EXISTENCE AND UNIQUENESS OF SOLUTIONS TO AN AGGREGATION EQUATION WITH DEGENERATE DIFFUSION

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Abstract. We present an energy-methods-based proof of the existence and uniqueness of solutions of a nonlocal aggregation equation with degenerate diffusion. The equation we study is relevant to models of biological aggregation.

1. Introduction

We consider the existence and uniqueness of solutions of an aggregation equation with degenerate diffusion. Such nonlocal equations appear in models of biological aggregation that have been introduced by Topaz, Bertozzi, and Lewis [25] and Burger, Capasso, and Morale [9]. The model has been further studied by Burger and Di Francesco [10]. We refer the reader to these papers for further references on the well studied problem of mathematical modeling of large-scale collective behavior in living organisms. Related model, without the diffusion, has been studied by Topaz and Bertozzi [24], Bodnar and Velazquez [6], Laurent [16], Bertozzi and Brandman [3], and others.

In this paper we provide a proof of the existence and uniqueness of weak solutions of the equation:

$$\rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0$$

where A is such that the equation is (degenerate) parabolic and K is smooth, nonnegative, and integrable. The precise conditions on A and K are given in Section 2. We consider the problem with no-flux boundary conditions on bounded convex domains and on $\mathbb{R}^N$. In applications to biology, ρ represents the population density, while K is the sensing (interaction) kernel that models the long-range attraction. The therm containing $A(\rho)$ models the local repulsion (dispersal mechanism).

Burger, Capasso, and Morale [9] have already shown the existence and uniqueness of entropy solutions to the equation. Such solutions have an entropy condition as a part of the definition of a solution. They were developed by Carrillo [11] to study (among other problems) parabolic-hyperbolic problems, in particular degenerate parabolic equations with lower order terms that include conservation-law-type terms. For more on entropy

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solutions we refer to works of Karlsen and Risebro [13, 14] and references therein. The uniqueness of solutions relies on L^1 stability estimates.

Here we show that the standard notion of a weak solutions is sufficient for uniqueness of solutions. Heuristically, this is possible since the nonlocal term does not create shocks. The proof of uniqueness relies on the stability of solutions in the H^{-1} sense. We also provide a detailed proof of the existence of solutions. The proof is based on energy methods and relies only on basic facts of theory of uniformly parabolic equations and some functional analysis. The main technical difficulty comes from the the degeneracy of the diffusion term. Note that without the nonlocal term the equation is the well studied filtration equation (generalized porous medium equation). For the wealth of information on the filtration equation and further references we refer to the book by Vazquez [26]. Our approach to existence uses a number of tools from the paper by Alt and Luckhaus [1].

Our manuscript considers the case where K is smooth enough to guarantee that solutions stay bounded on any finite time interval. We mention for completeness that there is significant activity on the blowup problem for the case where K is not smooth and indeed finite time blowup can occur with mildly singular K (e. g. Lipschitz continuous). Some recent work on this problem includes [5, 3, 4] for the inviscid case [21, 22, 20] for the problem with fractional diffusion and [17] for the problem in 1D with nonlinear diffusion.

Associated to the equation is a natural Lyapunov functional, the energy:

$$E(\rho) := \int_\Omega G(\rho) - \frac{1}{2}\rho K * \rho dx$$

where G is such that $G''(z) = A'(z)/z$ for $z > 0$. We prove the energy dissipation inequality (28), which is important for applications.

The energy is not just a dissipated quantity; the equation is a gradient flow of the energy with respect to the Wasserstein metric. This fact was used by Burger and Di Francesco [10] to show the existence and uniqueness of solutions in one dimension. They used the theory of gradient flows in Wasserstein metric developed by Ambrosio, Gigli, and Savaré [2]. The theory applies to several dimensions as well. However the present approach is applicable to a wider class of nonlinearities and directly provides better regularity of solutions. On the other hand for equations for which the gradient flow theory of [2] is applicable (i.e. when the energy is geodesically λ-convex), one obtains the energy-dissipation equality (instead of an inequality in (28)). Let us point out that, in one dimension, Burger and Di Francesco [10] also obtained further properties of solutions that do not follow from [2].

The paper is organized as follows. In Section 2 we prove the uniqueness and existence of solutions on bounded convex domains. In Section 3 we prove the existence and uniqueness on $\mathbb{R}^N$. In Section 4 we introduce the energy and prove the energy-dissipation inequality.

2. Solutions on a bounded domain

Let Ω be a bounded convex domain in $\mathbb{R}^N$. Let us remark that the convexity assumption is used only in Lemma 5. The assumption can be removed for many of the nonlinearities A, in particular for the ones where A'' is bounded.

We consider the equation on $\Omega_T := \Omega \times [0,T]$ with no-flux boundary conditions:

$$\begin{aligned} \rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0 \qquad & \text{in } \Omega_T \\ \rho(0) = \rho_0 \qquad & \text{in } \Omega \\ (-\nabla A(\rho) + \rho \nabla K * \rho) \cdot \nu = 0 \qquad & \text{on } \partial\Omega \times [0,T) \end{aligned} \quad . \tag{E1}$$

Above, and in the remainder of the paper, $\rho(t)$ refers to the function $\rho(\,\cdot\,,t) : \Omega \to \mathbb{R}$. In the convolution term, ρ is extended by zero, outside of Ω. More precisely $\nabla K * \rho(x) = \int_\Omega \nabla K(x-y)\rho(y)dy$.

We make the following assumptions on A and K:

(A1) A is a C^1 function on $[0,\infty)$ with $A' > 0$ on $(0,\infty)$. We also require $A(0) = 0$.

