small we can continue the solution to the whole time interval $[0, T]$ so that the bounds (7.13) and (7.14) still hold. ∎

REMARK 3. *Note that even though, as we mentioned previously, the embedding $L^\infty \subset H^1$, which holds in one dimension, fails in two dimensions, we can prove the pointwise convergence in* (7.14). *This fact is a consequence of the second order consistency of the numerical scheme* (7.1) *with the PDE* (7.2).

REMARK 4. *The convergence theorem shows that we can improve the bound* (7.5) *in the case where the scheme is approximating a positive bounded solution of the PDE. That is, the solution of the numerical scheme has a pointwise upper bound that does not depend strongly on the grid size, improving estimate* (7.5). *Likewise, for the EDS with $n \geq 2$, we can improve upon the pointwise lower bound* (7.8) *or* (7.9) *when the scheme approximates a positive bounded solution of the PDE.*

REMARK 5. *We can extend the modified entropy dissipating scheme of section* 6 *to the* 2*D case.*

REMARK 6. *If $\Delta x = \Delta y$, then one can obtain the following pointwise bound on the error:*

$$\sup_{t \in [0,T]} \sup_{i,j} |z_{ij}(t) - h(x_i, y_j, t)| \leq C(a)(\Delta x)^{2-a}$$

for arbitrary small a.

To show this we first note that second order convergence in discrete H_1 norm implies the second order convergence in discrete L_q norm for arbitrary large q (see the appendix for details). Thus

$$\max_{l,m} |e_{lm}| (\Delta x)^{2/q} \leq \left(\sum_{i,j} |e_{ij}|^q (\Delta x)^2 \right)^{1/q} \leq C(q)(\Delta x)^2,$$

which implies that

$$\max_{l,m} |e_{lm}| \leq C(q)(\Delta x)^{2 - \frac{2}{q}}.$$

8. Positivity-preserving finite element methods. So far, the schemes presented in one and two space dimensions have been for fixed uniform Cartesian grids. Equations of the type (5.1) often arise in the context of moving contact lines in thin films. For such problems, local mesh refinement must be used to resolve structures that can develop near the contact line [15]. Also, when schemes of this type are used to compute finite time singularities describing things such as film rupture [10] or neck rupture in a Hele–Shaw cell [12], local mesh refinement is crucial to resolve small scale structures near the point of singularity. Thus, a natural question to consider is whether one can generalize the positivity-preserving methods, discussed in previous sections of this paper, to more complicated grid structures.

A natural framework to work in is that of finite elements. We begin by showing that the 1D EDS ((3.1) with (5.3)) finite difference scheme is equivalent to a finite element approximation in which a nonlinear function of the solution is represented in the element basis. This representation suggests a general abstract finite element approximation of the problem (in any space dimension). We then show that the 2D scheme is another special case of the general finite element approximation. We present a third example of a new difference scheme that results as a special case of this approximation. We hope that the methods outlined below will be useful for the design of schemes on more complicated grids.

Example 1 (1D EDS). First introduce the pressure $p = -\partial_x^2 h$. The original PDE

$$h_t + \partial_x(f(h)\partial_x^3 h) = 0 \tag{8.1}$$

can be rewritten as

$$h_t - \partial_x(f(h)\partial_x p) = 0,\ p = -\partial_x^2 h. \tag{8.2}$$

We now present a finite element spatial discretization of (8.2) that we show is equivalent to the 1D EDS of section 3. Let (η_1, η_2) denote the standard inner product on $L^2(S^1)$, $G(y)$ be a function satisfying $G''(y) = 1/f(y)$, $T^{\Delta x}$ be a space of piecewise linear periodic functions on the spatial grid of size Δx. We introduce the interpolation operator $\pi^{\Delta x} : C(S^1) \to T^{\Delta x}$ such that $(\pi^{\Delta x}\eta)(x_j) = \eta(x_j)\ \forall j$. Define a discrete inner product on $C(S^1)$ by

$$(\eta_1, \eta_2)^{\Delta x} \stackrel{\text{def}}{=} \int_{S^1} \pi^{\Delta x}(\eta_1(x)\eta_2(x))dx \equiv \sum_j \eta_1(x_j)\eta_2(x_j)\Delta x.$$

We now show that the EDS is equivalent to the following finite element approximation of (8.2):

Find $y, w : G'(y) \in T^{\Delta x}, w \in T^{\Delta x}$ such that

$$(y_t, \chi)^{\Delta x} + (f(y)\partial_x w, \partial_x \chi) = 0 \quad \forall \chi \in T^{\Delta x}, \tag{8.3}$$

$$(\partial_x y, \partial_x \chi) = (w, \chi)^{\Delta x} \quad \forall \chi \in T^{\Delta x}. \tag{8.4}$$

Consider the standard basis $\{\chi_i(x)\}_{i=1}^N$ for $T^{\Delta x}$:

$$\chi_i(x) = \begin{cases} \frac{x - x_{i-1}}{\Delta x} & \text{if } x_{i-1} \le x \le x_i, \\ \frac{x_{i+1} - x}{\Delta x} & \text{if } x_i \le x \le x_{i+1}. \end{cases}$$

