EXISTENCE AND UNIQUENESS OF SOLUTIONS TO AN AGGREGATION EQUATION WITH DEGENERATE DIFFUSION

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ABSTRACT. We present an energy-methods-based proof of the existence and uniqueness of solutions of a nonlocal aggregation equation with degenerate diffusion. The equation we study is relevant to models of biological aggregation.

1. **Introduction.** A number of nonlocal continuum models have been proposed in order to understand aggregation in biological systems, see [8, 17, 19, 30, 31, 36] and references therein. Several of such models lead to nonlocal equations with degenerate diffusion. We consider the existence and uniqueness of solutions of nonlocal equations with degenerate diffusion which are relevant for models that have been introduced by Boi, Capasso, and Morale [8] and Topaz, Bertozzi, and Lewis [36]. These models have been further studied by Burger, Capasso, and Morale [11] and Burger and Di Francesco [12]. It is also important to note that nonlocal equations of this type have arisen in other contexts and that a number of results on existence, uniqueness and asymptotic behavior is known; see the work of Carrillo, McCann, Villani [16] and references therein.

A related model of aggregation, without the diffusion, has been studied by Topaz and Bertozzi [35], Bodnar and Velazquez [7], Laurent [23], Bertozzi and Brandman [3], Bertozzi, Laurent, and Rosado [6], Li and Zhang [27], Carrillo, Di Francesco, Figalli, Laurent, Slepčev [14] and others. Of further interest are models with asymmetric interaction kernels that have been studied by Milewski and Yang [30].

In this paper we provide a proof of the existence and uniqueness of weak solutions of the equation:

$$\rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0 \tag{1}$$

where A is such that the equation is (degenerate) parabolic and K is smooth, nonnegative, and integrable. The precise conditions on A and K are given in Section 2. We consider the problem with no-flux boundary conditions on bounded convex domains and periodic solutions in any dimension. We also consider the Cauchy

problem on $\mathbb{R}^N$ for $N \geq 3$. In applications to biology, ρ represents the population density, while K is the sensing (interaction) kernel that models the long-range attraction. The term containing $A(\rho)$ models the local repulsion (dispersal mechanism).

Burger, Capasso, and Morale [11] have already shown the existence and uniqueness of entropy solutions to the equation. Such solutions have an entropy condition as a part of the definition of a solution. They were developed by Carrillo [13] to study (among other problems) parabolic-hyperbolic problems, in particular degenerate parabolic equations with lower order terms that include conservation-law-type terms. For more on entropy solutions we refer to works of Karlsen and Risebro [20, 21] and references therein. The uniqueness of solutions relies on L^1 stability estimates.

Here we show that the standard notion of a weak solutions is sufficient for uniqueness of solutions. Heuristically, the entropy condition is not needed since the nonlocal term does not create shocks. The proof of uniqueness relies on the stability of solutions in the H^{-1} sense. We also provide a detailed proof of the existence of solutions. The proof is based on energy methods and relies only on basic facts of the theory of uniformly parabolic equations and some functional analysis. The main technical difficulty comes from the degeneracy of the diffusion term. Note that without the nonlocal term the equation is the well studied filtration equation (generalized porous medium equation). For the wealth of information on the filtration equation and further references we refer to the book by Vazquez [37]. Our approach to existence relies on a number of tools from the paper by Alt and Luckhaus [1].

We consider the case where K is smooth enough to guarantee that solutions stay bounded on any finite time interval. We mention for completeness that there is significant activity on the blowup problem for the case where K is not smooth and indeed finite time blowup can occur with mildly singular K (e. g. Lipschitz continuous). Some recent work on this problem includes [5, 3, 4] for the inviscid case [26, 25, 24] for the problem with fractional diffusion and [27] for the problem in 1D with nonlinear diffusion.

Provided that $K(x) = K(-x)$, associated to the equation is a natural Lyapunov functional, the energy:

$$E(\rho) := \int_\Omega G(\rho) - \frac{1}{2}\rho K * \rho\, dx$$

where G is such that $G''(z) = A'(z)/z$ for $z > 0$.

The energy is not just a dissipated quantity; the equation is a gradient flow of the energy with respect to the Wasserstein metric. This fact was used by Burger and Di Francesco [12] to show the existence and uniqueness of solutions in one dimension. They used the theory of gradient flows in Wasserstein metric developed by Ambrosio, Gigli, and Savaré [2]. The theory applies to several dimensions as well. However the approach we take does not require K to be even, applies to a wider class of nonlinearities and directly provides better regularity of solutions. Let us point out that, in one dimension, Burger and Di Francesco [12] also obtained further properties of solutions that do not follow from [2]. Optimal transportation methods have also been used by Carrilo and Rosado [15] to show uniqueness of solutions of aggregation equations with linear diffusion or with no diffusion.

The paper is organized as follows. In Section 2 we prove the uniqueness and existence of solutions on bounded convex domains. Analogous results for periodic solutions are established in Section 3. In Section 4 we prove the existence and

uniqueness of solutions on $\mathbb{R}^N$ when $N \geq 3$. In Section 5 we introduce the energy and prove the energy-dissipation inequality.

2. **Solutions on a bounded domain.** Let Ω be a bounded convex set in $\mathbb{R}^N$. We consider the equation on $\Omega_T := \Omega \times [0, T]$ with no-flux boundary conditions:

$$\rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0 \qquad \text{in } \Omega_T,$$
$$\rho(0) = \rho_0 \qquad \text{in } \Omega, \qquad \textbf{(E1)}$$
$$(-\nabla A(\rho) + \rho \nabla K * \rho) \cdot \nu = 0 \qquad \text{on } \partial\Omega \times [0, T].$$

Above, and in the remainder of the paper, $\rho(t)$ refers to the function $\rho(\,\cdot\,, t) : \Omega \to \mathbb{R}$. In the convolution term, ρ is extended by zero, outside of Ω. More precisely $\nabla K * \rho(x) = \int_\Omega \nabla K(x - y)\rho(y)dy$.

We make the following assumptions on A and K:

(A1) A is a C^1 function on $[0, \infty)$ with $A' > 0$ on $(0, \infty)$. Furthermore $A(0) = 0$.

(K1) $K \in W^{2,1}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ is a nonnegative function with $\|K\|_{C^2(\mathbb{R}^N)} < \infty$.

(K2) $\int_{\mathbb{R}^N} K(x)dx = 1$.

(KN) K is radial, $K(x) = k(|x|)$ and k is nonincreasing.

Since A and $A + c$ yield the same equation, the requirement that $A(0) = 0$ does not reduce generality. Note that $A(s) = s^m$ for $m \geq 1$ satisfies the above conditions. The requirement **(K2)** is nonessential and made only for convenience. The fact that the function k in condition **(KN)** is nonincreasing encodes the fact that the nonlocal term models attraction. The condition is needed when we consider the problem with no-flux boundary conditions, thus the symbol **(KN)**.

We are interested in existence of bounded, nonnegative weak solutions. By $\tilde{H}^{-1}(\Omega)$ we denote the dual of $H^1(\Omega)$.

Definition 2.1 (**Weak solution**)**.** Consider A which satisfies the assumption **(A1)** and K that satisfies the assumptions **(K1)**, **(K2)**, and **(KN)**. Assume $\rho_0 \in L^\infty(\Omega)$ is nonnegative. A function $\rho : \Omega_T \longrightarrow [0, \infty)$ is a weak solution of **(E1)** if $\rho \in L^\infty(\Omega_T)$, $A(\rho) \in L^2(0, T, H^1(\Omega))$, $\rho_t \in L^2(0, T, \tilde{H}^{-1}(\Omega))$ and for all test functions $\phi \in H^1(\Omega)$ for almost all $t \in [0, T]$

$$\langle \rho_t(t), \phi \rangle + \int_\Omega \nabla A(\rho(t)) \cdot \nabla \phi - \rho(t)(\nabla K * \rho(t)) \cdot \nabla \phi \, dx = 0. \qquad (2)$$

Here $\langle , \rangle$ denotes the dual pairing between $\tilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$. We furthermore require initial conditions to be satisfied in the $\tilde{H}^{-1}$ sense:

$$\rho(\,\cdot\,, t)) \to \rho_0 \quad \text{in } \tilde{H}^{-1}(\Omega) \text{ as } t \to 0.$$

Observe that $\rho \in H^1(0, T, \tilde{H}^{-1}(\Omega))$ implies that $\rho \in C(0, T, \tilde{H}^{-1}(\Omega))$. Below we show that in fact $\rho \in C(0, T, L^p(\Omega))$ for all $p \in [1, \infty)$, so that the initial conditions are taken in the L^p sense.

By density of piecewise constant functions in L^2 the condition (2) is equivalent to requiring

$$\int_0^T \langle \rho_t, \phi \rangle dt + \int_0^T \int_\Omega \nabla A(\rho) \cdot \nabla \phi - \rho \left(\nabla K * \rho \right) \cdot \nabla \phi \, dx \, dt = 0. \qquad (3)$$

for all $\phi \in L^2(0, T, H^1(\Omega))$.

