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Loss and gain of regularity in a lubrication equation for thin viscous films

This article is dedicated to my collaborators: Michael Brenner, Todd Dupont, Leo Kadanoff and Mary Pugh.

Fourth order degenerate diffusion equations arise in the study of in the motion of thin films and and membranes of viscous liquid in which the surface tension of the liquid/air interface determines the evolution. These equations are derived via a lubrication approximation in which the fluid velocity is depth averaged in the direction transverse to the film [16, 14]. One typically obtains an evolution equation for the thickness h of the film of the form

$$h_t + \nabla \cdot (h^n \nabla \Delta h) = 0, \quad x \in \mathbb{R}^d \quad h(x,0) = h_0(x) \quad n > 0 \tag{1}$$

where n depends on the geometry of the problem and any boundary conditions relevant for the fluid. Examples include a thin neck in the Hele-Shaw cell for which $n = 1$ and $d = 1$ [11] and a thin film on a solid surface with no slip on the fluid/solid interface for which $n = 3$ and $d = 2$ [14]. The problem that I consider here is the evolution from strictly positive initial data. For a discussion of evolution from nonnegative initial data see [7, 2, 3].

Topological transitions in the thin film have a simple mathematical interpretation in the context of the lubrication approximation. They correspond to finite time singularities in the PDE of the form

$$h(x,t) \to 0 \quad \text{as} \quad t \to t_c. \tag{2}$$

The recent work [8] is the first comprehensive study of the effect of the nonlinearity n on finite time singularities of the form (2) for (1) in one dimension. One feature of this work was the observation of various ‘bifurcations’ or transitions in the structure of the solutions at many critical values of n. This is in sharp contrast to the analogous second order ‘porous medium’ equation for which singularities of the form (2) are not possible due to a maximum principle [15]. Furthermore, equation (1) is known to have transitions at critical values of n for some known exact solutions. For example, ‘source type’ solutions with compact support only exist for $n < 3$ and have a transition in their behavior at the edge of the support at $n = 3/2$ [4]. The lack of source type solutions for the case $n = 3$ is particularly relevant to the problem of a spreading drop on a solid surface [10]. Furthermore, traveling wave solutions have critical transitions at $n = 3/2, 2,$ and 3 [9]. Recent work on the weak existence theory [7, 2] suggests that 3/2 and 3 are true critical exponents for the evolution problem.

A striking feature of the behavior of the singularities computed in [8] was that most of the isolated point singularities of the form (2) were of 'second type' [1] in the sense that the scaling observed was not what one predicts from the dimensional analysis of the equation. Instead the solution has a leading order behavior that does not solve the full equation and hence requires higher order corrections. In problems with a fixed, small parameter [1], the higher order corrections produce a 'nonlinear eigenvalue problem' that determines time dependence of the solution. In [8] the time dependences resulted from matching the the behavior of the local singularity to boundary conditions or to an 'outer' solution.

In this paper, I discuss the structure of the singularity as well as continuation of the solution past the singularity.

In Section 1 I present some rigorous theory for the equation on a periodic domain. The main features of the theory are (1) lack of singularity from positive data for n sufficiently large and (2) existence of a nonnegative weak solution in a sense of distributions for $3/8 < n < 3$ (and in a weaker sense for $0 < n < 3/8$) that decays to its mean as $t \to \infty$. The main tool used to prove all results are a class of dissipative entropies. This is in sharp contrast to similar second order problems (e.g. the 'porous media equation') which possess a maximum principle. One outcome of this theory is that even if a finite time singularity does occur from smooth positive initial data, one can continue that solution after the initial singularity time as a nonnegative weak solution and there must be a second critical time after which the solution is again strong and positive and furthermore decays to its mean in the infinite time limit. The weak continuation theory is particularly relevant to the droplet problem with a moving contact line is modeled by a slip condition on the liquid solid interface (see e.g. [7] or the references contained therein for a discussion).

In Section 2 I show numerical simulations of such an occurrence. I consider $n = 0.5$ on a periodic domain with specific positive initial data. This value of n is sufficiently small to produce a finite time singularity with this special choice of initial condition. First, I show that the singularity has the same self-similar structure observed in [8] with different boundary conditions. Some features of the singularity are that $h_{min} \to 0 \sim (t_c - t)$, where t_c is the critical time, the third derivative forms a bounded jump discontinuity at the critical time, and the singularity is symmetric about its singular point. For this value of n, Theorem 1.2 guarantees a weak distribution solution that describes the evolution past the singularity time. I numerically compute this weak solution using the same approximation scheme used to prove the existence theorem. The simulation suggests that shortly after the singularity time the weak solution becomes zero on a set of positive measure. At an even later time, the solution becomes strong a positive again and decays to its mean as dictated by the rigorous theory.