(K1) $K \in W^{2,1}(\mathbb{R}^N)$ is a smooth nonnegative function with $\|K\|_{C^2(\mathbb{R}^N)} < \infty$.
(K2) $\int_{\mathbb{R}^N} K(x)dx = 1$.

Since A and $A + c$ yield the same equation, the requirement that $A(0) = 0$ does not reduce generality. Note that $A(s) = s^m$ for $m \geq 1$ satisfies the above conditions. The requirement (K2) is nonessential and made only for convenience.

We are interested in existence of bounded, nonnegative weak solutions. By $\tilde{H}^{-1}(\Omega)$ we denote the dual of $H^1(\Omega)$.

Definition 1 (**Weak solution**). *Assume $\rho_0 \in L^\infty(\Omega)$ is nonnegative. A function $\rho : \Omega_T \longrightarrow [0,\infty)$ is a weak solution of* (E1) *if $\rho \in L^\infty(\Omega_T)$, $A(\rho) \in L^2(0,T,H^1(\Omega))$, $\rho_t \in L^2(0,T,\tilde{H}^{-1}(\Omega))$ and for all test functions $\phi \in H^1(\Omega)$ for almost all $t \in [0,T]$*

$$\langle \rho_t(t), \phi \rangle + \int_\Omega \nabla A(\rho(t)) \cdot \nabla \phi - \rho(t)(\nabla K * \rho(t)) \cdot \nabla \phi \, dx = 0. \tag{1}$$

Here $\langle , \rangle$ denotes the dual pairing between $\tilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$. We furthermore require initial conditions to be satisfied in $\tilde{H}^{-1}$ sense:

$$\rho(\,\cdot\,,t)) \to \rho_0 \quad \text{in } \tilde{H}^{-1}(\Omega) \text{ as } t \to 0.$$

Recall that $\rho \in H^1(0,T,\tilde{H}^{-1}(\Omega))$ implies that $\rho \in C(0,T,\tilde{H}^{-1}(\Omega))$. Below we show that in fact $\rho \in C(0,T,L^p(\Omega))$ for all $p \in [1,\infty)$, so that the initial conditions are taken in the L^p sense.

By density of piecewise constant functions in L^2 the condition (1) is equivalent to requiring

$$\int_0^T \langle \rho_t, \phi \rangle dt + \int_0^T \int_\Omega \nabla A(\rho) \cdot \nabla \phi - \rho(\nabla K * \rho) \cdot \nabla \phi \, dx\, dt = 0. \tag{2}$$

for all $\phi \in L^2(0,T,H^1(\Omega))$.

Furthermore, it is a simple exercise to check that the above definition is equivalent with the following statement

Definition 2. *Assume $\rho_0 \in L^\infty(\Omega)$ is nonnegative. A function $\rho : \Omega_T \longrightarrow [0,\infty)$ is a weak solution of* (E1) *if $\rho \in L^\infty(\Omega_T)$, $A(\rho) \in L^2(0,T,H^1(\Omega))$, and for all test functions $\phi \in C^\infty(\overline{\Omega}_T)$ such that $\phi(T) = \phi(0) = 0$*

$$\int_0^T \int_\Omega -\rho\phi_t + \nabla A(\rho)\cdot\nabla\phi - \rho(\nabla K * \rho)\cdot\nabla\phi\, dx = 0. \tag{3}$$

Initial conditions are required in $\tilde{H}^{-1}$ sense:

$$\rho(\,\cdot\,,t)) \to \rho_0 \quad \text{in } \tilde{H}^{-1}(\Omega) \text{ as } t \to 0.$$

Important property of weak solutions is that the total population is preserved in time.

Lemma 3. *Let u be a weak solution of* (E1). *Then for all $t \in [0,T]$*

$$\int_\Omega \rho(x,t) = \int_\Omega \rho_0(x)dx.$$

To prove this lemma it suffices to take the test function $\phi \equiv 1$ and integrate in time.

2.1. **Uniqueness.** We now establish the uniqueness of weak solutions.

Theorem 4. *Let $\rho_0 \in L^\infty(\Omega)$ be nonnegative. There exists at most one weak solution to problem* (E1).

Proof. Assume that there are two solutions to the problem: u and v. To prove uniqueness we use a version of the standard argument which is based on estimating the $\tilde{H}^{-1}$ norm of the difference $u(t) - v(t)$. Since $u, v \in C(0,T,\tilde{H}^{-1}(\Omega))$ we can define $\phi(t)$ to be the solution of

$$\begin{aligned} \Delta\phi(t) &= u(t) - v(t) && \text{in } \Omega \\ \nabla\phi(t)\cdot\nu &= 0 && \text{on } \partial\Omega\times(0,T) \end{aligned} \tag{4}$$

for which $\int_\Omega \phi(t)dx = 0$. Due to Lemma 3, $\int_\Omega u(t) - v(t)dx = 0$ for all $t \in [0,T]$ and thus the Neumann problem above has a solution. Note that $\phi(0) = 0$. Due to regularity of $u - v$, the basic regularity theory yields: $\phi \in L^2(0,T,H^2(\Omega))$ and $\phi \in H^1(0,T,H^1(\Omega))$. Thus $\nabla\phi \in C(0,T,L^2(\Omega))$. Also ϕ_t solves (in the weak sense)

$$\begin{aligned} \Delta\phi_t &= u_t - v_t && \text{in } \Omega_T \\ \nabla\phi_t\cdot\nu &= 0 && \text{on } \partial\Omega\times(0,T) \end{aligned} \tag{5}$$