Furthermore, it is a simple exercise to check that the above definition is equivalent to the following statement

Definition 2.2. Assume $\rho_0 \in L^\infty(\Omega)$ is nonnegative. A function $\rho : \Omega_T \longrightarrow [0, \infty)$ is a weak solution of **(E1)** if $\rho \in L^\infty(\Omega_T)$, $A(\rho) \in L^2(0, T, H^1(\Omega))$, and for all test functions $\phi \in C^\infty(\overline{\Omega}_T)$ such that $\phi(T) = \phi(0) = 0$

$$\int_0^T \int_\Omega -\rho\phi_t + \nabla A(\rho) \cdot \nabla\phi - \rho(\nabla K * \rho) \cdot \nabla\phi \, dxdt = 0. \tag{4}$$

Initial conditions are required in the $\tilde{H}^{-1}$ sense:

$$\rho(\,\cdot\,, t)) \to \rho_0 \quad \text{in } \tilde{H}^{-1}(\Omega) \text{ as } t \to 0.$$

An important property of weak solutions is that the total population is preserved in time.

Lemma 2.3. *Let u be a weak solution of **(E1)**. Then for all $t \in [0, T]$*

$$\int_\Omega \rho(x, t) = \int_\Omega \rho_0(x) dx.$$

To prove this lemma it suffices to take the test function $\phi \equiv 1$ and integrate in time.

2.1. **Uniqueness.** We now establish the uniqueness of weak solutions.

Theorem 2.4. *Let $\rho_0 \in L^\infty(\Omega)$ be nonnegative. There exists at most one weak solution to problem **(E1)**.*

Proof. Assume that there are two solutions to the problem: u and v. To prove uniqueness we use a version of the standard argument which is based on estimating the $\tilde{H}^{-1}$ norm of the difference $u(t) - v(t)$. Since $u, v \in C(0, T, \tilde{H}^{-1}(\Omega))$ we can define $\phi(t)$ to be the solution of

$$\begin{aligned} \Delta\phi(t) &= u(t) - v(t) & \text{in } \Omega \\ \nabla\phi(t) \cdot \nu &= 0 & \text{on } \partial\Omega \times (0, T) \end{aligned} \tag{5}$$

for which $\int_\Omega \phi(t) dx = 0$. Due to Lemma 2.3, $\int_\Omega u(t) - v(t) dx = 0$ for all $t \in [0, T]$ and thus the Neumann problem above has a solution. Note that $\phi(0) = 0$.

Due to regularity of $u - v$, the basic regularity theory yields: $\phi \in L^2(0, T, H^2(\Omega))$ and $\phi \in H^1(0, T, H^1(\Omega))$. Thus $\nabla\phi \in C(0, T, L^2(\Omega))$. Also ϕ_t solves (in the weak sense)

$$\begin{aligned} \Delta\phi_t &= u_t - v_t & \text{in } \Omega_T \\ \nabla\phi_t \cdot \nu &= 0 & \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{6}$$

Thus

$$\begin{aligned} -\int_0^\tau \langle u_t - v_t, \phi\rangle dt &= \int_0^\tau \int_\Omega \nabla\phi_t \cdot \nabla\phi \, dxdt = \frac{1}{2}\int_\Omega |\nabla\phi(\tau)|^2 - |\nabla\phi(0)|^2 dx \\ &= \frac{1}{2}\int_\Omega |\nabla\phi(\tau)|^2 dx \end{aligned} \tag{7}$$

Subtracting the weak formulations (3) satisfied by u and v we obtain for all $\tau \in [0, T]$:

$$\int_\Omega |\nabla\phi(\tau)|^2 dx = -\int_0^\tau \langle u_t - v_t, \phi\rangle dt$$
$$= \int_0^\tau \int_\Omega \nabla(A(u) - A(v)) \cdot \nabla\phi \, dxdt \qquad \Big\} I$$
$$- \int_0^\tau \int_\Omega ((\nabla K * u)u - (\nabla K * v)v) \cdot \nabla\phi \, dxdt \quad \Big\} II$$

From (5) follows, since $A(u) - A(v) \in L^2(0, T, H^1(\Omega))$ and A is increasing,

$$I = -\int_0^\tau \int_\Omega (A(u) - A(v))(u - v) \, dxdt \leq 0 \tag{8}$$

We now consider

$$II = -\int_0^\tau \int_\Omega (u - v)(\nabla K * u) \cdot \nabla\phi \, dxdt \qquad \Big\} III$$
$$+ \int_0^\tau \int_\Omega v(\nabla K * (u - v)) \cdot \nabla\phi \, dxdt \qquad \Big\} IV$$

Using (5) and the summation convention for repeated indices

$$III = \int_0^\tau \int_\Omega \partial_i\phi(\partial_{ij}^2 K * u)\partial_j\phi \, dxdt \qquad \Big\} V$$
$$+ \int_0^\tau \int_\Omega \partial_i\phi(\partial_j K * u)\partial_{ij}^2\phi \, dxdt \qquad \Big\} VI.$$

Integration by parts gives

$$VI = -\int_0^\tau \int_\Omega \partial_{ij}^2\phi(\partial_j K * u)\partial_i\phi \, dxdt - \int_0^\tau \int_\Omega \partial_i\phi(\partial_{jj}^2 K * u)\partial_i\phi \, dxdt$$
$$+ \int_0^\tau \int_{\partial\Omega} (\partial_i\phi)^2 \, \partial_j K * u \, \nu_j \, dSdt$$

where ν is the unit outward normal vector to Ω. To control the boundary term, we note $\nabla(K * u) \cdot \nu \leq 0$ on $\partial\Omega$ since Ω is convex, and K is radially decreasing. More precisely, this follows from the fact that for any $x \in \partial\Omega$ and any outward normal vector ν (that a vector orthogonal to a supporting hyperplane) and $s > 0$

$$K * u(x + s\nu) - K * u(x) = \int_\Omega (K(x + s\nu - y) - K(x - y))u(y)dy$$
$$= \int_\Omega (k(|x + s\nu - y|) - k(|x - y|))u(y)dy \leq 0 \tag{9}$$

since k is nonincreasing, $|x + s\nu - y| \geq |x - y|$, and u is nonnegative. This is the only step in the uniqueness argument that requires the condition **(KN)**. It follows that

$$\int_0^\tau \int_{\partial\Omega} (\partial_i\phi)^2 \, \partial_j K * u \, \nu_j \, dSdt \leq 0$$

and thus

$$VI \leq -\frac{1}{2}\int_0^\tau \int_\Omega (\Delta K * u)|\nabla\phi|^2 \, dxdt.$$

The expressions for V and VI imply

$$III \leq \sum_{i,j} \|\partial_{ij}^2 K\|_{L^\infty(\mathbb{R}^N)} \int_0^\tau \int_\Omega |\nabla\phi|^2 dxdt \tag{10}$$

Using the definition of solution of (5) in the inner-most integral gives

$$IV = \int_0^\tau \int_\Omega v(x)\nabla\phi(x) \cdot \int_\Omega \nabla K(x-y)(u(y,t)-v(y,t))dydxdt \tag{11}$$

$$= \int_0^\tau \int_\Omega v(\partial_{ij}^2 K * \partial_j\phi)\partial_i\phi \, dxdt$$

and thus

$$|IV| \leq \|v\|_{L^\infty(\Omega_T)} \int_0^\tau \int_\Omega \sum_{i,j} |(\partial_{ij}^2 K * \partial_j\phi)\partial_i\phi| \, dxdt \tag{12}$$

$$\leq \|\partial_{ij}^2 K * \partial_j\phi\|_{L^2(\Omega_T)}^{1/2} \|\partial_i\phi\|_{L^2(\Omega_T)}^{1/2} \leq C\|\nabla\phi\|_{L^2(\Omega_T)}$$

The last inequality is a consequence of Young's inequality for convolutions (see for example [18]). The constant C can be taken independent of Ω. Let $\eta(t) := \int_\Omega |\nabla\phi(t)|^2 dt$. Combining (7), (8), (10), and (12) gives that

$$\eta(\tau) \leq C \int_0^\tau \eta(s)ds.$$

Since $\eta(0) = 0$ from Gronwall's inequality follows that $\eta(t) = 0$ for all $T \geq t \geq 0$. Therefore $u \equiv v$. $\square$

2.2. **Existence.** To establish the existence of solutions, we carry out two approximating procedures. While this is not entirely necessary, it separates handling the nonlocality and the degeneracy of the equation. It is thus transparent which tools are necessary to handle each.

One approximation is to perturb the equation to make it uniformly parabolic: Let

$$a := A'.$$

For $\varepsilon > 0$ let $a_\varepsilon(z)$ be smooth and even, and such that

$$a(z) + \varepsilon \leq a_\varepsilon(z) \leq a(z) + 2\varepsilon \quad \text{for } z \geq 0. \tag{13}$$

Let

$$A_\varepsilon(z) := \int_0^z a_\varepsilon(s)ds.$$

Consider

$$\partial_t \rho_\varepsilon - \Delta A_\varepsilon(\rho_\varepsilon) + \nabla \cdot [(\rho_\varepsilon \nabla K * \rho_\varepsilon)] = 0 \quad \text{on } \Omega_T \tag{E2}$$

with no-flux boundary conditions and initial conditions as in (E1). The notion of weak solution for (E2) is analogous to the one for (E1).

To show the existence of solutions of the nonlocal equation (E2) we utilize the following local equation: For $\tilde{a} \in C^\infty(\mathbb{R}, [0,\infty))$ let $\tilde{A}(s) := \int_0^s \tilde{a}(z)dz$. We assume

(A2) There exists $\lambda > 0$ such that $\tilde{a}(z) > \lambda$ for all $z \in \mathbb{R}$.

