Swarming by Nature and by Design

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- Ira Schwartz, Naval Research Lab
- Jose Carrillo, Barcelona
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- 8 Journal Publications
- 13 Refereed Conference Proceedings

- Physics, Mathematics, Control Theory, Robotics literature.
Properties of biological aggregations

- Large-scale coordinated movement
- No centralized control
- Interaction length scale (sight, smell, etc.) << group size
- Sharp boundaries and “constant” population density
- Observed in insects, fish, birds, mammals…
- Also important for cooperative control robotics.
Propagation of constant density groups in 2D


- Conserved population
- Velocity depends nonlocally, linearly on density

\[ \rho_t + \nabla \cdot (\bar{v} \rho) = 0, \quad \bar{v}(\bar{x}, t) = \tilde{K} \ast \rho = \int_{\mathbb{R}^2} \tilde{K}(|\bar{x} - \bar{y}|) \rho(\bar{y}, t) \, d\bar{y} \]

\[ K = \nabla^\perp \Psi + \nabla \Phi \]

- incompressibility
- Potential flow
Incompressible flow dynamics


Assumptions:

- Conserved population
- Velocity depends nonlocally, linearly on density

\[ \rho_t + \nabla \cdot (\bar{v}\rho) = 0 \]

\[ \bar{v}(\vec{x}, t) = \vec{K} \ast \rho = \int_{\mathbb{R}^2} \vec{K}(|\vec{x} - \vec{y}|) \rho(\vec{y}, t) \, d\vec{y} \]

Incompressibility leads to rotation in 2D
\[
\rho_t + \nabla \cdot \{ \rho \left[ \nabla (K \ast \rho) - r \rho \nabla \rho \right] \} = 0
\]

Social attraction:

- Sense averaged nearby pop.
- Climb gradients
- K spatially decaying, isotropic
- Weight 1, length scale 1
Mathematical model

\[
\rho_t + \nabla \cdot \left\{ \rho \left[ \nabla (K * \rho) - r \rho \nabla \rho \right] \right\} = 0
\]

**Social attraction:**
- Sense averaged nearby pop.
- Climb gradients
- \( K \) spatially decaying, isotropic
- Weight 1, length scale 1

**Dispersal (overcrowding):**
- Descend pop. gradients
- Short length scale (local)
- Strength ~ density
- Characteristic speed \( r \)
Coarsening dynamics

Example

box length $L = 8\pi$ • velocity ratio $r = 1$ • mass $M = 10$
Example

box length $L = 2\pi$ • velocity ratio $r = 1$ • mass $M = 2.51$

Steady-state density profiles

Energy

\[ \int (r/3)\rho^3 - \rho K \star \rho \, d\vec{x} \]
How to understand? Minimize energy

\[ E(\rho) = \int_D \frac{r}{3} \rho^3 - \rho K * \rho \ d\mathbf{x} \]

over all possible rectangular density profiles.

**Results:**

- Energetically preferred swarm has density $1.5r$
- Preferred size is $L/(1.5r)$
- Independent of particular choice of $K$
- Generalizes to 2d – currently working on coarsening and boundary motion
Example

velocity ratio $r = 1$
Finite time singularities - pointy potentials

\[
\rho_t + \nabla \cdot (\rho \nabla K \ast \rho) = 0
\]

**Previous Results**
- For smooth \( K \) the solution blows up in infinite time.
- For \( n=1 \), and `pointy' \( K \) (biological kernel: \( K = e^{-|x|} \)) blows up in finite time due to delta in \( K_{xx} \).

**2007 result:**
For `pointy’ kernel one can have smooth initial data that blows up in finite time *in any space dimension*.

Proof: uses Lyapunov function and some potential theory estimates.

Finite time singularities - general potentials

\[ \rho_t + \nabla \cdot (\rho \nabla K * \rho) = 0 \]

- Previous Results
- For smooth \( K \) the solution blows up in infinite time
- For `pointy' \( K \) (biological kernel such as \( K = e^{-|x|} \)) blows up in finite time for special radial data in any space dimension.

New result:
Osgood condition
\[ \int_0^L \frac{1}{K'(r)} dr < \infty \]
is a necessary and sufficient condition for finite time blowup in any space dimension (under mild monotonicity conditions).

Moreover - finite time blowup for pointy potential can not be described by 'first kind' similarity solution in dimensions \( N=3,5,7,... \)
Similarity solution of form

\[ \rho(x, t) = \frac{1}{(T - t)^{\alpha}} w\left( \frac{x}{(T - t)^{\beta}} \right) \]

- The equation implies \( \alpha = (n - 1)\beta + 1 \)
- Conservation of mass would imply \( \alpha = n\beta \) - NO
- Second kind similarity solution - no mass conservation
- Experimentally, the exponents vary smoothly with dimension of space, and there is no mass concentration in the blowup....