There are a number of unsolved problems connected with this study. For example, within a highest regularity class, is the weak distribution solution of theorem 1.2 unique? Also, there is no rigorous proof of an initial finite time singularity with nonforcing (e.g. periodic) boundary conditions for any $n > 0$. I discuss these questions in more detail in Section 3.

Self similar singularities and second type scaling. The numerical singularity shown in Section 2 is typical of the ones seen in [8] in that it has only a leading order self similar behavior. That is the leading order behavior is given by

$$h(x,t) = \delta(t) H\Big(\frac{x}{\delta(t)^q}\Big). \tag{3}$$

An exact solution to the equation of the form (3) is called a similarity solution of 'first type'[1] in which equations for δ and H decouple into respective ODE's. In particular, fixing the scaling value q fixes the power law time dependence of δ. Typically q is determined from the particular problem at hand. For example, 'source type' solutions conserve mass which in 1-D fixes $q = -1$ and therefore $\delta \sim t^{-1/(n+4)}$. Consider a solution for which (3) is at least a leading order behavior. Plugging (3) into (1) gives

$$\frac{\delta_t}{\delta}(1 - q\eta\partial_\eta)H + \delta^{n-4q}(H^n H_{\eta\eta\eta})_\eta = 0. \tag{4}$$

An exact solution to the equation would have both terms in (4) of equivalent magnitude. Separation of variables then gives separate equations for δ and for H. If q is known, the equation for δ fixes its time dependence. If q is not known then the self-similarity is deemed of 'second type' [1] and the determination of q comes from solving a 'nonlinear eigenvalue problem'.

The typical situation for the self-similar singularities in [8] is different from the above scenario. In this case the self-similarity is only a leading order behavior, and one of the terms in (4) dominates the other. A commonly observed behavior in [8] was for the time dependence δ to cause the second term on the left hand side of (4) to dominate the first. This means that the function H describing the leading order behavior satisfies the ODE $(H^n H_{\eta\eta\eta})_\eta = 0$. The time dependence of δ is not determined by this leading order structure. However the scaling exponent q is determined simply by the parabolic scaling of the singularity[1] (seen for example in the fact that when $h \to 0$, the second derivative remains bounded away from zero and infinity).

The example computed in Section 2 has exactly this structure. Moreover, it is symmetric about its the singular point. Hence the only choice for H is the function $H(\eta) = A_0 + A\eta^2$. Since the singularity is of the form (2) with positive initial data, both A_0 and A must be positive.

[1]which fixes $q = \frac{1}{2}$

If the self similarity is only a leading order behavior it is natural to look for an infinite expansion. In [5] I discuss power series expansions in terms of rescaled variables to describe the local behavior of solutions near a singular point. I discuss the determination of δ for symmetric singularities in the lubrication approximation and in a related equation, $h_t + h^n h_{xxxx} = 0$.

1. Mathematical Theory for the Equations

I consider the one dimensional lubrication equation

$$h_t + (h^n h_{xxx})_x = 0 \tag{5}$$

on the circle, S^1. The degeneracy of this equation as $h \to 0$ requires h to be bounded away from zero for standard parabolic theory to ensure well–posedness. An a priori bound on the H^1 norm of h ensures that h is bounded from above $h < M$ on any time interval of its existence. Thus, given appropriate boundary conditions, the initial value problem with smooth initial $h_0(x) > 0$ has a unique smooth solution on any time interval on which the solution is bounded away from zero. Hence, the only possible finite time singularities are of the form $h \to 0$. This section presents some rigorous existence results for global strong and weak solutions.