Subtracting the weak formulations (2) satisfied by u and v we obtain for all $\tau \in [0,T]$:

$$\begin{aligned} \int_\Omega |\nabla\phi(t)|^2 dx = -\int_0^\tau \langle u_t - v_t, \phi\rangle dt = &\int_0^\tau \int_\Omega \nabla(A(u) - A(v))\cdot\nabla\phi\, dxdt && \Big\}\, I \\ &- \int_0^\tau \int_\Omega ((\nabla K * u)u - (\nabla K * v)v)\cdot\nabla\phi\, dxdt && \Big\}\, II \end{aligned}$$

□

From (5) follows

$$
-\int_0^\tau \langle u_t - v_t, \phi \rangle dt = \int_0^\tau \int_\Omega \nabla \phi_t \cdot \nabla \phi \, dxdt = \frac{1}{2} \int_\Omega |\nabla \phi(T)|^2 - |\nabla \phi(0)|^2 dx
$$
(6)
$$
= \frac{1}{2} \int_\Omega |\nabla \phi(T)|^2 dx
$$

From (4) follows, using that $A(u) - A(v) \in L^2(0, T, H^1(\Omega))$ and the monotonicity of A

$$
I = -\int_0^\tau \int_\Omega (A(u) - A(v))(u - v) \, dxdt \leq 0 \tag{7}
$$

We now consider

$$
\begin{aligned}
II = &-\int_0^\tau \int_\Omega (u - v)(\nabla K * u) \cdot \nabla \phi \, dx && \Big\} III \\
&+ \int_0^\tau \int_\Omega v(\nabla K * (u - v)) \cdot \nabla \phi \, dx && \Big\} IV
\end{aligned}
$$

Using (4)

$$
III = \int_0^\tau \int_\Omega \nabla \phi \cdot ((\Delta K * u) \nabla \phi + (\nabla K * u) \Delta \phi) dxdt
$$

and hence

$$
|III| = \left| \frac{1}{2} \int_0^\tau \int_\Omega (\Delta K * u) |\nabla \phi|^2 dxdt \right| \leq C \int_0^\tau \int_\Omega |\nabla \phi|^2 dxdt \tag{8}
$$

Using the definition of solution of (4) in the inner-most integral gives

$$
\begin{aligned}
IV &= \int_0^\tau \int_\Omega v(x) \nabla \phi(x) \cdot \int_\Omega \nabla K(x - y)(u(y, t) - v(y, t)) dydxdt \\
&= \int_0^\tau \int_\Omega \sum_{i,j} v(\partial_i \partial_i K * \partial_j \phi) \partial_i \phi \, dxdt
\end{aligned} \tag{9}
$$

and thus

$$
\begin{aligned}
|IV| &\leq \|v\|_{L^\infty(\Omega_T)} \int_0^\tau \int_\Omega \sum_{i,j} |(\partial_i \partial_i K * \partial_j \phi) \partial_i \phi \,| \, dxdt \\
&\leq \sum_{i,j} \|\partial_i \partial j K * \partial_j \phi\|^{1/2}_{L^2(\Omega_T)} \|\partial_i \phi\|^{1/2}_{L^2(\Omega_T)} \leq C \|\nabla \phi\|_{L^2(\Omega_T)}
\end{aligned} \tag{10}
$$

The last inequality is a consequence of Young's inequality for convolutions (see for example [12]). The constant C can be taken independent of Ω. Let $\eta(t) := \int_\Omega |\nabla \phi(t)|^2 dt$.

Combining (6), (7), (8), and (10) gives that

$$\eta(\tau) \le C \int_0^\tau \eta(s) ds.$$

Since $\eta(0) = 0$ from Gronwall's inequality follows that $\eta(t) = 0$ for all $T \ge t \ge 0$. Therefore $u \equiv v$.

2.2. **Existence.** To establish the existence of solutions, we carry out two approximating procedures. While this is not entirely necessary, it separates handling the nonlocality and the degeneracy of the equation. It is thus transparent which tools are necessary to handle each.

One approximation is to perturb the equation to make it uniformly parabolic: Let

$$a := A'.$$

For $\varepsilon > 0$ let $a_\varepsilon(z)$ be smooth and even, and such that

$$a(z) + \varepsilon \le a_\varepsilon(z) \le a(z) + 2\varepsilon \quad \text{for } z \ge 0. \tag{11}$$

Let

$$A_\varepsilon(z) := \int_0^z a_\varepsilon(s) ds.$$

Consider

$$\partial_t \rho_\varepsilon - \Delta A_\varepsilon(\rho_\varepsilon) + \nabla \cdot [(\rho_\varepsilon \nabla K * \rho_\varepsilon)] = 0 \quad \text{on } \Omega_T \tag{E2}$$

with no-flux boundary conditions and initial conditions as in (E1). The notion of weak solution for (E2) is analogous to the one for (E1).

To show the existence of solutions of the nonlocal equation (E2) we utilize the following local equation: For $\tilde{a} \in C^\infty(\mathbb{R}, [0, \infty))$ let $\tilde{A}(s) := \int_0^s \tilde{a}(z) dz$. We assume

(A2) There exists $\lambda > 0$ such that $\tilde{a}(z) > \lambda$ for all $z \in \mathbb{R}$.

Let V be a smooth vector field on Ω_T with bounded divergence. Consider the equation

$$\partial_t u - \Delta \tilde{A}(u) + \nabla \cdot (uV) = 0 \quad \text{on } \Omega_T \tag{E3}$$

with no-flux boundary condition

$$(-\nabla \tilde{A}(u) + uV) \cdot \nu = 0 \quad \text{on } \partial\Omega \times [0, T].$$

The initial data are taken in the $\tilde{H}^{-1}$ sense.