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Finite time blowup for 'pointy' potential, \( K = |x| \), can not be described by 'first kind' similarity solution in dimensions \( N=3,5,7,... \)” - Bertozzi, Carrillo, Laurent
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Huang and Bertozzi
preprint 2009-radially symmetric numerics

Shape of singularity-

pointy potential
Simulations by Y. Huang

Figure 3: The exponents characterizing the blowup in different spatial dimensions: $\beta$ (Left) and $\alpha$ (right). The comparison of the estimated $\alpha$ is in perfect agreement with the relation (11).

Figure 4: The convergence of the normalized profiles in dimension three. (a) Near the origin, all the profiles are indistinguishable. (b) Far away from the origin, the blowup dynamics adjusts the algebraic decay of the tail.
• CONNECTION TO BURGERS SHOCKS

• In one dimension, \( K(x) = |x| \), even initial data, the problem can be transformed exactly to the Burgers equation for the integral of \( u \).

\[
\psi = \int_0^x u(x') \, dx', \quad \phi = C - 2\psi, \quad \phi_t + \phi\phi_x = 0.
\]

• Burgers equation for odd initial data has an exact similarity solution for the blowup - it is an initial shock formation, with a \( 1/3 \) power singularity at \( x=0 \).

• There is no jump discontinuity at the initial shock time, which corresponds to a zero-mass blowup for the aggregation problem. However immediately after the initial shock formation a jump discontinuity opens up - corresponds to mass concentration in the aggregation problem instantaneously after the initial blowup.

• This Burgers solution is (a) self-similar, (b) of `second kind', and (c) generic for odd initial data. There is a one parameter family of such solutions (also true in higher D).

• For the original \( u \) equation, this corresponds to beta = \( 3/2 \).

\[ \phi(x, t) \]

Shape of singularity—pointy potential
• Local existence of solutions in $L^p$ provided that
  \[ \nabla K \in W^{1,q}(\mathbb{R}^N) \]
  where $q$ is the Holder conjugate of $p$ (characteristics).
• Existence proof constructs solutions using characteristics, in a similar fashion to weak $L^\infty$ solutions (B. and Brandman Comm. Math. Sci. - special issue).
• Global existence of the same solutions in $L^p$ provided that $K$ satisfies the Osgood condition (derivation of a priori bound for $L^p$ norm - similar to refined potential theory estimates in BCL 2009).
• When Osgood condition is violated, solutions blow up in finite time - implies blowup in $L^p$ for all $p>p_c$. 
• *Ill-posedness* of the problem in $L^p$ for $p$ less than the Holder-critical $p_c$ associated with the potential $K$.

• *Ill-posedness* results because one can construct examples in which mass concentrates instantaneously (for all $t>0$).

• For $p > p_c$, uniqueness in $L^p$ can be proved for initial data also having bounded second moment, the proof uses ideas from *optimal transport*.

• The problem is *globally well-posed* with measure-valued data (preprint of Carrillo, DiFrancesco, Figalli, Laurent, and Slepcev - using optimal transport ideas).

• Even so, for *non-Osgood* potentials $K$, there is loss of information as time increases.

• Analogous to information loss in the case of compressive shocks for scalar conservation laws.
Discrete Swarms: A simple model for the mill vortex

M. D’Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. Chayes,
Physical Review Letters 2006

Discrete:

\[ m_i \frac{\partial v_i}{\partial t} = (\alpha - \beta |v_i|^2)v_i - \nabla_i \sum_j V(|x_i - x_j|) \]  
Rayleigh friction  \( \beta |v|^2 = \alpha \)

\[ V(|x_i - x_j|) = -C_a^{-}|x_i - x_j|/l_a + C_r^{-}|x_i - x_j|/l_r \]  
Morse potential

Adapted from Levine, Van Rappel Phys. Rev. E 2000

- without self-propulsion and drag this is Hamiltonian
- Many-body dynamics given by statistical mechanics
- Proper thermodynamics in the H-stable range
- Additional Brownian motion plays the role of a temperature
- What about the non-conservative case?
  - Self-propulsion and drag can play the role of a temperature
  - non-H-stable case leads to interesting swarming dynamics
H-Stability for thermodynamic systems

Thermodynamic Stability for N particle system

Mathematically treat the limit \( N \to \infty \)

So that free energy per particle:

\[
\lim_{N \to \infty} \frac{E(N)}{N} \quad \text{exists}
\]

NO PARTICLE COLLAPSE IN ONE POINT

(H-stability)

NO INTERACTIONS AT TOO LARGE DISTANCES

(Tempered potential : decay faster than \( r^{-3+\varepsilon} \))

H-instability ‘catastrophic’ collapse regime

D. Ruelle, Statistical Mechanics, Rigorous results
A. Procacci, Cluster expansion methods in rigorous S.M.
Morse Potential 2D

$$r \equiv \frac{l_r}{l_a}$$

Catastrophic:
particle collapse as

$$N \to \infty$$

$$\beta |v_i|^2 = \alpha$$

$$\frac{r}{v_i} \text{ rotation!}$$

Stable:
particles occupy macroscopic volume as

$$N \to \infty$$

$$C = \frac{C_r}{C_a}$$

always stable

catastrophic

catastrophic

catastrophic

catastrophic

r=C

r=1/C^{1/2}
FEATURES OF SWARMING IN NATURE:

- Large-scale coordinated movement,
- No centralized control
- Interaction length scale (sight, smell, etc.) $\ll$ group size
- Sharp boundaries and “constant” population density, Observed in insects, fish, birds, mammals…

Interacting particle models for swarm dynamics

Catastrophic vs H Stable

Discrete:

H Stable
\[ \alpha = \beta = 0.5 \]
\[ C_a = 1.0, \ C_r = 40.0 \]
\[ l_a = 0.6, \ l_r = 0.1 \]

Catastrophic
\[ \alpha = 0.8, \beta = 0.5 \]
\[ C_a = 0.5, C_r = 1.0 \]
\[ l_a = 2.0, l_r = 0.5 \]
Catastrophic vs H Stable

Discrete:

H Stable

$\alpha = \beta = 0.5$
$C_a = 1.0, C_r = 40.0$
$l_a = 0.6, l_r = 0.1$

Catastrophic

$\alpha = 0.8, \beta = 0.5$
$C_a = 0.5, C_r = 1.0$
$l_a = 2.0, l_r = 0.5$
Catastrophic vs H Stable

Discrete:

H Stable
\[ \alpha = \beta = 0.5 \]
\[ C_a = 1.0, \ C_r = 40.0 \]
\[ l_a = 0.6, \ l_r = 0.1 \]

Catastrophic
\[ \alpha = 0.8, \beta = 0.5 \]
\[ C_a = 0.5, C_r = 1.0 \]
\[ l_a = 2.0, l_r = 0.5 \]
Double spiral

Discrete:

\[ \alpha = 3, \beta = 0.5 \]
\[ C_a = 0.5, C_r = 1.0 \]
\[ l_a = 2.0, l_r = 0.5 \]

Run3
H-unstable

\[ \beta \left| \frac{r}{v_i} \right|^2 = \alpha \]
\[ \frac{r}{v_i} \text{ rotation!} \]
H-stable dynamics

Well defined spacings

Large alpha fly apart (infinite b.c)

$N = 200, \; \beta = 0.5$

$C_a = 1.0, \; C_r = 37.0$

$I_a = 0.7, \; I_r = 0.1$

Constant angular velocity?
Ring and Clump formation

\[ \alpha = 0.8, \beta = 0.5 \]
\[ C_a = 2.0, C_r = 0.5 \]
\[ l_a = 2.0, l_r = 0.5 \]

\[ \alpha = 0.8, \beta = 0.5 \]
\[ C_a = 2.0, C_r = 0.6 \]
\[ l_a = 2.0, l_r = 0.5 \]
Ring formation
Continuum limit of particle swarms

set rotational velocities

\[ \mathbf{v} = \sqrt{\frac{\alpha}{\beta}} (-\sin \theta, \cos \theta) \]

Steady state: Density implicitly defined

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \alpha \mathbf{u} - \beta |\mathbf{u}|^2 \mathbf{u} - \frac{1}{m^2} \nabla \int V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t) d\mathbf{y} \]

\[ \rho(r) \]

Dynamic continuum model may be valid for catastrophic case but not H-stable.

Constant speed, not a constant angular velocity

\[ \int_{0}^{\infty} \rho(R) U(r - R) dR = D - \frac{\alpha}{\beta} \ln r \]
Comparison continuum vs discrete for catastrophic potentials
Why H-unstable for continuum limit?

• H-unstable - for large swarms, the characteristic distance between neighboring particles is much smaller than the interaction length of the potential so that

\[ \nabla \int V(\vec{x} - \vec{y}) \rho(\vec{y}, t) d\vec{y} \sim \nabla \sum_i V(x - y_i) \cdot \]

• In H-stable regime the two lengthscales are comparable (by definition).

Obstacle is camouflaged from overhead vision tracking system – cars use crude onboard sensors to detect location and avoid obstacle while maintaining some group cohesion. Their goal is to end up at target location on far side of obstacle.

Algorithm by D. Morgan and I. Schwartz, NRL, related to discrete swarming model.

Processed IR sensor output for obstacle avoidance shown along path.

box is blind to overhead tracking cameras
Biomimetic Boundary Tracking  

Joshi et al ACC 2009.

- boundary tracking like ants following pheremone trails.
- uses sensor data.
- geometric motion returns vehicle to the path.
- cooperative steering (convoy).
- statistical signal filtering is important.
- idea also applied to edge detection in hyperspectral imagery and AFM.
Papers-Swarm Models and Analysis

• Andrea L. Bertozzi, Jose A. Carrillo, and Thomas Laurent, Nonlinearity, 2009.
• Andrea L. Bertozzi, Thomas Laurent, Jesus Rosado, in preparation.
• Yanghong Huang, Andrea L. Bertozzi, in preparation.
Papers-Robotics and Control