Then

$$(y_t, \chi_i)^{\Delta x} = y_{i,t}\Delta x,$$

$$(f(y)\partial_x w, \partial_x \chi_i) = \int_{x_{i-1}}^{x_i} f(y)\,\frac{w_i - w_{i-1}}{\Delta x}\,\frac{1}{\Delta x}\,dx - \int_{x_i}^{x_{i+1}} f(y)\,\frac{w_{i+1} - w_i}{\Delta x}\,\frac{1}{\Delta x}\,dx.$$

Note that the integral

$$\int_{x_{i-1}}^{x_i} f(y)dx = \int_{x_{i-1}}^{x_i} \frac{dx}{G''(y)} = \int_{x_{i-1}}^{x_i} \frac{\partial_x y dx}{G''(y)\partial_x y}$$
$$= \int_{x_{i-1}}^{x_i} \frac{dy(x)}{s'(x)} = \frac{\Delta x}{s_i - s_{i-1}}(y_i - y_{i-1}) = \frac{\Delta x(y_i - y_{i-1})}{G'(y_i) - G'(y_{i-1})}.$$

In the last computation we used the substitution $s(x) = G'(y(x))$ and the fact that $s(x)$ is a piecewise linear function. From (8.4) it follows that $w_i = -y_{\bar{x}x,i}$. Therefore,

$$(f(y)\partial_x w, \partial_x \chi_i) = y_{\bar{x}x\bar{x},i+1}\frac{y_{i+1} - y_i}{G'(y_{i+1}) - G'(y_i)} - y_{\bar{x}x\bar{x},i}\frac{y_i - y_{i-1}}{G'(y_i) - G'(y_{i-1})}.$$

Now (8.3) gives

$$y_{i,t} + \frac{1}{\Delta x}\left(y_{\bar{x}x\bar{x},i+1}\frac{y_{i+1} - y_i}{G'(y_{i+1}) - G'(y_i)} - y_{\bar{x}x\bar{x},i}\frac{y_i - y_{i-1}}{G'(y_i) - G'(y_{i-1})}\right) = 0, \tag{8.5}$$

which indeed is the EDS (3.1) with (3.3).

Remark 7. *The same finite element approximation on a nonuniform mesh produces an analogous scheme* (8.5) *with nonuniform differences. A variant of this scheme was used recently in* [20] *to compute finite time singularities in a long-wave unstable generalization of* (8.1).

Remark 8. *The finite element representation* (8.3)–(8.4) *has a structure similar to that introduced in* [17]. *The main difference is that the method in* [17] *takes the solution y to be in the subspace $T^{\Delta x}$ spanned by the element basis, resulting in a method that requires a Lagrange multiplier to ensure nonnegativity of the solution in cases where it might otherwise become negative. Here, we take the nonlinear function $G'(y) \in T^{\Delta x}$, resulting in a scheme that preserves positivity. Our choice of element representation has one consequence: we need to ensure that $\nabla y \in L^2$ in order to make sense of the inner product on the left-hand side of* (8.4). *We address this point in Remark* 9 *below.*

The finite element approach above generalizes to positivity-preserving schemes in higher dimensions $(S^1)^d$ and to more complicated finite element subspaces of $H^1((S^1)^d)$. Consider the following form of the lubrication equation in d space dimensions:

$$h_t - \nabla \cdot (f(h)\nabla p) = 0, \quad p + \Delta h = 0, \quad f(h) \sim h^n,\ h \to 0. \tag{8.6}$$

Now let (η_1, η_2) denote an inner product in $L^2((S^1)^d)$, $T^{\Delta\vec{x}}$ be a finite dimensional subspace of $H^1((S^1)^d)$, $\pi^{I_1} : C((S^1)^d) \to T^{\Delta\vec{x}}$ be an interpolation operator, and $(\eta_1, \eta_2)^{I_1} \stackrel{\text{def}}{=} \int_{(S^1)^d} \pi^{I_1}(\eta_1(x)\eta_2(x))dx$ be an associated inner product on $C((S^1)^d)$. For any d-dimensional vectors $\xi_1, \xi_2 \in C((S^1)^d, R^d)$ let $(\xi_1, \xi_2)^{I_2}$ denote a numerical integration rule replacing an $L^2((S^1)^d, R^d)$ inner product.

Consider the following general finite element approximation of (8.6):

Find $z, w : G'(z) \in T^{\Delta\vec{x}}, w \in T^{\Delta\vec{x}}$ such that

$$(z_t, \chi)^{I_1} + (f(z)\nabla w, \nabla\chi)^{I_2} = 0 \qquad \forall \chi \in T^{\Delta\vec{x}}, \tag{8.7}$$

$$(\nabla z, \nabla \chi)^{I_2} = (w, \chi)^{I_1} \quad \forall \chi \in T^{\Delta\vec{x}}. \tag{8.8}$$

Taking $\chi = G'(z)$ in (8.7) and $\chi = w$ in (8.8), we obtain

$$(z_t, G'(z))^{I_1} + (\nabla w, \nabla z)^{I_2} = 0,$$
$$(\nabla z, \nabla w)^{I_2} = (w, w)^{I_1},$$

or

$$\frac{d}{dt} \int_{(S^1)^d} \pi^{I_1}(G(z))dx = -(w, w)^{I_1} \leq 0.$$

This yields the following a priori bound:

$$\int_{(S^1)^d} \pi^{I_1}(G(z(x,t)))dx \leq C.$$

Now by using an explicit form of the interpolation operator π^{I_1} we can rewrite the left-hand side of the above inequality as $\sum_i a_i G(z_i(t))$ with $a_i > 0\ \forall i$. As before, this gives a positive (dependent on $\Delta\vec{x}$) lower bound on $z_i(t)$ for $n \geq 2$.