Let V be a smooth vector field on Ω_T with bounded divergence. Consider the equation

$$\partial_t u - \Delta \tilde{A}(u) + \nabla \cdot (uV) = 0 \quad \text{on } \Omega_T \tag{E3}$$

with no-flux boundary condition

$$(-\nabla \tilde{A}(u) + uV) \cdot \nu = 0 \quad \text{on } \partial\Omega \times [0, T].$$

The initial data are taken in the $\tilde{H}^{-1}$ sense.

For $\tilde{a}$ satisfying the condition **(A2)** the equation **(E3)** is a uniformly parabolic quasi-linear equation with smooth coefficients. Thus, by standard theory [22, 29], there exist a unique classical short time solution to the equation on Ω_{T_0} for some $T_0 > 0$.

Lemma 2.5 (L^∞ **bound**). *Consider $\tilde{A}$ such that $\tilde{a} = \tilde{A}'$ satisfies the condition (A2). Assume $u \in C^2(\Omega_T)$ is a solution of (E3) with smooth, nonnegative bounded initial data u_0. Assume further that*

$$V \cdot \nu \leq 0 \qquad on \ \partial\Omega \times (0, T) \tag{14}$$

Then u is nonnegative and for all $t \in [0, T]$

$$\|u(\,\cdot\,, t)\|_{L^\infty(\Omega)} \leq e^{\mu t} \|u_0\|_{L^\infty(\Omega)}$$

where $\mu = \|(\nabla \cdot V)_-\|_{L^\infty(\Omega_T)}$.

Above z_- is the negative part of $z \in \mathbb{R}$, that is $z_- = \max\{-z, 0\}$.

Proof. The claim of the lemma follows directly from the comparison principle. Consider $v(x, t) := e^{\mu t} \|u(\,\cdot\,, 0)\|_{L^\infty(\Omega)}$. It is a supersolution in the interior of Ω_T. On the lateral boundary, $\partial\Omega \times (0, T)$, we use $V \cdot \nu \leq 0$ to establish that $(-\nabla A(v) + vV) \cdot \nu \leq 0$. Thus v is a supersolution to the problem. To show that u is nonnegative, note that $w(x, t) \equiv 0$ is a subsolution. $\square$

The condition (14) is satisfied for equations of our interest, **(E1)** and **(E2)**, with $V = \nabla K * \rho$ on convex domains. If the condition (14) does not hold the construction of a supersolution is still possible for a number of nonlinearities A, but is more intricate. In particular the supersolutions need to be x-dependent.

The L^∞ bounds above ensure that, when **(A2)** holds, the equation **(E3)** is uniformly parabolic, with smooth and bounded coefficients. By classical theory [22, 29] it then has smooth solutions for all $t > 0$.

The next four lemmas contain the compactness and continuity results we need. The result of Lemma 2.6 is well known, in particular it is analogous to one obtained by Alt and Luckhaus [1], who studied a family of equations that includes **(E3)**. Thus we omit the proof.

Lemma 2.6. *Let F be a convex C^1 function and $f = F'$. Assume*

$$f(u) \in L^2(0, T, H^1(\Omega)), \quad u \in H^1(0, T, \tilde{H}^{-1}(\Omega)), \quad and \ F(u) \in L^\infty(0, T, L^1(\Omega)).$$

Then for almost all $0 \leq s, \tau \leq T$

$$\int_\Omega F(u(x, \tau)) - F(u(x, s))dx = \int_s^\tau \langle u_t, f(u(t)) \rangle dt. \tag{15}$$

Lemma 2.7. *Assume that A satisfies the condition (A1). Let $M > 0$. Let U be a bounded measurable set. There exists $\omega_s : [0, \infty) \to [0, \infty)$ nondecreasing, with $\lim_{z \to 0} \omega_s(z) = 0$ such that for all nonnegative functions $f_1, f_2 \in L^\infty(U)$ for which $\|f_1\|_{L^\infty} \leq M$ and $\|f_2\|_{L^\infty} \leq M$*

$$\|f_2 - f_1\|_{L^1(U)} \leq \omega_s(\|A(f_2) - A(f_1)\|_{L^1(U)}).$$

We use this lemma with either $U = \Omega$ or $U = \Omega \times [0, T]$.

Proof. Let for $x \geq 0$ and $y \geq 0$

$$\sigma(x,y) := \begin{cases} \frac{A(y)-A(x)}{y-x} & \text{if } x \neq y \\ A'(x) & \text{if } x = y. \end{cases}$$

Note that σ is continuous. Consider $\delta > 0$. Let $C(\delta) = \min_{[\delta,M] \times [0,M]} \sigma(x,y)$. Since $A'(x) > 0$ for all $x > 0$, $C(\delta) > 0$ for all $\delta > 0$. Given f_1 and f_2 in $L^\infty(U)$, let $U_1 := \{x \in U \,:\, f_1(x) < \delta \text{ and } f_2(x) < \delta\}$ and let $U_2 := U \backslash U_1$. Then

$$\int_U |f_2(x) - f_1(x)|dx = \int_{U_1} |f_2(x) - f_1(x)|dx + \int_{U_2} |f_2(x) - f_1(x)|dx$$

$$\leq \delta|U| + \frac{1}{C(\delta)} \int_U |A(f_2(x)) - A(f_1(x))|dx.$$

Defining $\omega_s(z) := \inf_{\delta > 0} \left\{ \frac{1}{C(\delta)} z + |U|\delta \right\}$ completes the proof. $\square$

The following lemma is used in conjunction with the estimates of Lemma 2.11 to prove L^1 precompactness in time of approximate solutions to **(E1)**. It represents a version of Lemma 1.8 by Alt and Luckhaus [1].

Lemma 2.8. *Assume that A satisfies the condition **(A1)**. Let $M > 0$ and $\delta > 0$. Let $\mathcal{F}$ be a family of nonnegative $L^\infty(\Omega)$ functions such that for all $f \in \mathcal{F}$*

$$\|A(f)\|_{H^1(\Omega)} \leq M, \quad and \quad \|f\|_{L^\infty(\Omega)} \leq M. \tag{16}$$

There exists a nondecreasing function $\omega_M : [0,\infty) \to [0,\infty)$ satisfying $\omega_M(\delta) \to 0$ as $\delta \to 0$, such that if for $f_1, f_2 \in \mathcal{F}$

$$\int_\Omega (A(f_2) - A(f_1))(f_2 - f_1)dx \leq \delta$$

then

$$\|f_1 - f_2\|_{L^1(\Omega)} \leq \omega_M(\delta).$$

Proof. Assume that the claim does not hold. Then there exists $\kappa > 0$ and sequences $f_{1,n}$ and $f_{2,n}$ in $\mathcal{F}$ such that

$$\int_\Omega (A(f_{2,n}) - A(f_{1,n}))(f_{2,n} - f_{1,n})dx \leq \frac{1}{n} \quad \text{and} \quad \int_\Omega |f_{2,n} - f_{1,n}|dx \geq \kappa.$$

The bounds in (16) imply that there exist $f_1, f_2 \in L^2(\Omega)$, and a subsequence of $(A(f_{1,n}), A(f_{2,n}))$, which we can assume to be the whole sequence, such that

$$A(f_{1,n}) \to A(f_1) \text{ and } A(f_{2,n}) \to A(f_2) \text{ in } L^2(\Omega) \text{ as } n \to \infty$$

and furthermore

$$f_{1,n} \rightharpoonup f_1 \text{ and } f_{2,n} \rightharpoonup f_2 \text{ in } L^2 \text{ as } n \to \infty.$$

Therefore

$$\int_\Omega (A(f_2) - A(f_1))(f_2 - f_1)dx = 0.$$

Thus $f_1 = f_2$ a.e. Consequently $\|A(f_{2,n}) - A(f_{1,n})\|_{L^1} \to 0$ as $n \to \infty$. Lemma 2.7 implies that $\|f_{2,n} - f_{1,n}\|_{L^1} \to 0$ as $n \to \infty$. This contradicts the assumption we made when constructing the sequences. $\square$

The following lemma is needed for proving the continuity in time (in L^p topology) of solutions. It is a special case of results of Visintin [38] and Brezis [10]. Since in this special case there exists a simple proof, we present it.