For $\tilde{a}$ satisfying the condition (A2) the equation (E3) is a uniformly parabolic quasilinear equation with smooth coefficients. Thus, by standard theory [15, 18], there exist a unique classical short time solution to the equation on Ω_{T_0} for some $T_0 > 0$.

Lemma 5 (L^∞ **bound**). *Assume $u \in C^2(\Omega_T)$ is a solution of* (E3) *with smooth, nonnegative bounded initial data u_0. Assume further that*

$$V \cdot \nu \le 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{12}$$

Then u is nonnegative and for all $t \in [0,T]$

$$\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} \le e^{\mu t}\|u_0\|_{L^\infty(\Omega)}$$

where $\mu = \|\nabla \cdot V\|_{L^\infty(\Omega_T)}$.

Proof. The claim of the lemma follows directly from the comparison principle. Consider $v(x,t) := e^{\mu t}\|u(\,\cdot\,,0)\|_{L^\infty(\Omega)}$. It is a supersolution in the interior of Ω_T. On the lateral boundary, $\partial\Omega \times (0,T)$, we use $V \cdot \nu \le 0$ to establish that $(-\nabla A(v) + vV)\cdot \nu \le 0$. Thus v is a supersolution to the problem. To show that u is nonnegative, note that $w(x,t) \equiv 0$ is a subsolution. □

The condition (12) is satisfied for equations of our interest, (E1) and (E2), with $V = \nabla K * \rho$ on convex domains. If the condition (12) does not hold the construction of a supersolution is still possible for a number of nonlinearities A, but is more intricate. In particular the supersolutions need to be x-dependent. Let us also point out that the above lemma is the only claim that, as we apply it, relies on the convexity of Ω.

The L^∞ bounds above ensure that, when (A2) holds, the equation (E3) is uniformly parabolic, with smooth and bounded coefficients. By classical theory [15, 18] it then has smooth solutions for all $t > 0$.

The next four lemmas contain the compactness and continuity results we need. Lemma is analogous to one obtained by Alt and Luckhaus [1], who studied a family of equations that includes (E3). We present a proof here, for completeness.

Lemma 6. *Let F be a convex C^1 function and $f = F'$. Assume*

$$f(u) \in L^2(0,T,H^1(\Omega)), \quad u \in H^1(0,T,\tilde{H}^{-1}(\Omega)), \quad \text{and } F(u) \in L^\infty(0,T,L^1(\Omega)).$$

Then for almost all $0 \le s, \tau \le T$

$$\int_\Omega F(u(x,\tau)) - F(u(x,s))dx = \int_s^\tau \langle u_t, f(u(t))\rangle dt. \tag{13}$$

Proof. Let $t \in (0,T)$ and $h > 0$ small. Convexity of F implies that for all $x \in \Omega$

$$F(u(x,t)) - F(u(x,t-h)) \ge f(u(x,t-h))(u(x,t) - u(x,t-h)) \tag{14}$$

$$F(u(x,t)) - F(u(x,t-h)) \le f(u(x,t))(u(x,t) - u(x,t-h)) \tag{15}$$

Let $0 < s < \tau < T$. Since $u \in H^1(0,T,\tilde{H}^{-1}(\Omega))$

$$\frac{u(\,\cdot\,) - u(\,\cdot\, - h)}{h} \to u_t \quad \text{in } L^2(s,\tau,\tilde{H}^{-1}(\Omega)) \text{ as } h \to 0.$$

Convergence of translates gives

$$f(u(\,\cdot\, - h) \to f(u(\,\cdot\,)) \quad \text{in } L^2(s,\tau,H^1(\Omega)).$$

Using (14) and the claims above we obtain for $0 < h \leq s < \tau \leq T$

$$\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega F(u(t))dt - \frac{1}{h}\int_{s-h}^{s}\int_\Omega F(u(t))dt = \frac{1}{h}\int_{s}^{\tau}\int_\Omega F(u(t)) - F(u(t-h))dxdt$$
$$\geq \int_s^\tau \int_\Omega f(u(t-h))\,\frac{u(t)-u(t-h)}{h}dxdt$$

Taking the limit $h \to 0$ and using the Lebesgue differentiation theorem we obtain for a.e. $0 < h \leq s < \tau \leq T$

$$\int_\Omega F(u(x,\tau)) - F(u(x,s))dx \geq \int_s^\tau \langle u_t, f(u(t))\rangle dt.$$

Using the inequality (15) in analogous fashion one can obtain the opposite inequality. $\square$

Lemma 7. *Assume that A satisfies the condition (A1). Let $M > 0$. Let U be a bounded measurable set. There exists $\omega_s : [0,\infty) \to [0,\infty)$ nondecreasing, with $\lim_{z\to 0}\omega_s(z) = 0$ such that for all nonnegative functions $f_1, f_2 \in L^\infty(U)$ for which $\|f_1\|_{L^\infty} \leq M$ and $\|f_2\|_{L^\infty} \leq M$*

$$\|f_2 - f_1\|_{L^1(U)} \leq \omega_s(\|A(f_2) - A(f_1)\|_{L^1(U)}).$$

We use this lemma with either $U = \Omega$ or $U = \Omega \times [0,T]$.