Before stating the theorems I review some elementary properties of the equations for smooth solutions: One can use these properties and other properties to prove results for weak solutions. The first property is conservation of mass,

$$\int_{S^1} h(x,t)dx = \int_{S^1} h_0(x)dx.$$

Second, dissipation of surface tension energy,

$$\int_{S^1} |h_x(x,T)|^2 + \iint_{S^1\times[0,T]} f(h)h_{xxx}^2 = \int_{S^1} |h_x(x,0)|^2. \tag{6}$$

In addition to the above two quantities, there is a basic entropy dissipation: consider a function $G(y)$ satisfying $G''(y) = 1/f(y)$. The convexity of G and mass conservation allow a choice of G satisfying $\int_{S^1} G(h(x,t))\, dx \geq 0$ for all t. Integration by parts yields

$$\int G(h(x,T))dx + \iint_{S^1\times[0,T]} h_{xx}^2 = \int G(h(x,0))dx. \tag{7}$$

For $n = 0$, the linear problem, the entropy is merely the L^2 norm. Bernis and Friedman first introduced these entropies in [3]. In fact, a larger class of entropies also dissipate [8, 7, 2]. Namely,

$$\frac{d}{dt}\int G^s(h) \leq -C_s \int (h^{1-s/2})^2_{xx}, \quad C_s > 0.$$

For all G_s satisfying $G_s''(h) = h^{-(s+n)}$, and $s \in [0, 1/2)$. This fact allows one to prove

Theorem 1.1. *(Global strong solution for all time [8])*
Let h be a solution to (1) with periodic boundary conditions and smooth initial data $h_0(x) > 0$. Then if $n \geq 3.5$ then there exists a unique smooth solution $h(x,t)$ for all time that satisfies $h(x,t) > 0$. (2) If $n \geq 2$, then the above is true on any time interval on which $h_{xx}(x,t)$ remains bounded.

In lieu of a proof, I refer the reader to [8]. The proof of this theorem follows from the same argument used by Bernis and Friedman, [3], for $n \geq 4$. The sharper result is proved via a more refined class of entropies.

In addition, if a finite time singularity intervenes, one can continue the solution as a weak solution for all time. Moreover, there exists a critical time after which the solution is guaranteed to be strong and positive:

Theorem 1.2. *(Global existence of a weak solution with positive initial data [7])*
Let $0 < m \leq h_0 \leq M$, $h_0 \in H^1(S^1)$, $n > 0$ and $T > 0$. Given $0 \leq s < \frac{1}{2}$ and $n > \alpha \geq \frac{1}{2} - \frac{s}{4}$ then there exists a weak nonnegative solution to the lubrication approximation in the following sense of distributions

(8) *for* $n > 1$,
$$\iint_{Q_T} h\varphi_t dxdt - \iint_{Q_T} nh^{n-1} h_x h_{xx} \varphi_x dxdt - \iint_{Q_T} h^n h_{xx} \varphi_{xx} dxdt = 0,$$

(9) *for* $\dfrac{3}{8} < n \leq 1$,
$$\iint_{Q_T} h\varphi_t dxdt - \iint_{Q_T} h^n h_{xx} \varphi_{xx} dxdt - \iint_{Q_T} h^{n-\alpha} \left(\frac{h^\alpha}{\alpha}\right)_x h_{xx} \varphi_x dxdt = 0$$

and in the weaker sense

(10) *for* $0 < n < \dfrac{3}{8}$,
$$\iint_{Q_T} h\varphi_t dxdt + \iint_{P} h^n h_{xxx} \varphi_x dxdt = 0, \quad P = \{(x,t) | h(x,t) > 0\}$$

for all $\varphi \in C_0^\infty(Q_T)$. In all cases the solution has the additional regularity

$$h^{1-s/2} \in L^2(0,T; H^2(S^1)), \tag{11}$$

$$(h^\alpha)_x \in L^4(Q_T). \tag{12}$$

Moreover, in all cases there exist positive A and c so that

$$\|h(\cdot,t) - \bar{h}\|_{L^\infty} \leq Ae^{-ct}. \tag{13}$$

A depends on M, m, $\bar{h}$, and n, and c depends on n and $\bar{h}$. In particular, there exists a critical time T^* after which the solution is guaranteed to be strong and positive.

This theorem is proved in [7]. Similar existence and long time results are also proved in [7] for the case of *nonnegative* initial data for $0 < n < 3$.[2] A striking feature of the evolution from nonnegative data is the sharpness of the existence theory given a most regular class of 'source type' solutions for the problem.

The above result strengthens previously known weak existence result due to Bernis and Friedman [3]. One feature of the entropies used to prove this result is that when $n \geq 3/2$, if a singularity forms from positive initial data, the resulting weak solution can only be zero on a set of zero measure. No numerical evidence exists for the occurrence of finite time singularities from positive initial data with periodic boundary conditions when $n \geq 3/2$.