Lemma 2.9. *Let $F \in C^2([0,\infty),[0,\infty))$ be convex with $F(0) = 0$ and $F'' > 0$ on $(0,\infty)$. Let f_n, for $n = 1, 2, \ldots$, and f be nonnegative functions on Ω bounded from above by $M > 0$. Furthermore assume*

$$f_n \rightharpoonup f \quad in \ L^1(\Omega) \quad and \quad \|F(f_n)\|_{L^1(\Omega)} \to \|F(f)\|_{L^1(\Omega)}$$

as $n \to \infty$. Then

$$f_n \to f \quad in \ L^2(\Omega) \ as \ n \to \infty.$$

Proof. Since $F'' > 0$ on $(0,\infty)$ for each $\delta > 0$ there exists $\theta > 0$ such that for all $y \in [\delta, M]$ and all $h \in [0, y]$

$$F(y + h) + F(y - h) > 2F(y) + \theta h^2 \tag{17}$$

Let $\varepsilon > 0$. For $\delta \geq 0$ let $\Omega^\delta := \{x \in \Omega : \ f(x) > \delta\}$. Let us consider $\|f_n - f\|_{L^2(\Omega)}$:

$$\int_\Omega |f - f_n|^2 dx = \int_{\{f=0\}} f_n^2 dx + \int_{\Omega^0 \setminus \Omega^\delta} |f - f_n|^2 dx + \int_{\Omega^\delta} |f - f_n|^2 dx.$$

The first term

$$\int_{\{f=0\}} f_n^2 dx \leq M \int_{\{f=0\}} f_n dx < \frac{\varepsilon}{3}$$

for n large enough, by the weak $L^1(\Omega)$ convergence. For $\delta > 0$, small enough, $|\Omega^0 \setminus \Omega^\delta| < \frac{\varepsilon}{3M^2}$ and thus

$$\int_{\Omega^0 \setminus \Omega^\delta} |f - f_n|^2 dx \leq M^2 \frac{\varepsilon}{3M^2} = \frac{\varepsilon}{3}$$

for all n. Regarding the third term: Using (17) when integrating on Ω^δ and the fact that F is convex when integration on $\Omega \setminus \Omega^\delta$ one obtains

$$2 \int_\Omega F\left(\frac{f_n + f}{2}\right) dx \leq \int_\Omega F(f_n) + F(f) dx - \frac{\theta}{4} \int_{\Omega^\delta} |f - f_n|^2 dx.$$

Since F is convex the functional $w \mapsto \int F(w) dx$ is weakly lower-semicontinuous with respect to L^1 topology. Using the assumption of the lemma and taking $\liminf_{n\to\infty}$ gives

$$2 \int_\Omega F(f) dx \leq 2 \int_\Omega F(f) - \frac{\theta}{4} \limsup_{n\to\infty} \int_{\Omega^\delta} |f - f_n|^2 dx.$$

Therefore

$$\int_{\Omega^\delta} |f - f_n|^2 dx < \frac{\varepsilon}{3}$$

for all n large enough. Combining the bounds establishes the L^2 convergence. $\square$

The following is the standard gradient bound; we state it for weak solutions.

Lemma 2.10 (gradient bound). *Let $u \in L^\infty(\Omega_T)$ be a weak solution of **(E3)**. There exists a constant C depending only on T, $\| |V| \|_{L^\infty(\Omega_T)}$, $\|u\|_{L^\infty(\Omega_T)}$, $\|u_0\|_{L^1}$, and $\tilde{A}(\|u\|_{L^\infty(\Omega_T)})$ such that*

$$\|\nabla \tilde{A}(u)\|_{L^2(0,T,L^2(\Omega))} < C.$$

Proof. Let us use $\tilde{A}(u)$ as the test function in the formulation of a weak solution (3).

$$\int_0^T \langle u_t, \tilde{A}(u) \rangle dt = - \int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt + \int_{\Omega_T} uV \cdot \nabla \tilde{A}(u) dx dt.$$

Note that

$$\left| \iint_{\Omega_T} uV \cdot \nabla \tilde{A}(u) dx dt \right| \leq \int_{\Omega_T} |u|^2 |V|^2 dx dt + \frac{1}{4} \int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt.$$

Let $F(z) := \int_0^z \tilde{A}(s) ds$. Using Lemma 2.6 and $F(z) \leq \tilde{A}(z)z$ we obtain

$$\frac{3}{4} \int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt \leq \liminf_{t \to 0} \int_{\Omega} F(u(t)) dx + \int_{\Omega_T} |u|^2 |V|^2 dx dt$$

$$\leq \tilde{A}(\|u\|_{L^\infty(\Omega)}) \|u_0\|_{L^1(\Omega)} + T \|V\|_{L^\infty(\Omega_T)}^2 \|u\|_{L^\infty(\Omega_T)} \|u_0\|_{L^1(\Omega)}$$

which implies the desired bound. $\square$

The following lemma is a version of a claim proven in Subsection 1.7 of Alt and Luckhaus [1].

Lemma 2.11. *Let V be an L^∞ vector field on Ω_T. Assume $\tilde{A}$ satisfies conditions (A1)-(A2). Let u be a weak solution of (E3) with no-flux boundary conditions and initial data in $L^\infty(\Omega)$. There exists a constant C depending only on T, $\|u\|_{L^\infty(\Omega_T)}$, $\|A(u)\|_{L^\infty(\Omega_T)}$, and $\|A(u)\|_{L^2(0,T,H^1(\Omega))}$ such that*

$$\int_0^{T-h} \int_\Omega \left(u(x, t+h) - u(x,t) \right) \left(A(u(x, t+h)) - A(u(x,t)) \right) dx dt \leq Ch$$

for all $h \in [0, T]$.

Proof. Consider the test function

$$\phi_1(t) := \frac{1}{h} \int_t^{t+h} A(u(s)) ds$$

on the time interval $[0, T-h]$ and the test function

$$\phi_2(t) := \phi_1(t - h)$$

on the time interval $[h, T]$. Subtracting the equalities resulting from the definition of the weak solution yields the desired bound, via straightforward calculations. $\square$

Theorem 2.12 (**Existence for (E2)**). *Consider A which satisfies the assumption (A1) and K that satisfies the assumptions (K1), (K2), and (KN). Let $\varepsilon > 0$ and ρ_0 a nonnegative smooth function on $\overline{\Omega}$. The equation (E2) has a weak solution ρ on Ω_T. Furthermore ρ is smooth on $\overline{\Omega} \times (0, T]$.*

Proof. Let $\tilde{a} := a_\varepsilon$. We employ the following iteration scheme: Let $u^1(x,t) := \rho_0(x)$ for all $(x,t) \in \Omega_T$. For $k \geq 1$ let u^{k+1} be the solution of

$$u_t^{k+1} - \nabla \cdot (\tilde{a}(u^{k+1}) \nabla u^{k+1}) + \nabla \cdot (u^{k+1} \nabla(K * u^k)) = 0 \qquad (18)$$

with initial data $u^{k+1}(\cdot, 0) = \rho_0$ and no-flux boundary conditions.

Since the equations preserve the nonnegativity and the "mass" of the solutions we have $\|u_k(\cdot, t)\|_{L^1} = \int_\Omega u_k(x,t) dx = \int_\Omega \rho_0(x) dx$. By the L^∞ estimates of Lemma 2.5, $\|u_k\|_{L^\infty(\Omega_T)} \leq e^{M_k T} \|\rho_0\|_{L^\infty}$, where $M_k = \|(\Delta K) * u_{k-1}(\cdot, t)\|_{L^\infty(\Omega_T)}$. Thus

$$M_k \leq \sup_{t \in [0,T]} \|\Delta K\|_{L^\infty} \|u_{k-1}(\cdot, t)\|_{L^1} = \|\Delta K\|_{L^\infty} \|\rho_0\|_{L^1}. \qquad (19)$$

Hence M_k have an upper bound independent of k. Consequently, Lemma 2.10 produces a bound on $\|\tilde{a}(u_k) \nabla u_k\|_{L^2(0,T,L^2(\Omega))}$ which is independent of k. Since $\tilde{a} \geq \varepsilon > 0$ this implies bounds on $\|\nabla u_k\|_{L^2(0,T,L^2(\Omega))}$. The $L^2(0, T, L^2(\Omega))$ bound on u_k and $L^2(0, T, L^\infty(\Omega))$ bound on $\nabla K * u_{k-1}$ that follows via Young's inequality, imply

a bound on $u_k \nabla K * u_{k-1}$ in $L^2(0, T, L^2(\Omega))$ independent of k. Weak formulation of the equation then yields that u_t^k is a bounded sequence in $L^2(0, T, \tilde{H}^{-1}(\Omega))$.

Repeated application of the Lions-Aubin Lemma (see [33][pg. 106], for example) yields that there exists a subsequence of u_k, which for convenience we assume to be the whole sequence, and a function $\rho \in L^2(0, T, L^2(\Omega))$ such that

$$u_k \to \rho \quad \text{and} \quad \tilde{A}(u_k) \to \tilde{A}(\rho) \text{ in } L^2(0, T, L^2(\Omega)). \tag{20}$$

The L^∞ bound of Lemma 2.5, implies a bound on L^∞ norm of ρ. The gradient bound of Lemma 2.10 now implies that, along a subsequence, which we again assume to be the whole sequence,

$$\nabla \tilde{A}(u_k) \rightharpoonup \nabla \tilde{A}(\rho) \qquad \text{in } L^2(0, T, L^2(\Omega)).$$

By Cauchy-Schwarz inequality

$$\|\nabla K * u_{k-1} - \nabla K * \rho\|_{L^2(0,T,L^\infty(\Omega))} \leq \|\nabla K\|_{L^2(\mathbb{R}^N)} \|u_{k-1} - \rho\|_{L^2(\Omega_T)}. \tag{21}$$

It follows that

$$u_k \nabla K * u_{k-1} \to \rho \nabla K * \rho \quad \text{in } L^2(0, T, L^2(\Omega)).$$

Therefore

$$\int_0^T \int_\Omega \rho \phi_t - \nabla A(\rho) \cdot \nabla \phi + \rho(\nabla K * \rho) \cdot \nabla \phi \, dx dt = 0.$$

By the estimates above u_k are bounded in the space $H^1(0, T, \tilde{H}^{-1}(\Omega))$. Since $H^1(0, T, \tilde{H}^{-1}(\Omega))$ continuously embeds in $C^{1/2}(0, T, \tilde{H}^{-1}(\Omega))$ and thus compactly in $C(0, T, \tilde{H}^{-1}(\Omega))$ there exists a subsequence which converges in $C(0, T, \tilde{H}^{-1}(\Omega))$. We assume for notational simplicity that the subsequence is the whole sequence. Thus

$$u_k \to \rho \text{ in } C(0, T, \tilde{H}^{-1}(\Omega)) \text{ as } k \to \infty.$$

Therefore $\rho(t) \to \rho_0$ in $\tilde{H}^{-1}(\Omega)$ as $t \to 0$.