Proof. Let for $x \geq 0$ and $y \geq 0$

$$\sigma(x,y) := \begin{cases} \frac{A(y)-A(x)}{y-x} & \text{if } x \neq y \\ A'(x) & \text{if } x = y. \end{cases}$$

Note that σ is continuous. Consider $\delta > 0$. Let $C(\delta) = \min_{[\delta,M]\times[0,M]}\sigma(x,y)$. Since $A'(x) > 0$ for all $x > 0$, $C(\delta) > 0$ for all $\delta > 0$. Given f_1 and f_2 in $L^\infty(U)$, let $U_1 := \{x \in U \,:\, f_1(x) < \delta \text{ and } f_2(x) < \delta\}$ and let $U_2 := U\backslash U_1$. Then

$$\int_U |f_2(x) - f_1(x)|dx = \int_{U_1} |f_2(x) - f_1(x)|dx + \int_{U_2} |f_2(x) - f_1(x)|dx$$
$$\leq \delta|U| + \frac{1}{C(\delta)}\int_U |A(f_2(x)) - A(f_1(x))|dx.$$

Defining $\omega_s(z) := \inf_{\delta>0}\left\{\frac{1}{C(\delta)}\,z + |U|\delta\right\}$ completes the proof. $\square$

The following lemma is used in conjunction with the estimates of Lemma 11 to prove L^1 precompactness in time of approximate solutions to (E1). It represents a version of Lemma 1.8 by Alt and Luckhaus [1].

Lemma 8. *Assume that A satisfies the condition (A1). Let $M > 0$ and $\delta > 0$. Let $\mathcal{F}$ be a family of nonnegative $L^\infty(\Omega)$ functions such that for all $f \in \mathcal{F}$*

$$\|A(f)\|_{H^1(\Omega)} \leq M, \quad \textit{and} \quad \|f\|_{L^\infty(\Omega)} \leq M. \tag{16}$$

There exists a nondecreasing function $\omega_M : [0,\infty) \to [0,\infty)$ satisfying $\omega_M(\delta) \to 0$ as $\delta \to 0$, such that if for $f_1, f_2 \in \mathcal{F}$

$$\int_\Omega (A(f_2) - A(f_1)(f_2 - f_1)dx \le \delta$$

then

$$\|f_1 - f_2\|_{L^1(\Omega)} \le \omega_M(\delta).$$

Proof. Assume that the claim does not hold. Then there exists $\kappa > 0$ and sequences $f_{1,n}$ and $f_{2,n}$ in $\mathcal{F}$ such that

$$\int_\Omega (A(f_{2,n}) - A(f_{1,n})(f_{2,n} - f_{1,n})dx \le \frac{1}{n} \text{ and } \int_\Omega |f_{2,n} - f_{1,n}|dx \ge \kappa.$$

The bounds in (16) imply that there exist $f_1, f_2 \in L^2(\Omega)$, and a subsequence of $(A(f_{1,n}), A(f_{2,n}))$, which we can assume to be the whole sequence, such that

$$A(f_{1,n}) \to A(f_1) \text{ and } A(f_{2,n}) \to A(f_2) \text{ in } L^2(\Omega) \text{ as } n \to \infty$$

and furthermore

$$f_{1,n} \rightharpoonup f_1 \text{ and } f_{2,n} \rightharpoonup f_2 \text{ in } L^2 \text{ as } n \to \infty.$$

Therefore

$$\int_\Omega (A(f_2) - A(f_1))(f_2 - f_1)dx = 0.$$

Thus $f_1 = f_2$ a.e. Consequently $\|A(f_{2,n}) - A(f_{1,n})\|_{L^1} \to 0$ as $n \to \infty$. Lemma 7 implies that $\|f_{2,n} - f_{1,n}\|_{L^1} \to 0$ as $n \to \infty$. This contradicts the assumption we made when constructing the sequences. $\square$

The following lemma is needed for proving the continuity in time (in L^p topology) of solutions. It is a special case of results of Visintin [27] and Brezis [8]. Since in this special case there exists a simple proof, we present it.

Lemma 9. *Let $F \in C^2([0,\infty),[0,\infty))$ be convex with $F(0) = 0$ and $F'' > 0$ on $(0,\infty)$. Let f_n, for $n = 1,2,\dots,$ and f be nonnegative functions on Ω bounded from above by $M > 0$. Furthermore assume*

$$f_n \rightharpoonup f \quad \text{in } L^1(\Omega) \quad \text{and} \quad \|F(f_n)\|_{L^1(\Omega)} \to \|F(f)\|_{L^1(\Omega)}$$

as $n \to \infty$. Then

$$f_n \to f \quad \text{in } L^2(\Omega) \text{ as } n \to \infty.$$

Proof. Since $F'' > 0$ on $(0,\infty)$ for each $\delta > 0$ there exists $\theta > 0$ such that for all $y \in [\delta, M]$ and all $h \in [0, y]$

$$F(y+h) + F(y-h) > 2F(y) + \theta h^2 \tag{17}$$

Let $\varepsilon > 0$. For $\delta \ge 0$ let $\Omega^\delta := \{x \in \Omega : \ f(x) > \delta\}$. Let us consider $\|f_n - f\|_{L^2(\Omega)}$:

$$\int_\Omega |f - f_n|^2 dx = \int_{\{f=0\}} f_n^2 dx + \int_{\Omega^0 \setminus \Omega^\delta} |f - f_n|^2 dx + \int_{\Omega^\delta} |f - f_n|^2 dx.$$