2. Numerical Results

In this section I present a numerical case study showing both the onset of the singularity and the evolution of the weak solution past the singularity time. Specifically I solve the equation

$$h_t + (h^n h_{xxx})_x = 0, x \in [-1, 1], n = 0.5,$$

on the periodic domain $[-1, 1]$ with positive initial data

$$h_0(x) = 0.8 - \cos(\pi x) + 0.25 \cos(2\pi x). \tag{14}$$

Since the initial data is symmetric about $x = 0$ and the equation preserves the symmetry, I compute the solution on $[0, 1]$ and use reflection symmetry at the boundary.

The numerical method used is an adaptation of a code used in [6, 8] and earlier papers [11, 12]. It is a conventional finite difference method using an implicit, two level scheme based on central differences. The implicit time step is crucial to remove stiffness associated with the parabolicity of the equation. On each time step, the fully nonlinear difference equations are solved via Newton's method. The interested reader can read [6, 8] for details on the difference scheme.

The main difference between the code used to compute solutions in [6, 8] and the results presented here is the implementation of a *self similar dynamically adaptive mesh scheme* to resolve the singularity to arbitrary order. This method is also used in [5] to study a family of singularities of this type.

2.1. Self similar adaptive mesh scheme. The main advantage of this regridding scheme is that (1) it is straightforward to implement and (2) it is very efficient in that the total number of mesh points required depends only logarithmically on the smallest length scale resolved. The computation described below took only a few minutes to run on a Sparc10.

The following scheme assumes a singularity at the origin. However, it can be generalized to a singularity at another point or a moving singularity. Initially

[2] Existence with the same degree of regularity (e.g. (11–12)) in the weaker sense (10) was proved in the recent work [2]).

start with a fixed mesh. The computation presented here uses an initial grid of $2^{10} = 1024$ uniform intervals. A second parameter is the number of grid points desired to resolve the singularity scale. In this calculation, this scale is $\sim \delta^{1/2}$ (for the remainder of this article I use the notation δ interchangeably with the minimum height, h_{min}: see below). The calculation presented here uses 64 mesh points resolving the smallest length scale. When $\delta^{1/2}$ decreases to 64 mesh points ($= 64/1024$) wide divide the first 64 intervals in half and define this to be the new mesh. When $\delta^{1/2}$ decreases to 64 of the new mesh points, repeat the division process. Each regridding introduces only 64 new mesh points. This can be repeated *ad infinitum* with only logarithmic dependence of the number of mesh points on the smallest scale resolved. Moreover, in this special case where the singularity occurs at $x = 0$, the fact that intervals are always halved exploits the binary arithmetic of of the computation and hence computes dx to infinite precision[3] (within the limits of machine storage of the exponent (e.g. approx. 10^{-300} for double precision)). The simulation presented here goes through 55 levels of regridding, producing 3520 new points. Hence the smallest dx at the end of the simulation is $dx = 2^{-65}$. The values of h at each new mesh point are computed by linearly interpolating h_{xx}.[4] The third derivative h_{xxx}, stored as a separate array throughout the calculation, is computed by linear interpolation.

2.2. Loss of regularity: form of the initial singularity. First I show that the structure of the solution around the singularity is locally described by the first two terms in an expansion.

First, to check that $h(x,t)$ to leading order behaves like a decreasing parabola, one simply needs to note that the minimum remains at $x = 0$ and that the second derivative remains bounded away from zero and infinity as the solution nears the critical time. In fact the second derivative at $x = 0$ approaches a constant value of 1.00 as $t \to t_c$. The simulation indicates that the singularity occurs at $t_c \sim 0.00073$.

The behavior of the second derivative suggests that locally, to leading order, the solution behaves like $\delta(t) + \frac{1}{2}x^2 + \ldots$. As in [8], a higher order correction to this simple form comes from writing $h(x,t) = \delta(t) + \frac{1}{2}x^2 + g(x,t)$ and solving for g as a perturbation. This procedure results in the following equation for g:

$$\dot{\delta} + g_t = -\Big[(\delta + \tfrac{1}{2}x^2 + g(x,t))^n g_{xxx}\Big]_x.$$

To leading order, neglect g_t on the left hand side and g in the ( ) on the right hand side to obtain a correction to the leading order parabola, $\delta + \frac{1}{2}x^2$, with a self

[3]There is also roundoff error introduced at the level of the spatial differencing. This error is damped out by the implicit time step for large dt but becomes significant as dt gets small. This calculation, run in double precision arithmetic, was stopped after the roundoff error became significant in the computation of $(h^n h_{xxx})_x$.