Smoothness of solution can now be shown using the standard theory (using test functions that approximate $\Delta A(\rho)$ and ρ_t to show improved regularity, differentiating the equation and iterating the procedure). $\qquad \square$

Theorem 2.13 (**Existence for (E1)**). *Consider A which satisfies the assumption (A1) and K that satisfies the assumptions (K1), (K2), and (KN). Let ρ_0 be a nonnegative function in $L^\infty(\Omega)$. The problem (E1) has a weak solution on Ω_T. Furthermore $\rho \in C(0, T, L^p(\Omega))$ for all $p \in [1, \infty)$.*

Proof. Let a_ε and $A_\varepsilon(z)$ be as in (13). Let ρ_0^ε be smooth approximations of ρ_0 such that $\|\rho_0^\varepsilon\|_{L^1} = \|\rho_0\|_{L^1}$, $\|\rho_0^\varepsilon\|_{L^\infty} \leq 2\|\rho_0\|_{L^\infty}$, and $\rho_0^\varepsilon \to \rho_0$ in L^p as $\varepsilon \to 0$, for all $p \in [1, \infty)$

By Theorem 2.12 there exists a nonnegative solution ρ_ε of (E2) with initial datum ρ_0^ε. The proof of the theorem provides uniform-in-ε bounds on

$$A_\varepsilon(\rho_\varepsilon) \text{ in } L^2(0, T, H^1(\Omega)), \quad \rho_\varepsilon \text{ in } L^\infty(\Omega_T) \text{ and } \partial_t \rho_\varepsilon \text{ in } L^2(0, T, \tilde{H}^{-1}(\Omega)).$$

Since $A_\varepsilon \geq A$ and $a_\varepsilon \geq a$ on $[0, \infty)$ uniform bounds on $L^2(0, T, H^1(\Omega))$ norm of $A(\rho_\varepsilon)$ hold. Therefore there exists $w \in L^2(0, T, H^1(\Omega))$ and a sequence ε_j converging to 0 such that

$$A(\rho_{\varepsilon_j}) \rightharpoonup w \qquad \text{(weakly) in } L^2(0, T, H^1(\Omega)). \tag{22}$$

Note that ρ_ε is a weak solution of **(E3)** with $V = \nabla K * \rho_\varepsilon$. Using the uniform-in-$\varepsilon$ bounds above and that $|A(z_1) - A(z_2)| \le |A_\varepsilon(z_1) - A_\varepsilon(z_2)|$ for all $\varepsilon > 0$ and $z_1, z_2 \ge 0$, by Lemma 2.11 there exists $C > 0$, independent of ε, such that

$$\int_0^{T-h} \int_\Omega (\rho_\varepsilon(x, t+h) - \rho_\varepsilon(x, t))\, (A(\rho_\varepsilon(x, t+h)) - A(\rho_\varepsilon(x, t)))dxdt \le Ch \quad (23)$$

for all $h \in [0, T]$. To show that the family $\{\rho_\varepsilon\}$ is precompact in $L^1(\Omega_T)$ it is enough to show that it satisfies the assumptions of the Riesz-Frechet-Kolmogorov compactness criterion [9][IV.26]. In particular, it suffices to show:

Claim 1° For all $\theta > 0$ there exists $0 < h_0 \le \theta$ such that for all $\varepsilon > 0$ and all $0 < h \le h_0$

$$\int_0^{T-\theta} \int_\Omega |\rho_\varepsilon(x, t+h) - \rho_\varepsilon(x, t)|dxdt \le \theta.$$

Claim 2° For all $\theta \in (0, T)$ there exists $0 < h_0 \le \theta$ such that for all $\varepsilon > 0$ and all $0 < h \le h_0$ and all $i = 1, \dots, N$

$$\int_0^T \int_{\Omega^\theta} |\rho_\varepsilon(x + he_i, t) - \rho_\varepsilon(x, t)|dxdt \le \theta$$

where $\Omega^\theta = \{x \in \Omega \,:\, d(x, \partial\Omega) > \theta\}$.

To prove the first claim, we recall that by the L^∞ bound of Lemma 2.5 and the L^2 gradient bound there exists $M > \|\rho_0\|_{L^1(\Omega)}$ such that for all $\varepsilon \in (0, 1)$

$$\|\rho_\varepsilon\|_{L^\infty(\Omega_T)} \le M \text{ and } \|A(\rho_\varepsilon)\|_{L^2(0,T,H^1(\Omega))} \le M.$$

Consider for $0 < h < \theta$ and $\gamma > 1$ the set of times for which "good" estimates hold:

$$E_\gamma(h) = \Big\{ t \in [0, T-\theta] \,:\, \|A(\rho_\varepsilon(t))\|_{H^1(\Omega)} \le M\sqrt{\gamma}, \|A(\rho_\varepsilon(t+h))\|_{H^1(\Omega)} \le M\sqrt{\gamma},$$

$$\text{and } \int_\Omega (\rho_\varepsilon(x, t+h) - \rho_\varepsilon(x, t))(A(\rho_\varepsilon(x, t+h)) - A(\rho_\varepsilon(x, t)))dx < C\gamma h \Big\}.$$

Let $E_\gamma^c(h) = [0, T-\theta] \backslash E_\gamma(h)$. Note that $|E_\gamma^c(h)| \le \frac{3}{\gamma}$, since each condition cannot be violated on a set of measure larger than $1/\gamma$. Let $\omega_{M\sqrt{\gamma}}$ be as in Lemma 2.8. Then

$$\int_0^{T-\theta} \int_\Omega |\rho_\varepsilon(x, t+h) - \rho_\varepsilon(x, t)|dxdt \le T\omega_{M\sqrt{\gamma}}(C\gamma h) + 2M\frac{3}{\gamma}.$$

Set $\gamma = \max\{\frac{12M}{\theta}, 1\}$. Taking $h_0 > 0$ such that $T\omega_{M\sqrt{\gamma}}(C\gamma h_0) < \frac{\theta}{2}$ completes the proof.

To show Claim 2° note that for $0 < h < \theta$

$$\int_0^T \int_{\Omega^\theta} |A(\rho_\varepsilon(x + he_i, t)) - A(\rho_\varepsilon(x, t))|dxdt$$

$$\le h \int_0^T \int_0^1 \int_{\Omega^\theta} |\nabla(A(\rho_\varepsilon)(x + she_i, t)|dxdsdt$$

$$\le h\sqrt{|\Omega|T} \|A(\rho_\varepsilon)\|_{L^2(0,T,H^1(\Omega))}.$$

Lemma 2.7, applied to $U = \Omega^\theta \times [0, T]$, implies that

$$\int_0^T \int_{\Omega^\theta} |\rho_\varepsilon(x + he_i, t) - \rho_\varepsilon(x, t)|dx \le \omega_s(h\sqrt{|\Omega|T}M).$$

The claim follows by taking h_0 small enough.

In conclusion, along a subsequence, which we still denote by ρ_{ε_j},

$$\rho_{\varepsilon_j} \to \rho \qquad \text{in } L^1(\Omega_T) \tag{24}$$

for some $\rho \in L^1(\Omega_T)$. Therefore $w = A(\rho)$. Furthermore

$$\|\nabla K * (\rho_{\varepsilon_j} - \rho)\|_{L^1(0,T,L^\infty(\Omega))} \le \|\nabla K\|_{L^\infty(\mathbb{R}^N)} \|\rho_{\varepsilon_j} - \rho\|_{L^1(\Omega_T)}.$$

Combining this claim with (24) gives

$$\rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \to \rho(\nabla K * \rho) \qquad \text{in } L^1(\Omega_T). \tag{25}$$

Since $\rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j})$ are uniformly bounded in $L^\infty(\Omega_T)$, by interpolating we have

$$\rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \to \rho(\nabla K * \rho) \quad \text{in } L^2(\Omega_T).$$

Therefore we can take the limit as $j \to \infty$ in the weak formulation of the equation **(E2)**: For $\phi \in C_0^\infty(\Omega \times (0,T))$

$$\int_0^T \int_\Omega \rho_{\varepsilon_j} \phi_t - \nabla A(\rho_{\varepsilon_j}) \cdot \nabla\phi + \rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \cdot \nabla\phi \, dxdt = 0.$$

to obtain that (3) holds. Note also that uniform L^∞ bound on ρ_ε and the L^1 convergence of ρ_{ε_j} yield that $\rho \in L^\infty(\Omega_T)$. The proof that $\rho \in C(0,T,\tilde{H}^{-1}(\Omega))$ and $\rho(t) \to \rho_0$ in $\tilde{H}^{-1}$ as $t \to 0$ is the same as before. It follows that $\rho(t) : [0,T] \to L^2(\Omega)$ is continuous with respect to weak $L^2(\Omega)$ topology. In particular, it suffices to establish that $\int_\Omega \rho(x,s)\psi(x)dx \to \int_\Omega \rho(x,t)\psi(x)dx$ as $s \to t$ for all $\psi \in L^2(\Omega)$. By a density argument it is enough to consider smooth ψ. Finally for smooth ψ the claim holds since $\rho \in C(0,T,\tilde{H}^{-1}(\Omega))$.