REMARK 9. *One has to insure that* $\nabla z \in L^2((S^1)^d, R^d)$ *to use the finite element method* (8.7)–(8.8). *This is guaranteed as long as all* $z_i(t)$ *are bounded from above. In the special cases of Example* 1 *above and Example* 2 *below, additional structure of the scheme allows us to prove an a priori bound on the discrete* H^1 *norm of the solution* z*, independent of the grid size. In general, however, discrete energy dissipation (analogous to* (1.5) *may not occur for* (8.7)–(8.8). *Note that since* $G'(z) \in T^{\Delta\vec{x}}$*, in general* $\nabla G'(z) = \frac{1}{f(z)}\nabla z \in L^2((S^1)^d)$. *Thus a sufficient condition for* $\nabla z \in L^2((S^1)^d, R^d)$ *is that* $f(z)$ *is bounded. This is also a sufficient condition for* $f(z)\nabla w \in L^2((S^1)^d, R^d)$ *which is needed to make sense of the nonlinear term in* (8.7).

Now let us consider some further examples of the general finite element method (8.7)–(8.8).

Example 2. In two space dimensions, let $\vec{x} = (x, y)$ and $T^{\Delta\vec{x}}$ be a space of piecewise bilinear functions on the rectangles of size $\Delta x \times \Delta y$. Let $\chi_{ij}(x, y)$ be a basis for this space such that $\chi_{ij}(x, y)$ is equal to 1 at node (x_i, y_j) and 0 at all other nodes. As before, let $\pi^{I_1} : C((S^1)2) \to T^{\Delta\vec{x}}$ be the interpolation operator such that for any continuous function $\eta(x, y)$ $(\pi^{I_1}\eta)(x_i, y_j) = \eta(x_i, y_j)\ \forall i, j$. For any 2D vectors $\xi_1, \xi_2 \in C((S^1)^2, R^2)$, let

$$(\xi_1, \xi_2)^{I_2} = \sum_{i,j} \int_{x_i}^{x_{i+1}} \frac{1}{2}\left(\xi_1^1(x, y_{j+1})\xi_2^1(x, y_{j+1}) + \xi_1^1(x, y_j)\xi_2^1(x, y_j)\right)\Delta y dx$$
$$+ \sum_{i,j} \int_{y_j}^{y_{j+1}} \frac{1}{2}\left(\xi_1^2(x_{i+1}, y)\xi_2^2(x_{i+1}, y) + \xi_1^2(x_i, y)\xi_2^2(x_i, y)\right)\Delta x dy.$$

On the product of first components this numerical integration rule performs integration exactly in the first variable but uses trapezoidal rule instead of integration in the second variable. Similarly, on the product of second components it performs integration exactly in the second variable but uses trapezoidal rule instead of integration in the first variable.

We show that this choice gives us the 2D positivity-preserving scheme (7.2) with

(7.6) from section 7. Taking $\chi = \chi_{ij}$ in (8.7) gives

$$
\begin{aligned}
(z_t, \chi_{ij})^{I_1} &= z_{ij,t}\Delta x \Delta y,\\
(f(z)\nabla w, \nabla \chi_{ij})^{I_2} &= \sum_{k=i-1}^{i}\sum_{l=j-1}^{j}\int_{x_k}^{x_{k+1}} \frac{1}{2}\left\{ f(z(x,y_{l+1}))\partial_x w\big|_{y=y_{l+1}}\ \partial_x\chi_{ij}\big|_{y=y_{l+1}}\right.\\
&\quad \left. + f(z(x,y_l))\ \partial_x w|_{y=y_l}\ \partial_x \chi_{ij}|_{y=y_l}\right\}\Delta y dx\\
&\quad + \sum_{k=i-1}^{i}\sum_{l=j-1}^{j}\int_{y_l}^{y_{l+1}} \frac{1}{2}\left\{ f(z(x_{k+1},y))\partial_y w\big|_{x=x_{k+1}}\ \partial_y\chi_{ij}\big|_{x=x_{k+1}}\right.\\
&\quad \left. + f(z(x_k,y))\ \partial_y w|_{x=x_k}\ \partial_y \chi_{ij}|_{x=x_k}\right\}\Delta x dy.
\end{aligned}
$$

To compute both double sums in the expression above we have to sum over four rectangles that have the node (x_i, y_j) as one of their vertices. Let us consider one of these rectangles, for instance, the rectangle $x_i \le x \le x_{i+1}, y_j \le y \le y_{j+1}$. Then