The first term

$$\int_{\{f=0\}} f_n^2 dx \le M \int_{\{f=0\}} f_n < \frac{\varepsilon}{3}$$

for n large enough, by the weak $L^1(\Omega)$ convergence. For $\delta > 0$, small enough, $|\Omega^0 \backslash \Omega^\delta| < \frac{\varepsilon}{3M^2}$ and thus

$$\int_{\Omega^0 \backslash \Omega^\delta} |f - f_n|^2 dx \le M^2 \frac{\varepsilon}{3M^2} = \frac{\varepsilon}{3}$$

for all n. Regarding the third term: Using (17) when integrating on Ω^δ and the fact that F is convex when integration on $\Omega \backslash \Omega^\delta$ one obtains

$$2 \int_\Omega F\left(\frac{f_n + f}{2}\right) dx \le \int_\Omega F(f_n) + F(f) dx - \frac{\theta}{4} \int_{\Omega^\delta} |f - f_n|^2 dx.$$

Since F is convex the functional $w \mapsto \int F(w)dx$ is weakly lower-semicontinuous with respect to L^1 topology. Using the assumption of the lemma and taking $\liminf_{n\to\infty}$ gives

$$2 \int_\Omega F(f) dx \le 2 \int_\Omega F(f) - \frac{\theta}{4} \limsup_{n\to\infty} \int_{\Omega^\delta} |f - f_n|^2 dx.$$

Therefore

$$\int_{\Omega^\delta} |f - f_n|^2 dx < \frac{\varepsilon}{3}$$

for all n large enough. Combining the bounds establishes the L^2 convergence. $\square$

The following is the standard gradient bound; we state it for weak solutions.

Lemma 10 **(gradient bound).** *Let $u \in L^\infty(\Omega_T)$ be a weak solution of* (E3)*. There exists a constant C depending only on T, $\|\,|V|\,\|_{L^\infty(\Omega_T)}$, $\|u\|_{L^\infty(\Omega_T)}$, $\|u_0\|_{L^1}$, and $\tilde{A}(\|u\|_{L^\infty(\Omega_T)})$ such that*

$$\|\nabla \tilde{A}(u)\|_{L^2(0,T,L^2(\Omega))} < C.$$

Proof. Let us use $\tilde{A}(u)$ as the test function in the formulation of a weak solution (2).

$$\int_0^T \langle u_t, \tilde{A}(u) \rangle dt = -\int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt + \int_{\Omega_T} uV \cdot \nabla \tilde{A}(u) dx dt.$$

Note that

$$\left| \int_{\Omega_T} uV \cdot \nabla \tilde{A}(u) dx dt \right| \le \int_{\Omega_T} |u|^2 |V|^2 dx dt + \frac{1}{4} \int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt.$$

Let $F(z) := \int_0^z \tilde{A}(s) ds$. Using Lemma 6 and $F(z) \le \tilde{A}(z) z$ we obtain

$$\begin{aligned} \frac{3}{4} \int_{\Omega_T} |\nabla \tilde{A}(u)|^2 dx dt &\le \liminf_{t\to 0} \int_\Omega F(u(t)) dx + \int_{\Omega_T} |u|^2 |V|^2 dx dt \\ &\le \tilde{A}(\|u\|_{L^\infty(\Omega)}) \|u_0\|_{L^1(\Omega)} + T \|V\|^2_{L^\infty(\Omega_T)} \|u\|_{L^\infty(\Omega_T)} \|u_0\|_{L^1(\Omega)} \end{aligned}$$

which implies the desired bound. $\square$

The following lemma is a version of a claim proven in Subsection 1.7 of Alt and Luckhaus [1].

Lemma 11. *Let V be an L^∞ vector field on Ω_T. Assume $\tilde{A}$ satisfies conditions (A1)-(A2). Let u be a weak solution of* (E3) *with no-flux boundary conditions and initial data in $L^\infty(\Omega)$. There exists a constant C depending only on T, $\|u\|_{L^\infty(\Omega_T)}$, $\|A(u)\|_{L^\infty(\Omega_T)}$, and $\|A(u)\|_{L^2(0,T,H^1(\Omega))}$ such that*

$$\int_0^{T-h}\int_\Omega (u(x,t+h)-u(x,t))\,(A(u(x,t+h))-A(u(x,t)))dxdt \leq Ch$$

for all $h \in [0,T]$.

Proof. Consider the test function

$$\phi_1(t) := \frac{1}{h}\int_t^{t+h} A(u(s))ds$$

on the time interval $[0, T-h]$ and the test function

$$\phi_2(t) := \phi_1(t-h)$$

on the time interval $[h, T]$. Subtracting the equalities resulting from the definition of the weak solution yields the desired bound, via straightforward calculations. □

Theorem 12 (**Existence for** (E2))**.** *Let $\varepsilon > 0$ and ρ_0 a nonnegative smooth function on $\overline{\Omega}$. The equation* (E2) *has a weak solution ρ on Ω_T. Furthermore ρ is smooth on $\overline{\Omega} \times (0,T]$.*

Proof. Let $\tilde{a} := a_\varepsilon$. We employ the following iteration scheme: Let $u^1(x,t) := \rho_0(x)$ for all $(x,t) \in \Omega_T$. For $k \geq 1$ let u^{k+1} be the solution of

$$u_t^{k+1} - \nabla\cdot(\tilde{a}(u^{k+1})\nabla u^{k+1}) + \nabla\cdot(u^{k+1}\nabla(K * u^k)) = 0 \tag{18}$$

with initial data $u^{k+1}(\,\cdot\,,0) = \rho_0$ and no-flux boundary conditions.