Since Ω is bounded, $\rho(t)$ is also continuous with respect to weak L^1 topology. Let $F(z) := \int_0^z A(s)ds$. Lemma 2.6 and Lemma 2.10 then imply that $t \mapsto \int_\Omega F(\rho(t))$ is continuous. Lemma 2.9 then implies that $\rho(t)$ is continuous with respect to $L^2(\Omega)$ topology. Using the boundedness of domain, and interpolating with L^∞ bound on ρ implies that $\rho \in C(0,T,L^p(\Omega))$ for all $p \in [1,\infty)$. $\qquad\square$

3. **Periodic solutions.** In this sections we consider the periodic solutions to the equation (1). Such solutions are useful in studies of the coarsening phenomena [34]. Let Q be the period cell. We consider the problem

$$\begin{aligned} \rho_t - \Delta A(\rho) + \nabla \cdot [(\rho\nabla K * \rho)] = 0 \qquad &\text{in } Q_T \\ \rho(0) = \rho_0 \qquad &\text{in } Q. \end{aligned} \qquad (\mathbf{E}_{per})$$

where both ρ_0 and ρ are periodic in space with period cell Q. Above $\nabla K * \rho(x) = \int_{\mathbb{R}^N} \nabla K(x-y)\rho(y)dy$.

Establishing the existence and uniqueness is similar to the case of the bounded domain treated in Section 2. However there are a few differences which we highlight below. In particular the condition **(KN)** is no longer needed. However, since $\rho(t)$ is no longer in $L^1(\mathbb{R}^N)$, a decay condition on K is needed. Thus we assume that K satisfies

(KP) Let $f(r) = \sup\{(\Delta K)_-(x) : |x| \ge r\}$. Assume that $\int_0^\infty f(r)r^{N-1}dr < \infty$.

In other words we assume that $f(|x|) \in L^1(\mathbb{R}^N)$. Note that this condition is only slightly stronger than $(\Delta K)_- \in L^1(\mathbb{R}^N)$, which is already assumed as part of the condition **(K1)**.

The definition of weak solutions is the same as before, only that Ω is replaced by Q and ρ_0 and ρ are periodic in space. The proof of uniqueness is slightly simpler than before, since there are no boundary terms. For this reason the condition **(KN)**

is no longer needed. The statement of Lemma 2.5 now holds without the assumption (14).

The first instance the issue that $\rho(t) \notin L^1(\mathbb{R}^N)$ is encountered is in the proof of the statement of Theorem 2.12 for periodic solutions. Namely the estimate (19) is no longer usable. To obtain a uniform estimate on $\|(\Delta K * u_{k-1})_-\|_{L^\infty(Q_T)}$ we proceed as follows. For arbitrary $t \in [0, T]$ let $w = u_{k-1}(\cdot, t)$. Consider $x \in Q$. We can assume, without the loss of generality, that $Q = [-l, l]^N$. Since when constructing solutions to Cauchy problem we need to take $l \to \infty$ we carefully consider how the terms behave for l large. To avoid irrelevant technicalities for l small, we assume $l \geq 1$.

$$
\begin{aligned}
|(\Delta K * w)_-(x)| \leq (\Delta K)_- * w(x) &= \int_{\mathbb{R}^N} (\Delta K)_-(y)w(x-y)dy \\
&= \sum_{a \in 2l\mathbb{Z}^N} \int_Q (\Delta K)_-(y+a)w(x-y-a)dy \\
&\leq \sum_{a \in 2l\mathbb{Z}^N} \|(\Delta K)_-\|_{L^\infty(Q+a)} \|w\|_{L^1(Q)} \\
&\leq \|w\|_{L^1(Q)} \sum_{a \in 2l\mathbb{Z}^N} f(|a| - \sqrt{N}l) \\
&\leq \|w\|_{L^1(Q)} \sum_{i=0}^\infty f((i-1)\sqrt{N}\,l)\,\sharp\{a \in 2l\mathbb{Z}^N \,:\, \sqrt{N}\,l\,i \leq a \leq \sqrt{N}\,l\,(i+1)\} \\
&\leq \|w\|_{L^1(Q)} \left(c_1(N)f(0) + \sum_{i=1}^\infty f((i-1)\sqrt{N}\,l)c_2(N)i^{N-1} \right) \\
&\leq \|w\|_{L^1(Q)} C(N) \left(f(0) + \int_0^\infty f(r)r^{N-1}dr \right).
\end{aligned}
$$

To estimate the number of "integer" points between two spherical shells we used the fact that all the cubes, $Q + a$ centered at the "integer" points, a, are contained in $B(0, \sqrt{N}\,l\,(i+2)) \backslash B(0, \sqrt{N}\,l\,(i-2))$. Thus their number can be estimated from above by the volume of the annulus.

This establishes the uniform bounds on $\|(\Delta K * u_{k-1})_-\|_{L^\infty(Q_T)}$ that replace the ones from (19).

The next issue is that the estimate of $\nabla K * (u_{k-1} - \rho)$ in (21) is no longer usable. Instead we note that, from the proof of the Theorem 2.12 follows that for any m, a positive, integer multiple of l, $u_{k-1} \to \rho$ in $L^1([-m, m]^N \times [0, T])$ as $k \to \infty$. For $t \in [0, T]$, we estimate for $x \in \mathbb{R}^N$

$$
\begin{aligned}
|\nabla K * (u_{k-1} - \rho)(x, t)| &= \left| \int_{\mathbb{R}^N} \nabla K(y)(u_{k-1}(x-y, t) - \rho(x-y, t))dy \right| \\
&\leq \left| \int_{[-m, m]^N} \nabla K(y)(u_{k-1}(x-y, t) - \rho(x-y, t))dy \right| \\
&\quad + \left| \int_{\mathbb{R}^N \backslash [-m, m]^N} \nabla K(y)(u_{k-1}(x-y, t) - \rho(x-y, t))dy \right| \\
&\leq \|\nabla K\|_{L^\infty(\mathbb{R}^N)} \|u_{k-1} - \rho\|_{L^1([-m, m]^N)} \\
&\quad + \|\nabla K\|_{L^1(\mathbb{R}^N \backslash [-m, m]^N)}(\|u_{k-1}\|_{L^\infty(\mathbb{R}^N)} + \|\rho\|_{L^\infty(\mathbb{R}^N)}).
\end{aligned}
$$

Therefore

$$\|\nabla K * (u_{k-1} - \rho)\|_{L^1(0,T,L^\infty(\mathbb{R}^N))} \leq \|\nabla K\|_{L^\infty(\mathbb{R}^N)} \|u_{k-1} - \rho\|_{L^1([-m,m]^N \times [0,T])}$$
$$+ T(\|\nabla K\|_{L^1(\mathbb{R}^N \setminus [-m,m]^N)} (\|u_{k-1}\|_{L^\infty(Q \times [0,T])} + \|\rho\|_{L^\infty(Q \times [0,T])})).$$

Since $\nabla K \in L^1(\mathbb{R}^N)$ by assumption **(K1)**, and given the L^∞ bounds on u_{k-1} and ρ, the second term can be made arbitrarily small by selecting m large enough. The first term can then be made arbitrarily small by considering k large enough. In conclusion

$$\nabla K * u_{k-1} \to \nabla K * \rho \quad \text{in } L^1(0,T,L^\infty(\mathbb{R}^N)).$$

Therefore

$$u_k \nabla K * u_{k-1} \to \rho \nabla K * \rho \quad \text{in } L^1(Q_T).$$

Since $u_k \nabla K * u_{k-1}$ are uniformly bounded in $L^\infty(Q_T)$, by interpolating we have

$$u_k \nabla K * u_{k-1} \to \rho \nabla K * \rho \quad \text{in } L^2(Q_T).$$

From this point on the proof of the statement Theorem 2.12 for periodic solutions is as before.

Finally the existence result analogous to Theorem 2.13 holds as well. The proof is analogous, only that the proof of (25) requires the modification we presented above for the proof of the statement of Theorem 2.12 for periodic solutions.

4. **Solution on $\mathbb{R}^N$ when $N \geq 3$.** We now consider the Cauchy problem on $\mathbb{R}^N$ for $N \geq 3$:

$$\rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0 \qquad \text{on } \mathbb{R}^N \times [0,T] \tag{26}$$
$$\rho(\cdot,0) = \rho_0 \quad \text{on } \mathbb{R}^N$$

To define the solution, utilize the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^N)$, that is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to norm generated by the inner product

$$(f,g)_{\dot{H}^1} := \int_{\mathbb{R}^N} \nabla f(x) \cdot \nabla g(x) dx.$$

When $N \geq 3$, then $\dot{H}^1$ embeds continuously in $L^{2^*}(\mathbb{R}^N)$ where $2^* = \frac{2N}{N-2}$. See [28][Theorem 8.3.1]. Furthermore $\dot{H}^1 = \{u \in L^1_{loc}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N), u \in L^{2^*}(\mathbb{R}^N)\}$.

The definition of weak solutions we present below is appropriate for $N \geq 3$. For $N = 2$ (unless decay of solutions at infinity is assumed), the solution $\rho \in L^1 \cap L^\infty$ may lie outside of $\dot{H}^{-1}$, in which case ρ_t may not be in the space required below.