$$
\begin{aligned}
w(x,y) &= w_{i+1,j+1}\,\frac{(x-x_i)(y-y_j)}{\Delta x\Delta y} + w_{i+1,j}\,\frac{(x-x_i)(y_{j+1}-y)}{\Delta x\Delta y}\\
&\quad + w_{i,j+1}\,\frac{(x_{i+1}-x)(y-y_j)}{\Delta x\Delta y} + w_{i,j}\,\frac{(x_{i+1}-x)(y_{j+1}-y)}{\Delta x\Delta y},\\
\partial_x w &= \frac{w_{i+1,j+1}-w_{i,j+1}}{\Delta x}\,\frac{y-y_j}{\Delta y} + \frac{w_{i+1,j}-w_{i,j}}{\Delta x}\,\frac{y_{j+1}-y}{\Delta y},\\
\partial_x \chi_{ij} &= -\frac{1}{\Delta x}\,\frac{y_{j+1}-y}{\Delta y}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{x_i}^{x_{i+1}} \frac{1}{2}\left\{ f(z(x,y_{j+1}))\partial_x w\big|_{y=y_{j+1}}\ \partial_x\chi_{ij}\big|_{y=y_{j+1}} + f(z(x,y_j))\partial_x w\big|_{y=y_j}\ \partial_x\chi_{ij}\big|_{y=y_j}\right\}\Delta y dx\\
&= \frac{\Delta y}{2}\partial_x w\big|_{y=y_j}\ \partial_x\chi_{ij}\big|_{y=y_j}\int_{x_i}^{x_{i+1}} f(z(x,y_j))dx = -\frac{\Delta y}{2}\,\frac{w_{i+1,j}-w_{i,j}}{\Delta x}\,a(z_{i,j}, z_{i+1,j}),
\end{aligned}
$$

where $a(z_{i,j}, z_{i+1,j})$ is defined in (7.6). It is easy to see that we get the same contribution from the rectangle $x_i \le x \le x_{i+1}, y_{j-1} \le y \le y_j$, while the contributions from two other rectangles, $x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j$ and $x_{i-1} \le x \le x_i, y_j \le y \le y_{j+1}$, are $\frac{\Delta y}{2}\ \frac{w_{i,j}-w_{i-1,j}}{\Delta x}\ a(z_{i-1,j}, z_{ij})$ each. Therefore, the first double sum is $-\Delta x\Delta y\left(w_{\bar{x},ij}a(z_{i-1,j}, z_{ij})\right)_x$. Similarly, the second double sum is $-\Delta x\Delta y\left(w_{\bar{y},ij}a(z_{i,j-1}, z_{ij})\right)_y$. Taking $\chi = \chi_{ij}$ in (8.8), we get that $w_{ij} = -z_{\bar{x}x,ij} - z_{\bar{y}y,ij}$. Taking this into account, we obtain

$$
z_{ij,t} + (a(z_{i-1,j}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{x}})_x + (a(z_{i,j-1}, z_{i,j})(z_{\bar{x}x,ij} + z_{\bar{y}y,ij})_{\bar{y}})_y = 0,
$$

which is the positivity-preserving 2D scheme (7.2) and (7.6) that we introduced in section 7.

Both of the examples considered above were introduced in previous sections in a finite difference framework. Now we introduce a new scheme that arises from a different choice of inner products in (8.7)–(8.8).

Example 3. In one dimension consider the same scheme as (8.3)–(8.4) except with trapezoidal rule instead of exact integration for computation of $(f(y)\partial_x w, \partial_x \chi)$ and

$(\partial_x y, \partial_x \chi)$. Then

$$\begin{aligned}
(z_t, \chi_i)^{I_1} &= z_{i,t}\Delta x,\\
(f(z)\partial_x z, \partial_x \chi_i)^{I_2} &= \frac{w_i - w_{i-1}}{\Delta x}\,\frac{f(z_{i-1}) + f(z_i)}{2} - \frac{w_{i+1} - w_i}{\Delta x}\,\frac{f(z_i) + f(z_{i+1})}{2},\\
(\partial_x z, \partial_x \chi_i)^{I_2} &= \frac{\partial_x z\big|_{x=x_{i-1}+0} + \partial_x z\big|_{x=x_i-0}}{2} - \frac{\partial_x z\big|_{x=x_i+0} + \partial_x z\big|_{x=x_{i+1}-0}}{2}.
\end{aligned}$$

To compute the last expression we introduce a new variable $s(x) = G'(z(x))$ which is linear on each of the intervals $[x_{i-1}, x_i]$. Differentiating and solving for $\partial_x z(x)$ gives

$$\partial_x z(x) = \partial_x s(x) f(z(x)) = \frac{G'(z_i) - G'(z_{i-1})}{\Delta x}\, f(z(x)) \qquad \text{for} \qquad x_{i-1} \le x \le x_i.$$

Therefore,

$$(\partial_x z, \partial_x \chi_i)^{I_2} = \frac{G'(z_i) - G'(z_{i-1})}{\Delta x}\,\frac{f(z_{i-1}) + f(z_i)}{2} - \frac{G'(z_{i+1}) - G'(z_i)}{\Delta x}\,\frac{f(z_i) + f(z_{i+1})}{2}.$$

Thus, we obtain the scheme

$$z_{i,t} - (a(z_{i-1}, z_i) w_{\bar{x},i})_x = 0$$

with

$$\begin{aligned}
a(z_{i-1}, z_i) &= \frac{f(z_{i-1}) + f(z_i)}{2} \qquad \text{and}\\
w_i &= -\left(\{G'(z_i)\}_{\bar{x}}\, a(z_{i-1}, z_i)\right)_x.
\end{aligned}$$

This shows that we can use a discretization of the diffusion coefficient of the form $a(z_{i-1}, z_i) = \frac{f(z_{i-1}) + f(z_i)}{2}$ and still have a positivity-preserving finite difference scheme as long as we also change the definition of the numerical second derivative.

9. A computational example. In this section we illustrate the advantage of using a positivity-preserving scheme over a generic one.