Since the equations preserve the nonnegativity and the "mass" of the solutions we have $\|u_k(\,\cdot\,,t)\|_{L^1} = \int_\Omega u_k(x,t)dx = \int_\Omega \rho_0(x)dx$. By the L^∞ estimates of Lemma 5, $\|u_k\|_{L^\infty(\Omega_T)} \leq e^{M_k T}\|\rho_0\|_{L^\infty}$, where $M_k = \|(\Delta K) * u_{k-1}(\,\cdot\,,t)\|_{L^\infty(\Omega_T)}$. Thus $M_k \leq \|\Delta K\|_{L^\infty}\,\|u_{k-1}(\,\cdot\,,t)\|_{L^1} = \|\Delta K\|_{L^\infty}\,\|\rho_0\|_L^1$. Hence M_k can be chosen independent of k. Consequently, Lemma 10 produces bounds on $\|\tilde{a}(u_k)\nabla u_k\|_{L^2(0,T,L^2(\Omega))}$ which are independent of k. Since $\tilde{a} \geq \varepsilon > 0$ this implies bounds on $\|\nabla u_k\|_{L^2(0,T,L^2(\Omega))}$. The $L^2(0,T,L^2(\Omega))$ bound on u_k and $L^2(0,T,L^\infty(\Omega))$ bound on $\nabla K * u_{k-1}$ that follows via Young's inequality, imply a bound on $u_k\,\nabla K * u_{k-1}$ in $L^2(0,T,L^2(\Omega))$ independent of k. Weak formulation of the equation then yields that u_t^k is a bounded sequence in $L^2(0,T,\tilde{H}^{-1}(\Omega))$.

Repeated application of the Lions-Aubin Lemma (see [23][pg. 106], for example) yields that there exists a subsequence of u_k, which for convenience we assume to be the whole sequence, and a function $\rho \in L^2(0,T,L^2(\Omega))$ such that

$$u_k \to \rho \quad \text{and} \quad \tilde{A}(u_k) \to \tilde{A}(\rho) \text{ in } L^2(0,T,L^2(\Omega)). \tag{19}$$

The L^∞ bound of Lemma 5, implies a bound on L^∞ norm of ρ. The gradient bound of Lemma 10 now implies that, along a subsequence, which we again assume to be the whole sequence,

$$\nabla\tilde{A}(u_k) \rightharpoonup \nabla\tilde{A}(\rho) \qquad \text{in } L^2(0,T,L^2(\Omega)).$$

By Cauchy-Schwarz inequality

$$\|\nabla K * u_{k-1} - \nabla K * \rho\|_{L^2(0,T,L^\infty(\Omega))} \leq |\Omega|^{1/2}\|\nabla K\|_{L^\infty(\mathbb{R}^N)}\|u_{k-1}-\rho\|_{L^2(\Omega_T)}.$$

It follows that

$$u_k\nabla K * u_{k-1} \to \rho\nabla K * \rho \quad \text{in } L^2(0,T,L^2(\Omega)).$$

Therefore

$$\int_0^T\int_\Omega \rho\phi_t - \nabla A(\rho)\cdot\nabla\phi + \rho(\nabla K * \rho)\cdot\nabla\phi\, dxdt = 0.$$

By the estimates above ρ_k are bounded in $H^1(0,T,\tilde{H}^{-1}(\Omega))$. Since $H^1(0,T,\tilde{H}^{-1}(\Omega))$ continuously embeds in $C^{1/2}(0,T,\tilde{H}^{-1}(\Omega))$ and thus compactly in $C(0,T,\tilde{H}^{-1}(\Omega))$ there exists a subsequence of ρ_k which converges in $C(0,T,\tilde{H}^{-1}(\Omega))$. We assume for notational simplicity that the subsequence is the whole sequence. Thus

$$\rho_k \to \rho \text{ in } C(0,T,\tilde{H}^{-1}(\Omega)) \text{ as } k\to\infty.$$

Therefore $\rho(t)\to\rho_0$ in $\tilde{H}^{-1}(\Omega)$ as $t\to 0$.

Smoothness of solution can now be shown using the standard theory (using test functions that approximate $\Delta A(\rho)$ and ρ_t to show improved regularity, differentiating the equation and iterating the procedure). □

Theorem 13 (**Existence for** (E1))**.** *Let ρ_0 be a nonnegative function in $L^\infty(\Omega)$. The problem* (E1) *has a weak solution on Ω_T. Furthermore $\rho \in C(0,T,L^p(\Omega))$ for all $p \in [1,\infty)$.*

Proof. Let a_ε and $A_\varepsilon(z)$ be as in (11). Let ρ_0^ε be smooth approximations of ρ_0 such that $\|\rho_0^\varepsilon\|_{L^1} = \|\rho_0\|_{L^1}$, $\|\rho_0^\varepsilon\|_{L^\infty} \leq 2\|\rho_0\|_{L^\infty}$, and $\rho_0^\varepsilon \to \rho_0$ in L^p as $\varepsilon\to 0$, for all $p\in[1,\infty)$

By Theorem 12 there exists a nonnegative solution ρ_ε of (E2) with initial datum ρ_0^ε. The proof of the theorem provides uniform bounds in ε on

$$A_\varepsilon(\rho_\varepsilon) \text{ in } L^2(0,T,H^1(\Omega)), \quad \rho_\varepsilon \text{ in } L^\infty(\Omega_T) \text{ and } \partial_t\rho_\varepsilon \text{ in } L^2(0,T,\tilde{H}^{-1}(\Omega)).$$

Since $A_\varepsilon \geq A$ and $a_\varepsilon \geq a$ on $[0,\infty)$ uniform bounds on $L^2(0,T,H^1(\Omega))$ norm of $A(\rho_\varepsilon)$ hold. Therefore there exists $w\in L^2(0,T,H^1(\Omega))$ and a sequence ε_j converging to 0 such that

$$A(\rho_{\varepsilon_j}) \rightharpoonup w \qquad \text{(weakly) in } L^2(0,T,H^1(\Omega)). \tag{20}$$