Definition 4.1 (**Weak solution**). Consider A which satisfies the assumption **(A1)** and K that satisfies the assumptions **(K1)**, **(K2)**, and **(KN)** or **(KP)**. Assume $\rho_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is nonnegative. A function $\rho : \mathbb{R}^N_T \longrightarrow [0,\infty)$ is a weak solution of (26) if $\rho \in L^\infty(\mathbb{R}^N_T) \cap L^\infty(0,T,L^1(\mathbb{R}^N))$, $A(\rho) \in L^2(0,T,\dot{H}^1(\mathbb{R}^N))$, $\rho_t \in L^2(0,T,\dot{H}^{-1}(\mathbb{R}^N))$ and for all test functions $\phi \in \dot{H}^1(\mathbb{R}^N)$ for almost all $t \in [0,T]$

$$\langle \rho_t(t), \phi \rangle + \int_{\mathbb{R}^N} \nabla A(\rho(t)) \cdot \nabla \phi - \rho(t)(\nabla K * \rho(t)) \cdot \nabla \phi \, dx = 0. \tag{27}$$

Here $\langle,\rangle$ denotes the dual pairing between $\dot{H}^{-1}(\mathbb{R}^N)$ and $\dot{H}^1(\mathbb{R}^N)$. We furthermore require initial conditions to be satisfied in $\dot{H}^{-1}$ sense:

$$\rho(\cdot,t)) \to \rho_0 \quad \text{in } \dot{H}^{-1}(\mathbb{R}^N) \text{ as } t \to 0.$$

Recall that $\rho \in H^1(0, T, \dot{H}^{-1}(\mathbb{R}^N))$ implies that $\rho \in C(0, T, \dot{H}^{-1}(\mathbb{R}^N))$. As before it turns out that $\rho \in C(0, T, L^p(\mathbb{R}^N))$ for all $p \in [1, \infty)$, so that the initial conditions are taken in the L^p sense. A reformulation of the definition of the solution analogous to one in Definition 2.2 also holds.

Theorem 4.2. *Assume $N \geq 3$. Consider A which satisfies the assumption **(A1)** and K that satisfies the assumptions **(K1)**, **(K2)**, and **(KN)** or **(KP)**. Assume that $\rho_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\rho_0 \geq 0$. Then there exists a unique weak solution of (26). Furthermore it preserves the integral $\int_{\mathbb{R}^N} \rho(x, t) dt$.*

Proof. If the condition **(KN)** is assumed the solution is obtained as the limit of solutions to the problem (E1) on a sequence of expanding domains. If the condition **(KP)** is assumed then a sequence of solutions with expanding period cell (e.g. $[-l, l]^N$) is considered.

Since the arguments are rather similar we only consider the former case. As there are no new essential estimates needed, we only sketch out the proof. Let $\Omega_n := B(0, n)$. Let ρ_n be the unique weak solutions of **(E1)** on Ω_n with initial data the restriction of ρ_0 to Ω_n. Note that the bounds of Lemma 2.5 and of Lemma 2.10 are independent of Ω_n. These are sufficient to extract a convergent subsequence, via a diagonal argument: There exist $\rho \in L^\infty(\mathbb{R}^N \times [0, T]) \cap L^2(0, T, L^2(\mathbb{R}^N))$ and $w \in L^2(0, T, H^1(\mathbb{R}^N))$ such that

$$\rho_n \rightharpoonup \rho \text{ in } L^2(U \times [0, T]) \quad \text{and} \quad A(\rho_n) \rightharpoonup w \text{ in } L^2(0, T, H^1(U))$$

for any compact set U. The estimate in Lemma 2.11 also does not depend on Ω. However obtaining compactness in L^1, (24), relies on estimates that are domain-size dependent. Thus, at this point, we only have $\rho \in L^1_{loc}(\mathbb{R}^N \times [0, T])$ and $\rho_n \to \rho$ in $L^1_{loc}(\mathbb{R}^N \times [0, T])$. That is, nevertheless, sufficient to establish that $w = A(\rho)$. Furthermore $\|\rho\|_{L^\infty(0, T, L^1(B(0,n)))} \leq \|\rho_0\|_{L^1(\mathbb{R}^N)}$ for every n. Therefore, since $\rho \chi_{B(0,n)} \to \rho$, monotone convergence theorem implies $\rho \in L^\infty(0, T, L^1(\mathbb{R}^N))$. Combining the L^∞ estimates and the fact that $\nabla K \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is enough to establish that $\rho_n(\nabla K * \rho_n) \to \rho(\nabla K * \rho)$ in $L^1_{loc}(\mathbb{R}^N \times [0, T])$. Since, as before, $\rho_n(\nabla K * \rho_n)$ is bounded in $L^2(\mathbb{R}^N \times [0, T])$ we can extract a weakly convergent subsequence in L^2 and identify the limit as $\rho(\nabla K * \rho)$. This is now enough to establish that ρ is a weak solution and that $\rho_t \in L^2(0, T, \dot{H}^{-1}(\mathbb{R}^N))$.

To show the conservation of $\int_{R^N} \rho(x, t) dx$ consider in the definition of a weak solution (2.1) test functions $\phi_n \in C^\infty(\mathbb{R}^N, [0, 1])$ supported on $B(0, n+1)$ and equal to 1 on $B(0, n)$ and such that their gradient and laplacian are bounded in L^∞ uniformly in n. We use the fact that $A(\rho)$ is in $L^1(\mathbb{R}^N \times [0, T])$ which follows from $A \in C^1([0, \infty))$ and $\rho \in L^\infty(\mathbb{R}^N) \cap L^\infty(0, T, L^1(\mathbb{R}^N))$. From (2.1) follows, via integrating in time and integrating by parts in space, that for $0 \leq s < t \leq T$

$$\left| \int_{\mathbb{R}^N} \rho(t) \phi_n dx - \int_{\mathbb{R}^N} \rho(s) \phi_n dx \right| = \left| \int_s^t \int_{\mathbb{R}^N} A(\rho(\tau)) \Delta \phi_n + \rho(\tau)(\nabla K * \rho(\tau)) \nabla \phi_n dx d\tau \right|$$

$$\leq C \int_s^t \int_{B(0,n+1) \setminus B(0,n)} A(\rho(\tau)) + \rho(\tau) dx d\tau.$$

Taking $n \to \infty$ and using monotone convergence theorem on the LHS and the fact that $A(\rho) + \rho \in L^1(\Omega_T)$ on the RHS completes the proof.

Uniqueness arguments given in Theorem 2.4 carry over to $\mathbb{R}^N$ with minor modifications when $N \geq 3$. A particular issue when $N = 2$ is that, since $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \not\subset \dot{H}^{-1}(\mathbb{R}^N)$, $u(t) - v(t)$ may not belong to $\dot{H}^{-1}$ and thus the Poisson

equation (5) (without boundary conditions) may not have a solution in $\dot{H}^1(\mathbb{R}^N)$. When $N \geq 3$ solution is in $\dot{H}^1$ and can be represented via the convolution with the fundamental solution to Laplace equation. Furthermore $D^2u \in L^2(\mathbb{R}^N)$. The fact that $\phi_t \in \dot{H}^{-1}$ and solves (6) (again on $\mathbb{R}^N$) follows using the Riesz representation theorem. One should also note that the integrations by parts are justified via approximations by smooth functions. $\qquad\square$

5. **Energy.** We consider kernels that are symmetric, that is satisfy the assumption
(K3) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N$.
Note that **(K3)** is satisfied whenever **(KN)** holds.

For symmetric kernels the equation possesses a dissipated quantity we call the energy. To define the energy, we first rewrite the equation in a slightly different form:

$$\rho_t = \nabla \cdot (\rho\nabla(g(\rho) - K * \rho)) \tag{28}$$

where g is smooth on $(0, \infty)$, and $a (= A')$ and g are related by

$$a(z) = zg'(z) \tag{29}$$

Let $G(z) := \int_0^z g(s)ds$. Integration by parts gives $A(z) = zg(z) - G(z)$.

We now define the energy:

$$E(\rho) := \int_\Omega G(\rho) - \frac{1}{2}\rho K * \rho \, dx. \tag{30}$$

The variational derivative of E in the direction $v \in L^2$, for which $\int_\Omega v = 0$

$$DE(\rho)[v] = \left\langle \frac{\delta E}{\delta \rho}, v \right\rangle_{L^2(\Omega)} = \int_\Omega (g(\rho) - K * \rho)v dx$$

Let $p := -\frac{\delta E}{\delta \rho}$ and flux $J = \rho\nabla p$. Then the equation can be written as

$$\rho_t = -\nabla \cdot J = -\nabla \cdot (\rho\nabla p) = \nabla \cdot \left(\rho\nabla\frac{\delta E}{\delta \rho}\right)$$

If the solution is smooth a simple calculation shows that the energy (30) is dissipated and

$$\frac{dE}{dt} = -\int_\Omega \rho|\nabla p|^2 dx = -\int_\Omega \frac{1}{\rho}|J|^2 dx.$$

For weak solutions we claim the following:

Lemma 5.1 (**Energy dissipation**). *Assume A satisfies **(A1)** and K satisfies **(K1)**, **(K2)**, and **(KN)**. Let ρ be a weak solution of **(E1)** on $\Omega \times [0, T]$. Then for almost all $\tau \in (0, T)$*

$$E(\rho(0)) - E(\rho(\tau)) \geq \int_0^\tau \int_\Omega \frac{1}{\rho}|J|^2 dx \tag{31}$$

where $J = \nabla A(\rho) - \rho\nabla K * \rho$.