We consider the following example. Solve the equation

$$h_t + \partial_x(f(h)\partial_x^3 h) = 0,\ f(h) = h^{1/2}, \tag{9.1}$$

with positive initial condition

$$h_0(x) = 0.8 - \cos(\pi x) + 0.25\cos(2\pi x).$$

It was computationally shown in [16] that the solution of this problem develops singularity in finite time. The solution past the singularity time can be defined by introducing, at all times, the regularization

$$h_{\epsilon t} + \partial_x(f_\epsilon(h_\epsilon)\partial_x^3 h_\epsilon) = 0,\ f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 f(h_\epsilon)}{\epsilon f(h_\epsilon) + h_\epsilon^4}, \tag{9.2}$$

finding the solution of the regularized problem, and taking the regularization parameter ϵ to zero [13, 14, 9]. The resulting "weak solution" is guaranteed to be C^1 almost every time and to have a bounded second derivative almost every time. Since $f_\epsilon(h_\epsilon) \sim h_\epsilon^4/\epsilon$ as $h_\epsilon \to 0$ we know that $\forall \epsilon > 0$ the analytical solution of the regularized

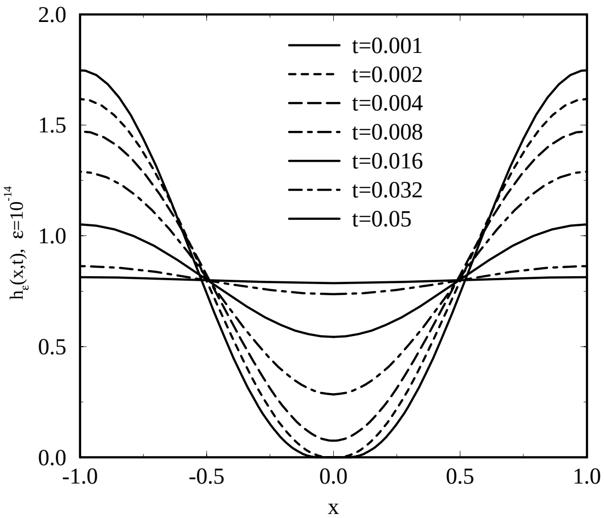

FIG. 9.1. *The solution of the regularized problem* (9.2) *for* $\epsilon = 10^{-14}$ *at the time* $t = 0.001$ *and subsequent later times. As* $\epsilon \to 0$ *at* $t = 0.001$ *there is a region, from roughly* -0.06 *to* 0.06, *for which the resulting weak solution is identically zero.*

problem is positive. In [16] a nonnegative weak solution was approximated numerically by positive strong solutions of the regularized problems with small ϵ. That paper used a generic scheme (2.2) with $a_\epsilon(s_1, s_2) = f_\epsilon(0.5(s_1 + s_2))$. The numerical solutions of the regularization suggested that after the initial singularity, the solution develops a region where it is identically zero (see Figure 9.1). The regularized solution stays positive and develops a lot of structure near the edge of the support of the weak solution. In [16] a fine grid was required in order to resolve the spatial structure and keep the numerical solution positive. Here we show that an entropy dissipating scheme does a much better job at computing this problem without requiring excessive spatial resolution.

The numerical schemes presented in this paper are all discrete in space and continuous in time. For a practical implementation of such a method, we need to discretize in time as well and choose a time-stepping method. We use the θ-weighted time step scheme in [10, 11, 18, 12] with $\theta = 1$ (backward Euler). The choice of θ was motivated by stability and monotonicity issues involving problems with diffusion. We take an adaptive time step where we ensure that the error over the step is below a threshold value, and that the minimum of the solution does not decrease below a percentage of its previous value. The final concern is that the time step is small enough to accurate approximate the system of ODEs (2.2) and in particular to still capture the positivity-preserving property when solving the EDS. While we have not derived any analytical measure of the time step constraint for positivity, in the case of the EDS, empirically, we find that it does not have to be especially small to achieve positivity on a coarse mesh. In particular, the adaptive time step algorithm (the same that was used in, e.g., [10]) ensures that the solution stays positive, and the size of the step chosen by this algorithm, coupled with the EDS, was not unreasonably small. All computations presented below were performed on a uniform grid. However, a special