[4]computed from simple three point central differencing

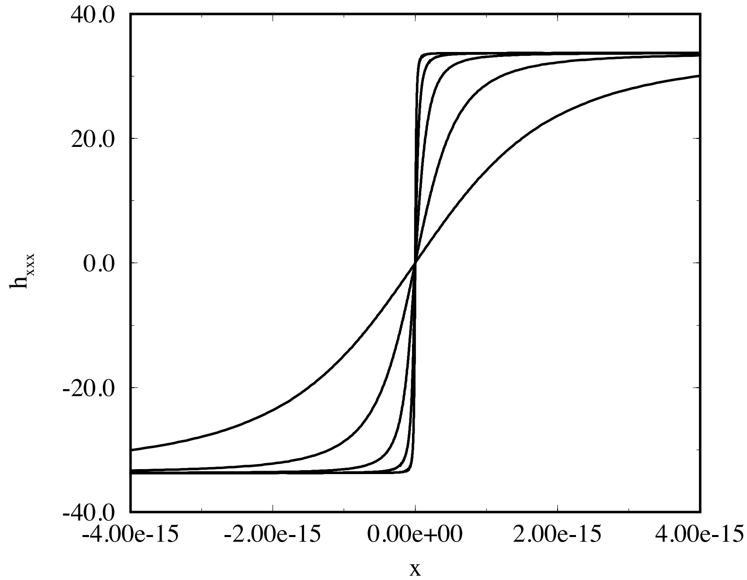

FIGURE 1. Onset of initial singularity. Formation of a jump discontinuity in the third derivative.

similar third derivative:

$$g_{xxx} \sim \dot{\delta}\delta^{1/2-n}\frac{y}{(1+\frac{1}{2}y^2)^n}, \quad y = \frac{x}{\delta^{1/2}}. \tag{15}$$

This functional form was verified in [8] for solutions with 'pressure' and 'pumping' boundary conditions for many values of $0 < n < 1$.

Note that when $n = 0.5$, $\delta \sim (t_c - t)$, the above dictates that the third derivative should form a jump discontinuity as $t \to t_c$. This phenomena is shown in Figure 1 which shows the third derivative at times very close to the singularity time. The different curves correspond to a minimum height, δ, of 2.1×10^{-30}, 1.9×10^{-31}, 1.8×10^{-32}, 1.7×10^{-33}, 1.6×10^{-34}. To check the functional form of the solution, note that (15) predicts the fourth derivative locally behaves as $\dot{\delta}\delta^{-n}/(1+\frac{1}{2}y^2)^{3/2}$. Figure 2 shows rescaled profiles of $h_{xxxx}(x,t)$. The plots are of

$$\frac{h_{xxxx}(x/\sqrt{h_{min}}, t)}{h_{xxxx}(0,t)}.$$

The data collapses onto the solid line $1/(1+\frac{1}{2}y^2)$. The five different profiles correspond to approximate values of δ: 10^{-7}, 10^{-12}, 10^{-20}, and 10^{-30}. The data shows that the simulated solutions agree with the leading order terms in the expansions quite well.

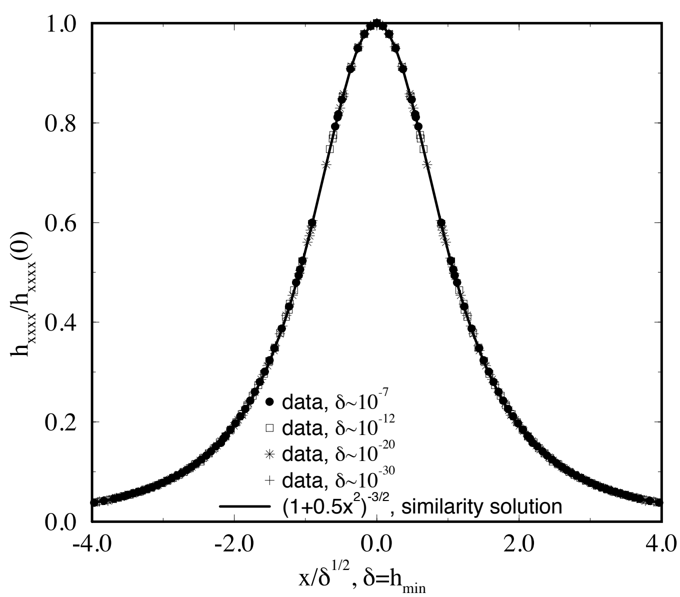

FIGURE 2. Self similarity in the fourth derivative. Rescaled data.