Proof. Let us regularize the equation as before by considering smooth a_ε such that $a + \varepsilon \leq a_\varepsilon \leq a + 2\varepsilon$. Define g and g_ε by using (29) and setting $g(1) = g_\varepsilon(1) = 0$ Then for $z > 0$

$$g'(z) \leq g_\varepsilon'(z) \leq g'(z) + \frac{2\varepsilon}{z}$$

$$\text{for } z \leq 1 \quad g(z) \geq g_\varepsilon(z) \geq g(z) + 2\varepsilon \ln z$$

$$\text{for } z \geq 1 \quad g(z) \leq g_\varepsilon(z) \leq g(z) + 2\varepsilon \ln z$$

integrating from 0 to z gives

$$G(z) - 2\varepsilon \leq G_\varepsilon(z) \leq G(z) + 2\varepsilon(z \ln z - z + 1)_+$$

Let ρ_ε be the (smooth) solutions of the regularized equation. Using the smoothness of ρ_ε one verifies via direct computation:

$$E_\varepsilon(\rho_\varepsilon(0)) - E_\varepsilon(\rho_\varepsilon(\tau)) = \int_0^\tau \int_\Omega \frac{1}{\rho_\varepsilon} |J_\varepsilon|^2 dx \tag{32}$$

We claim that for almost all $0 < \tau < T$

$$\lim_{\varepsilon \to 0} E_\varepsilon(\rho_\varepsilon(\tau)) \to E(\rho(\tau)) \tag{33}$$

From (25) follows that for almost all $\tau \in (0, T)$, along a subsequence as $\varepsilon \to 0$

$$\rho_\varepsilon(\tau) \nabla K * \rho_\varepsilon(\tau) \to \rho(\tau) \nabla K * \rho(\tau) \text{ in } L^1(\Omega).$$

Thus for almost all $\tau \in (0, T)$

$$\int_\Omega \rho_\varepsilon(\tau) \nabla K * \rho_\varepsilon(\tau) dx \to \int_\Omega \rho(\tau) \nabla K * \rho(\tau) dx. \tag{34}$$

along the subsequence as $\varepsilon \to 0$. Let us show that

$$\int_\Omega G_\varepsilon(\rho_\varepsilon(\tau)) dx \to \int_\Omega G(\rho(\tau)) dx \tag{35}$$

for almost all τ. Using the uniform L^∞ bound on ρ_ε

$$\left| \int_\Omega G_\varepsilon(\rho_\varepsilon(\tau)) - G(\rho_\varepsilon(\tau))) dx \right| \leq 2\varepsilon \int_\Omega 1 + (\rho_\varepsilon(\tau) \ln \rho_\varepsilon(\tau))_+ dx \leq C|\Omega|\varepsilon,$$

$$\int_\Omega |G(\rho_\varepsilon(\tau)) - G(\rho(\tau))| dx \leq \|G\|_{C^1([0, \max_\varepsilon \|\rho_\varepsilon\|_{L^\infty}])} \|\rho_\varepsilon(\tau) - \rho(\tau)\|_{L^1(\Omega)}$$

which, due to (24), for almost all τ converges to 0 along a further subsequence in ε. Thus (35) holds, and combined with (34) implies (33).

Regarding the right hand side of (31), we use the following weak lower-semicontinuity property, proven in Otto [32][pg. 165-166]: Assume that $\sigma_\varepsilon \geq 0$ are in $L^1(\Omega_\tau)$ and f_ε are L^1 vector fields on Ω_τ such that

$$\int_{\Omega_\tau} \sigma_\varepsilon \phi \, dx dt \to \int_{\Omega_\tau} \sigma \phi \, dx dt \quad \text{and}$$

$$\int_{\Omega_\tau} f_\varepsilon \cdot \xi \, dx dt \to \int_{\Omega_\tau} f \cdot \xi \, dx$$

for all $\phi \in C_0^\infty(\overline{\Omega}_\tau)$ and all $\xi \in C_0^\infty(\overline{\Omega}_\tau, \mathbb{R}^N)$. Then

$$\int_{\Omega_\tau} \frac{1}{\sigma} |f|^2 dx dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\tau} \frac{1}{\sigma_\varepsilon} |f_\varepsilon|^2 dx dt. \tag{36}$$

the proof is simple and relies on observation that

$$\int_{\Omega_\tau} \frac{1}{\sigma} |f|^2 dx dt = \sup_{\xi \in C_0^\infty(\overline{\Omega}_\tau, \mathbb{R}^N)} \int_{\Omega_\tau} 2f \cdot \xi - \sigma |\xi|^2 dx dt.$$

The bounds on ρ_ε and J_ε that stated in the proof of Theorem 2.13 imply that along a subsequence as $\varepsilon \to 0$

$$\rho_\varepsilon \rightharpoonup \rho \text{ in } L^2(\Omega_\tau) \quad \text{and} \quad J_\varepsilon \rightharpoonup J \text{ in } L^2(\Omega_\tau, \mathbb{R}^N).$$

Therefore the claim above implies

$$\int_{\Omega_\tau} \frac{1}{\rho} |J|^2 dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\tau} \frac{1}{\rho_\varepsilon} |J_\varepsilon|^2 dxdt. \tag{37}$$

Finally, claims (33) and (37), and observing that (33) holds when $\tau = 0$, imply (31). □

Let us remark that for equations for which the gradient flow theory of [2] is applicable (i.e. when the energy is geodesically λ-convex), one obtains the energy-dissipation equality (instead of an inequality in (31)).

The energy dissipation for periodic solutions is proven in the same way:

Lemma 5.2. *The claim of Lemma 5.1 also holds for periodic solutions, provided that the instead of (KN) the kernel K satisfies (K3) and (KP).*

Furthermore the energy dissipation holds for the solutions of the Cauchy problem:

Lemma 5.3. *The claim of Lemma 5.1 also holds when $\Omega = \mathbb{R}^N$, $N \geq 3$. It also holds if instead of (KN) the kernel K satisfies (K3) and (KP).*

Proof. We only provide the proof for the case that **(KN)** holds. Let, as in the proof of the existence of weak solutions on $\mathbb{R}^N$, ρ_n be the solutions of the problem **(E1)** on $\Omega_n = B(0, n)$. The available bounds imply that

$$\nabla A(\rho_n) \rightharpoonup \nabla A(\rho) \text{ and } \rho_n K * \rho_n \rightharpoonup \rho K * \rho \text{ in } L^2(\mathbb{R}^N \times [0, \tau])$$

along a subsequence, which for simplicity we assume to be the whole sequence. In the above claim the quantities defined on Ω_n have been extended by zero to $\mathbb{R}^N$. By the monotone convergence theorem

$$E(\rho_n(0)) \to E(\rho(0)) \quad \text{as } n \to \infty. \tag{38}$$

As in the proof of existence, we have $\rho_n(K * \rho_n) \to \rho(K * \rho)$ in $L^1_{loc}(\mathbb{R}^N \times [0, T])$. Using a diagonal procedure, for almost all $\tau \in [0, T]$, there exist a subsequence n_j (dependent on τ) such that $\rho_{n_j}(\tau) \to \rho(\tau)$ a.e. and $\rho_{n_j}(\tau)(K * \rho_{n_j}(\tau)) \to \rho(\tau)(K * \rho(\tau))$ in $L^1(B(0, k))$ for each integer $k > 0$.

To prove the convergence on the whole space we use the following uniform integrability: For every $\varepsilon > 0$ there exist k_0, j_0 such that for all $k > k_0$ and $j > j_0$

$$\int_{R^N \backslash B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) < \varepsilon \tag{39}$$

To show this note that using "mass" conservation

$$\int_{R^N \backslash B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) \leq C \int_{R^N \backslash B(0,k)} \rho_{n_j}(\tau) \leq C \left(\int_{R^N} \rho(\tau) - \int_{B(0,k)} \rho_{n_j}(\tau) \right)$$

For k_0 large enough $\int_{R^N \backslash B(0,k_0)} \rho(\tau) < \frac{\varepsilon}{2C}$. Choose j_0 so that for all $j > j_0$, $\int_{B(0,k_0)} |\rho_{n_j}(\tau) - \rho(\tau)| dx < \frac{\varepsilon}{2C}$, which can be achieved thanks to L^1_{loc} convergence of $\rho_{n_j}(\tau)$. This implies (39). Consequently

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) = \int_{\mathbb{R}^N} \rho(\tau) K * \rho(\tau).$$

Note also that since ρ_n are bounded in $L^\infty(0, T, L^2(\mathbb{R}^N))$ for almost all $\tau > 0$ $\rho_{n_j}(\tau) \rightharpoonup \rho(\tau)$ along a subsequence in $L^2(\mathbb{R}^N)$. Since G is convex the mapping

$u \mapsto \int_{R^N} G(u)dx$ is weakly lower-semicontinuous with respect to $L^2(\mathbb{R}^N)$ topology. Combining the two claims we conclude

$$\liminf_{n \to \infty} E(\rho_n(\tau)) \geq E(\rho(\tau)).$$

By Lemma 5.1

$$E(\rho_n(0)) - E(\rho_n(\tau)) \geq \int_0^\tau \int_{\Omega_n} \frac{1}{\rho_n} |\nabla A(\rho_n) - \rho_n \nabla K * \rho_n|^2 dx.$$

The claims we have proven, along with the lower-semicontinuity claim (36) are sufficient to pass to limit $n \to \infty$. □

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