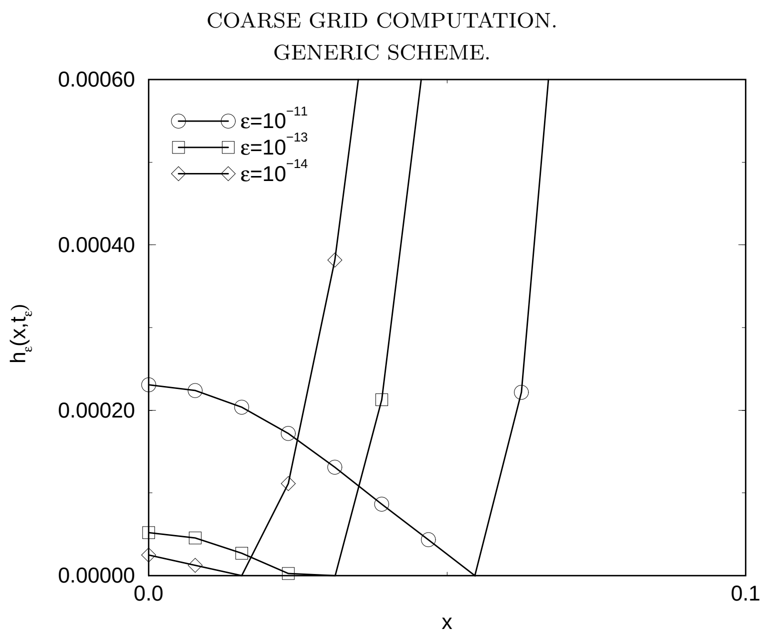

FIG. 9.2. *Failure to compute the solution of the approximating problem* (9.2) *up to the time* $t = 0.001$ *on a coarse grid (*128 *grid points on* $[0, 1]$*) using a "generic" numerical method. For each* $\epsilon = 10^{-11}$, 10^{-13}, *and* 10^{-14}, *the scheme developed a numerical singularity (shown above) at the respective times* $t \approx 0.00086$, 0.00076, *and* 0.00074. *We could not continue computing beyond these times since the numerical solution becomes negative.*

routine was used to plot only selected points.

Figure 9.2 shows the computational results obtained by the generic scheme for three values of the regularization parameter $\epsilon = 10^{-11}, 10^{-13}$, and 10^{-14} on a grid with 128 points on $[0, 1]$. For each value of ϵ in (9.2), we tried to compute the solution until $t = 0.001$. However, in each case, the solution of this generic scheme developed a singularity at a time t_c earlier, in which $\min_i y_i(t) \to 0$ as $t \to t_c < 0.001$. In each case, the smaller the value of ϵ, the earlier the time of the numerical singularity.

Figure 9.3 shows the results that the entropy dissipating scheme ((3.1) with (3.3)) gave for the same input. In this case we successfully computed the numerical solution at $t = 10^{-3}$. Note that in both cases we used the same purely implicit method for the time integration, choosing the time step Δt small enough to ensure that the discrete time system shows the same behavior as the continuous time one.

The final Figure 9.4 shows the results obtained by the entropy dissipating scheme on a much finer grid, one with 1024 points on [0,1]. Note that the positive approximation (9.2) is now well resolved near the edge of the support of the weak solution. This computation shows that the EDS does a good job on a coarser mesh (compare with Figure 9.3).

In this example we are in effect using a modified entropy dissipating scheme (as in section 6) to compute a weak solution of (9.1). Note that both Figure 9.3 and Figure 9.4 indicate that the method is converging in ϵ. Unfortunately, the results of section 6 do not help in identifying the order of convergence in ϵ since we are computing a nonnegative solution here. In fact, the computations suggest that convergence in ϵ is slower that the first order predicted for strong solutions in section 6. For tracking

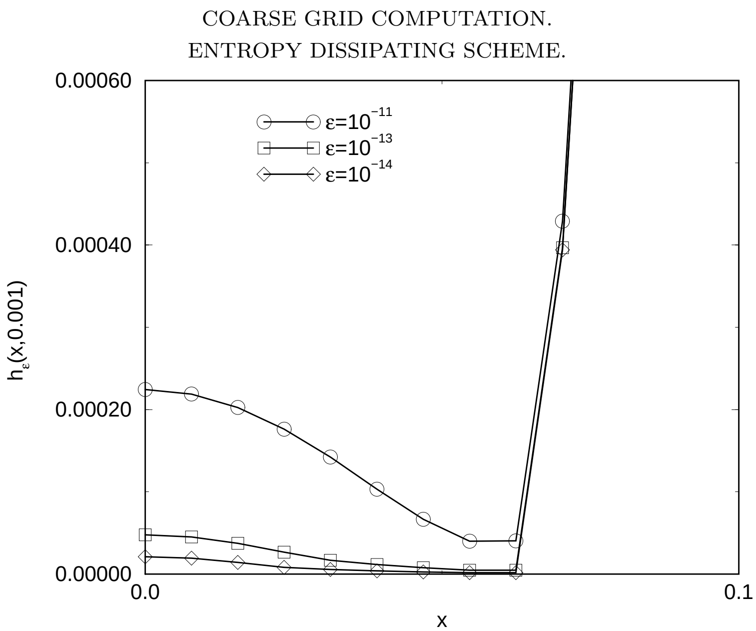

FIG. 9.3. *Successful computation of the solution of the approximating problem* (9.2) *at the time* $t = 0.001$ *on a coarse grid (*128 *grid points on* $[0, 1]$*) using the EDS of section* 3. *Shown are examples for* $\epsilon = 10^{-11}$, $\epsilon = 10^{-13}$, *and* $\epsilon = 10^{-14}$. *The minimum time step imposed by adaptive time-stepping code was* $\log_{10}(\min \Delta t) = -6.6$ *for* $\epsilon = 10^{-11}$, -7.2 *for* $\epsilon = 10^{-13}$, *and* -7.4 *for* $\epsilon = 10^{-14}$. *The solution on the course grid shows good agreement with the same computation (below in Figure* 9.4*) on a fine grid.*