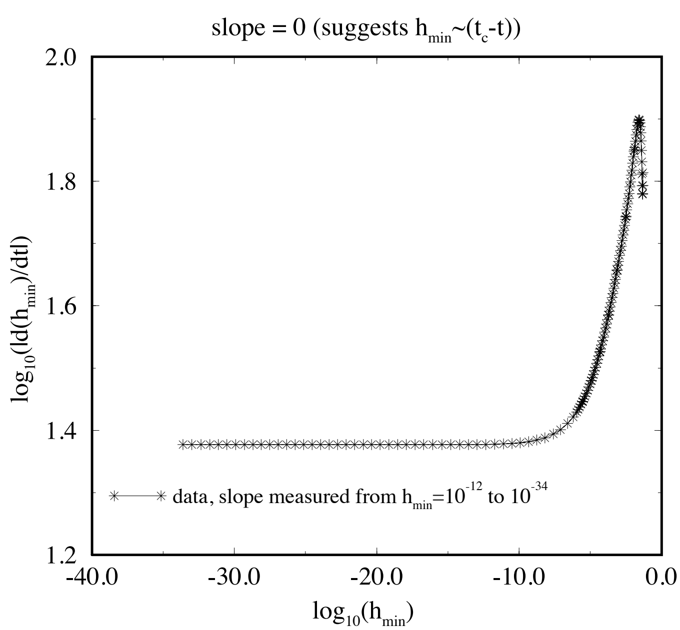

FIGURE 3. Time dependence of the minimum.

2.3. Time dependence of the minimum. I show that the solutions have the time dependence $\delta \sim (t_c - t)$. Figure 3 depicts $log_{10}(h_{min})$ versus $log_{10}(|dh_{min}/dt|)$ for 22 decades of scaling in h_{min}. The slope measured is $0.0000\ldots$ (to five decimal places) and is computed using data with h_{min} ranging from 10^{-12} to 10^{-34} and a standard linear regression fit of the data. dh_{min}/dt is computed by simply differencing $(h_{min}(t + \Delta t) - h_{min}(t))/dt$ over the step. This data clearly indicates a linear time dependence in $(t_c - t)$. Note that the solution must reach a fairly small minimum height before the linear time dependence is truly observed. This 'transient' time scale in the range $h_{min} > 10^{-10}$ has been observed in other fourth order computations with 'second type' scaling [12, 8, 5]. Hence much resolution is needed to see the actual power law in the time dependence.

2.4. After the initial singularity and eventual gain of regularity. I compute a nonnegative weak continuation of the solution after the singularity via the approximation scheme first introduced in [3] and used in [7] to prove Theorem 1.2 for existence of a distribution solution after the singularity time.

That is, regularize the equation by

$$h_{\epsilon t} + (f_\epsilon(h_\epsilon) h_{\epsilon xxx})_x = 0 \quad f_\epsilon(h_\epsilon) = \frac{{h_\epsilon}^4 f(h)}{\epsilon f(h) + {h_\epsilon}^4}. \tag{16}$$

Note that f_ϵ is still degenerate however for $n < 4$, $f_\epsilon \sim h^4/\epsilon$ as $h \to 0$. Bernis and Friedman [3] proved that this approximate problem has global, positive, smooth solutions:

Theorem 2.1. *(Global existence of smooth positive solutions for the regularized problem [3]) Let $h_0 \in H^1(S^1)$, $h_0 \geq 0$. Given an initial condition*

$$h_{\epsilon 0}(x) = h_0(x) + \delta(\epsilon)$$

there exists a unique positive solution to the regularized equation

$$h_{\epsilon t} + (f_\epsilon(h_\epsilon) h_{\epsilon xxx})_x = 0$$

$$f_\epsilon(h_\epsilon) = \frac{{h_\epsilon}^4 f(h)}{\epsilon f(h) + {h_\epsilon}^4}.$$

In the case of nonnegative initial data, the regularization of the equation must be accompanied by a 'lifting' of the data. However, since I consider strictly positive initial data in this example, the only change is in the equation. The weak solution of Theorem 1.2 is obtained as the uniform limit of strong positive solutions of (16).