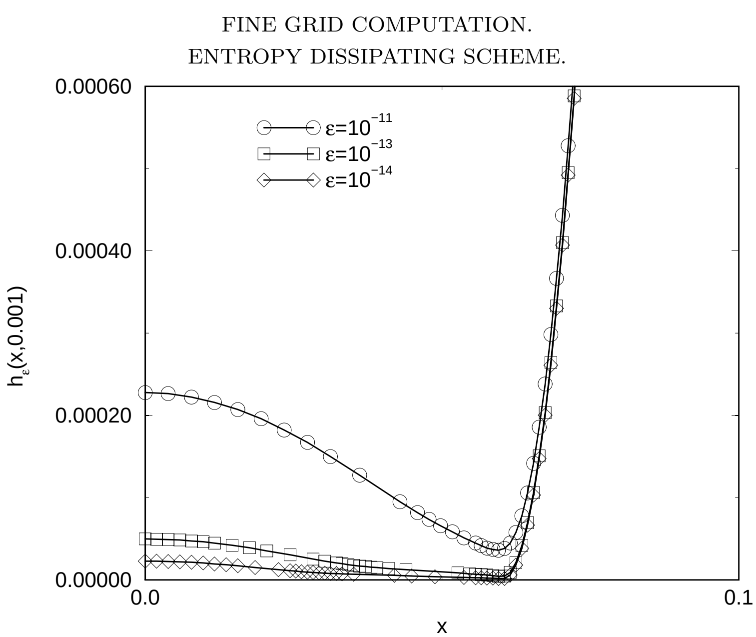

FIG. 9.4. *Numerical computation of* (9.2) *at fixed time* $t = 0.001$, $\epsilon = 10^{-11}, \epsilon = 10^{-13}$, *and* $\epsilon = 10^{-14}$, *on a fine grid of* 1024 *grid points on* $[0, 1]$. *For this run,* $\log_{10}(\min \Delta t) = -6.7$ *for* $\epsilon = 10^{-11}$, -7.0 *for* $\epsilon = 10^{-13}$, *and* -7.3 *for* $\epsilon = 10^{-14}$.

the numerical free boundary of the positive region one may choose a function $\delta(\epsilon) > 0$, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and then define a free boundary as a boundary of the region where the solution is less than the prescribed $\delta(\epsilon)$. The question of how to choose $\delta(\epsilon)$ is a nontrivial one since it is related to the order of convergence of numerical solutions in ϵ. Nevertheless, the computations shown on Figure 9.3 suggest that the method of section 6 may be very useful not only for computing strong solutions but also for approximating weak solutions by computing the solutions to regularized problems.

10. Conclusions. In this paper we discuss several kinds of finite difference schemes for solving fourth order degenerate diffusion equations arising in the study of viscous films and layers driven by surface tension. Previous computational work on this problem showed that much grid refinement is typically needed in order to resolve the solution when it approaches a "near" singularity. Underrefinement of the grid can lead to false premature rupture of the solution and also to inaccurate computations of weak solutions with moving contact lines.

We show that it is possible to design a finite difference scheme, in one and two space dimensions, that preserves positivity of the solution. This technique is useful both for computing solutions that have near singularities when forced (such as flow down an inclined plane with a precursor layer model (see [15]) and for computing weak solutions via regularizations, as we demonstrated in section 9 of this paper. We also showed that the idea for designing the scheme, that of dissipation of a discrete version of a nonlinear entropy function, can be generalized to an abstract finite element context which suggests ways for designing positivity-preserving schemes on more complicated grids. Note that in the case where the solution of the PDE has a finite time singularity in which $h \to 0$, a positivity-preserving approximation of the PDE will not see the singularity. However, it will resolve the solution, before the singularity happens, upon successive levels of mesh refinement. Thus such a scheme could be very useful for careful studies of singularity formation.

There are a number of questions that stem from this work. For example, it seems worthwhile to examine the question of convergence of schemes that arise from the general finite element model (8.7)–(8.8), where the elements are generated on nonuniform grids. In general, such a scheme may not be second order consistent (i.e., may not satisfy the equivalent of Lemma 4.1), so that the estimates derived here do not directly generalize. Another open question concerns the use of positivity-preserving schemes in the study of weak solutions. In section 7 we introduced a modified entropy dissipating scheme that has two small parameters involved, namely, grid size Δx and the regularization parameter ϵ. If we take $\epsilon = \epsilon(\Delta x)$ then the new scheme obtained can be considered as an alternative positive approximation of a solution of the PDE. The results of section 10 suggest that this method could be particularly useful for computing weak solutions. We also expect that the EDS will perform well on 2D computations since it has the potential to give good results for moving contact lines without requiring local mesh refinement, a procedure that is much more difficult in two dimensions than in one dimension.

Finally we note that it would be interesting to try to generalize the method to implement other standard boundary conditions as well as to apply it to other problems that possess positivity-preserving entropies, such as lubrication models with stabilizing or destabilizing forces such as gravity [21] or long range Van der Waals interactions [18] or driving (convective) terms [15].

11. Appendix. We derive the following estimate used to prove Remark 6 in section 7:

$$\|u\|_{L_q,\Delta x,\Delta y} \le C(q)\|u\|_{H_1,\Delta x,\Delta y}, \tag{11.1}$$

where $u_{ij} = u(i\Delta x, j\Delta y)$ is a grid function on $[0,1]\times[0,1]$; $\|\cdot\|_{L_q,\Delta x,\Delta y}$ and $\|\cdot\|_{H_1,\Delta x,\Delta y}$ are discrete H_1 and L_q norms, respectively; and q can be taken arbitrarily large. Since the domain of u is compact, it suffices to show the following statement similar to that of Theorem 2.1, p. 181, in [22].

LEMMA 11.1. *For an arbitrary grid function $u_{ij} = u(i\Delta x, j\Delta y)$ on $[0,1]\times[0,1]$*

$$\|u\|_{L_q,\Delta x,\Delta y} \le C(p)\left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{x}}\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y}\right), \tag{11.2}$$

where $p < 2$ is defined by $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

Proof. The proof follows that of Theorem 2.1, p. 181, in [22].

Consider the scalar function $g(s) = |s|^{\frac{p}{2-p}}$. This function is differentiable for $\frac{p}{2-p} > 1$,

$$g'(s) = \frac{p}{2-p} s^{\frac{2(p-1)}{2-p}} \operatorname{sign} s.$$

Taylor's formula gives

$$|g(s_1) - g(s_2)| = |s_1 - s_2| g'(\lambda s_1 + (1-\lambda)s_2), \lambda \in (0,1).$$

Therefore

$$|g(s_1) - g(s_2)| \le C(p)|s_1 - s_2| \left\{ |s_1|^{\frac{2(p-1)}{2-p}} + |s_2|^{\frac{2(p-1)}{2-p}} \right\}.$$

Take $s_1 = u_{ij}, s_2 = u_{i,j-1}$. Then

$$|u_{ij}|^{\frac{p}{2-p}} - |u_{i,j-1}|^{\frac{p}{2-p}} \le C(p)|u_{\bar{y},ij}|\Delta y \left\{ |u_{ij}|^{\frac{2(p-1)}{2-p}} + |u_{i,j-1}|^{\frac{2(p-1)}{2-p}} \right\}.$$

Summation over $j = k+1, \dots, l$ gives

$$|u_{il}|^{\frac{p}{2-p}} - |u_{ik}|^{\frac{p}{2-p}} \le C(p) \sum_j |u_{\bar{y},ij}|\Delta y \left\{ |u_{ij}|^{\frac{2(p-1)}{2-p}} + |u_{i,j-1}|^{\frac{2(p-1)}{2-p}} \right\}.$$

The sum on the right-hand side is taken over all indices l. Taking first a sup over all l, then multiplying by Δy and summing over k and finally multiplying by Δx and summing over i implies

$$\begin{aligned} \sum_i \sup_l |u_{il}|^{\frac{p}{2-p}} \Delta x \le{}& \sum_{i,k} |u_{ik}|^{\frac{p}{2-p}} \Delta x \Delta y \\ &+ C(p) \sum_{i,j} |u_{\bar{y},ij}| \left\{ |u_{ij}|^{\frac{2(p-1)}{2-p}} + |u_{i,j-1}|^{\frac{2(p-1)}{2-p}} \right\} \Delta x \Delta y. \end{aligned}$$

Applying Hölder's inequality to both terms on the right-hand side gives

$$\begin{aligned} \sum_i \sup_l |u_{il}|^{\frac{p}{2-p}} \Delta x &\le \|u\|_{L_p,\Delta x,\Delta y} \|u\|^{\frac{q(p-1)}{p}}_{L_q,\Delta x,\Delta y} + C_2(p)\|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y} \|u\|^{\frac{q(p-1)}{p}}_{L_q,\Delta x,\Delta y} \\ &\le C_3(p)\left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y}\right) \|u\|^{\frac{q(p-1)}{p}}_{L_q,\Delta x,\Delta y}. \end{aligned}$$

Similarly one can show that

$$\sum_l \sup_i |u_{il}|^{\frac{p}{2-p}} \Delta y \leq C_3(p) \left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{x}}\|_{L_p,\Delta x,\Delta y} \right) \|u\|_{L_q,\Delta x,\Delta y}^{\frac{q(p-1)}{p}}.$$

Therefore

$$\|u\|_{L_q,\Delta x,\Delta y}^q = \sum_{i,j} |u_{ij}|^q \Delta x \Delta y \leq \sum_{i,j} \sup_i |u_{ij}|^{\frac{p}{2-p}} \sup_j |u_{ij}|^{\frac{p}{2-p}} \Delta x \Delta y$$

$$= \left(\sum_i \sup_j |u_{ij}|^{\frac{p}{2-p}} \Delta x \right) \left(\sum_j \sup_i |u_{ij}|^{\frac{p}{2-p}} \Delta y \right)$$

$$\leq C_4(p) \left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y} \right) \left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{x}}\|_{L_p,\Delta x,\Delta y} \right) \|u\|_{L_q,\Delta x,\Delta y}^{\frac{2q(p-1)}{p}}.$$

Dividing both sides by $\|u\|_{L_q,\Delta x,\Delta y}^{\frac{2q(p-1)}{p}}$ and using the inequality $\sqrt{ab} \leq \frac{a+b}{2}$ we get

$$\|u\|_{L_q,\Delta x,\Delta y} \leq C_5(p) \left(\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{x}}\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y} \right). \qquad \square$$

Now to finish the proof of inequality (11.1) we note that since the domain of u is compact, the Hölder inequality implies

$$\|u\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{x}}\|_{L_p,\Delta x,\Delta y} + \|u_{\bar{y}}\|_{L_p,\Delta x,\Delta y} \leq C \|u\|_{H_1,\Delta x,\Delta y}$$

$\forall p < 2$. Taking $p \to 2-$ in (11.2) corresponds to taking $q \to +\infty$. Therefore, we can take q to be arbitrarily large in (11.1).

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