The numerics indicate that the initial singularity occurs at approximately $t = 0.00073$. To compute the weak solution I solve (16) with a fixed ϵ and initial data (14). The data presented here compares several values of ϵ. The simulations show that after the singularity time the weak solution is zero on finite interval around $x = 0$.

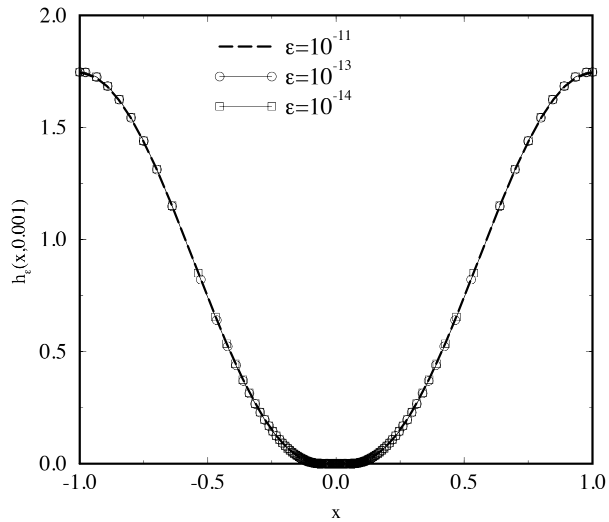

FIGURE 4. Regularized approximation of weak solution. Pictured are the $h_\epsilon(x,t)$ profiles at fixed time $t = 0.001$ for the three values of ϵ, 10^{-11}, 10^{-13}, and 10^{-14}.

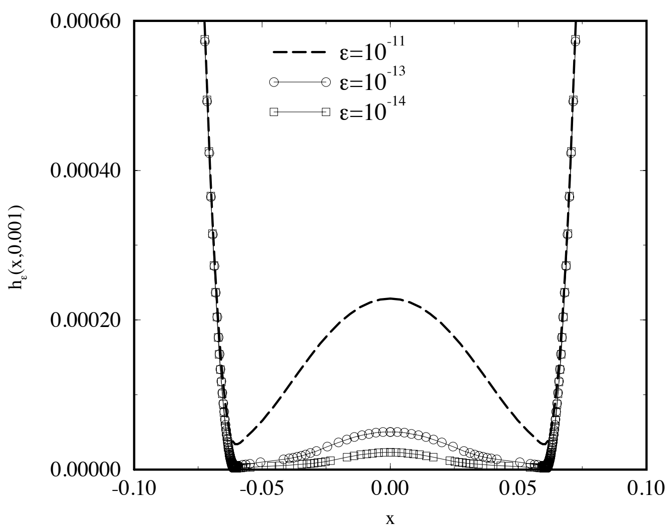

FIGURE 5. Regularized approximation of weak solution. At fixed time $t = 0.001$, $\epsilon = 10^{-11}$, 10^{-13}, and 10^{-14}. Regularizations converge to 0 on the interval $[-0.06, 0.06]$.

Figure 4 shows the height profiles for h at the fixed time $t = 0.001$ for three different values of the regularization parameter, $\epsilon = 10^{-11}$, 10^{-13}, and 10^{-14}. Figure 5

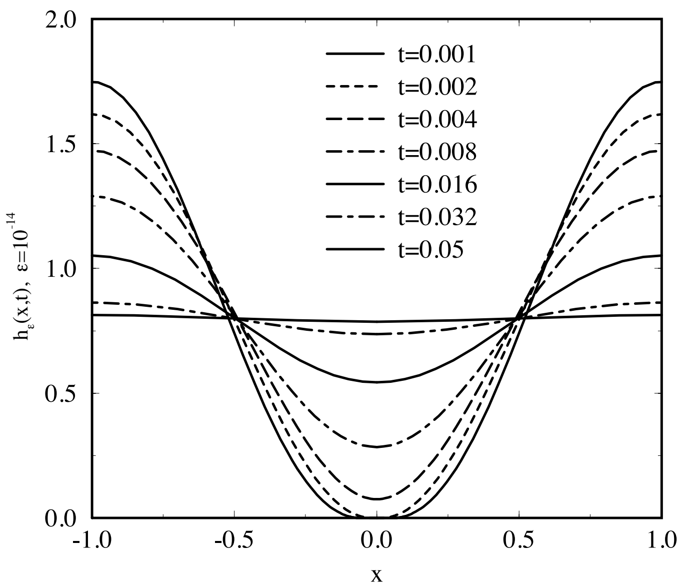

FIGURE 6. Eventual gain of regularity and convergence to the mean. 'Weak solution' computed from the ϵ regularization, $\epsilon = 10^{-14}$.

depicts the same data as Figure 4 at very small heights on the interval $[-0.1, 0.1]$. The trend in Figure 5 as $\epsilon \to 0$ indicates that the weak solution is zero on the interval $[-0.06, 0.06]$ (approximate). Note that the apparent 'edge' of the support of the weak solution has quite good dependence on ϵ. Similar data (not shown) of the second derivative indicates that at this time $t = 0.001$ the solution has a jump discontinuity in the second derivative at the edge of its support. This is characteristic of such solutions and is the weakest behavior possible given the regularity dictated by Theorem 1.2. This behavior is also consistent with local analysis of explicit traveling wave solutions to the equations [9]. Figure 6 shows the solution at later times $t = 0.001$ to 0.05. The data suggests that at approximately $t = 0.0023$ the weak solution becomes strong and positive again. It clearly decays to its mean in the infinite time limit as dictated by Theorem 1.2.

3. Conclusions and Remarks

This paper serves to discuss some issues involved in the singularity problem for a family of fourth order degenerate diffusion equations of the form $h_t + (h^n h_{xxx})_x = 0$. In addition to the discussion, I present a numerical case study of such a singularity when $n = 0.5$ and when the equation is solved with periodic boundary conditions. The numerics indicate that with a special choice of initial condition, the solution has a finite time singularity at the origin and that a nonnegative weak continuation of the solution past the singularity time shows the weak solution to be zero on a set of positive measure for a finite interval of time. Moreover, this weak solution eventually becomes strong a positive as dictated by rigorous theory.

This example illustrates just one of the ways in which fourth order degenerate diffusion equations show a richer class of behavior than their second order cousins. Finite time singularities of the type described here are only one facet of the problem. The weak solution theory also shows much more rich behavior than that indicated for the second order porous media equation. A common thread in both the singularity and weak solution problems is the appearance of 'critical exponents', values of n where the equation $h_t + (h^n h_{xxx})_x = 0$ has a characteristic change in its behavior. For a discussion of critical exponents and singularity formation in this equation with the 'pumping' boundary conditions, $h(\pm 1) = 1$, $h_{xxx}(\pm 1) = \pm c$ or 'pressure' boundary conditions, $h(\pm 1) = 1$, $h_{xx}(\pm 1) = p$, including some with non self similar complex structure, see [8]. For a discussion of critical exponents and singularity formation in the equation $h_t + h^n h_{xxxx} = 0$ see [5].

Regarding the case of periodic boundary conditions, singular solutions of the type (2) have not been observed for the lubrication equation with $n = 1$ (corresponds to the thin neck in the Hele-Shaw cell) unless one adds a destabilizing force such as gravity in the Hele-Shaw cell [13, 16]. Alternatively, with the pressure boundary conditions (a forcing at the boundary), finite time singularities have been observed for $n = 1$ [12, 8].

Very few rigorous results exist for the question of finite time singularities. With the 'pumping' boundary condition, one trivially obtains a singularity because $\int h$ decreases at a constant rate. However, we lack theory for whether this happens on the boundary or in the interior of the domain when $n \leq 3.5$. Numerics [8] indicates a transition from an interior singularity to a boundary singularity at $n \sim 1.5$.

Also, a rigorous result with the pressure boundary conditions, with $n < 0.5$ and $p > 2$ was recently obtained [2] by exploiting the nontrivial long time character of the system. However, no results exist proving singularity formation for the situation with unforced boundary conditions. Moreover no rigorous results with any boundary conditions exist to describe the local structure of the singularity.

The are also a number of open problems concerning the weak solutions. One such problem is uniqueness of a distribution solution within the higher regularity class. While the recent work [2] shows clear nonuniqueness of very weak solutions (in the sense (10)) the sharp existence results of [7] suggest a uniqueness result in a restricted class may be true. Furthermore, none of the above questions have been addressed in higher dimensions.

4. Acknowledgements

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