Note that ρ_ε is a weak solution of (E3) with $V = \nabla K * \rho_\varepsilon$. Using the uniform-in-$\varepsilon$ bounds above and that $|A(z_1) - A(z_2)| \leq |A_\varepsilon(z_1) - A_\varepsilon(z_2)|$ for all $\varepsilon > 0$ and $z_1, z_2 \geq 0$, by Lemma 11 there exists $C > 0$, independent of ε, such that

$$\int_0^{T-h} \int_\Omega (\rho_\varepsilon(x,t+h) - \rho_\varepsilon(x,t))\,(A(\rho_\varepsilon(x,t+h)) - A(\rho_\varepsilon(x,t)))dxdt \leq Ch \tag{21}$$

for all $h \in [0,T]$. To show that the family $\{\rho_\varepsilon\}$ is precompact in $L^1(\Omega_T)$ it is enough to show that it satisfies the assumptions of the Riesz-Frechet-Kolmogorov compactness criterion [7][IV.26]. In particular, it suffices to show:
Claim 1^o For all $\theta > 0$ there exists $0 < h_0 \leq \theta$ such that for all $\varepsilon > 0$ and all $0 < h \leq h_0$

$$\int_0^{T-\theta} \int_\Omega |\rho_\varepsilon(x,t+h) - \rho_\varepsilon(x,t)|dxdt \leq \theta.$$

Claim 2^o For all $\theta \in (0,T)$ there exists $0 < h_0 \leq \theta$ such that for all $\varepsilon > 0$ and all $0 < h \leq h_0$ and all $i = 1, \ldots, N$

$$\int_0^T \int_{\Omega^\theta} |\rho_\varepsilon(x + he_i, t) - \rho_\varepsilon(x,t)|dxdt \leq \theta$$

where $\Omega^\theta = \{x \in \Omega \,:\, d(x, \partial\Omega) > \theta\}$.

To prove the first claim, we recall that by the L^∞ bound of Lemma 5 and the L^2 gradient bound there exists $M > \|\rho_0\|_{L^1(\Omega)}$ such that for all $\varepsilon \in (0,1)$

$$\|\rho_\varepsilon\|_{L^\infty(\Omega_T)} \leq M \text{ and } \|A(\rho_\varepsilon)\|_{L^2(0,T,H^1(\Omega))} \leq M.$$

Consider for $0 < h < \theta$ and $\gamma > 1$ the set of times for which "good" estimates hold:

$$\begin{aligned} E_\gamma(h) := \Big\{ t \in [0, T-\theta] \,:\, &\|A(\rho_\varepsilon(t))\|_{H^1(\Omega)} \leq M\sqrt{\gamma}, \|A(\rho_\varepsilon(t+h))\|_{H^1(\Omega)} \leq M\sqrt{\gamma}, \\ &\text{and } \int_\Omega (\rho_\varepsilon(x,t+h) - \rho_\varepsilon(x,t))(A(\rho_\varepsilon(x,t+h)) - A(\rho_\varepsilon(x,t)))dx < Ch\gamma \Big\}. \end{aligned}$$

Let $E^c_\gamma(h) = [0, T-\theta] \backslash E_\gamma(h)$. Note that $|E^c_\gamma(h)| \leq \frac{3}{\gamma}$, since each condition cannot be violated on a set of measure larger than $1/\gamma$. Then by Lemma 8

$$\int_0^{T-\theta} \int_\Omega |\rho_\varepsilon(x,t+h) - \rho_\varepsilon(x,t)|dxdt \leq T\omega_{M\gamma}(C\gamma h) + 2M\,\frac{3}{\gamma}.$$

Set $\gamma = \max\{\frac{12M}{\theta}, 1\}$. Taking $h_0 > 0$ such that $T\omega_{M\gamma}(C\gamma h_0) < \frac{\theta}{2}$ completes the proof.

To show Claim 2^o note that for $0 < h < \theta$ and a.e. $t \in [0,T]$

$$\begin{aligned} \int_0^T \int_{\Omega^\theta} |A(\rho_\varepsilon(x + he_i, t)) - A(\rho_\varepsilon(x,t))|dxdt &\leq h \int_0^T \int_0^1 \int_{\Omega^\theta} |\nabla(A(\rho_\varepsilon)(x + she_i)|dxdsdt \\ &\leq h|\Omega|^{\frac{1}{2}} \|A(\rho_\varepsilon)\|_{L^2(0,T,H^1(\Omega))}. \end{aligned}$$

Lemma 7, applied to $U = \Omega^\theta \times [0,T]$, implies that

$$\int_0^T \int_{\Omega^\theta} |\rho_\varepsilon(x+he_i,t) - \rho_\varepsilon(x,t)| dx \le \omega_s(h|\Omega|^{\frac{1}{2}} M).$$

The claim follows by taking h_0 small enough.

In conclusion, along a subsequence, which we still denote by ρ_{ε_j},

$$\rho_{\varepsilon_j} \to \rho \qquad \text{in } L^1(\Omega_T) \tag{22}$$

for some $\rho \in L^1(\Omega_T)$. Therefore $w = A(\rho)$. Note that, using the Young's inequality

$$\|\nabla K * (\rho_{\varepsilon_j} - \rho)\|_{L^1(0,T,L^\infty(\Omega))} \le |\Omega| \, \|\nabla K\|_{L^\infty(\mathbb{R}^N)} \|\rho_{\varepsilon_j} - \rho\|_{L^1(\Omega_T)}.$$

Combining this claim with (22) gives

$$\rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \to \rho(\nabla K * \rho) \qquad \text{in } L^1(\Omega_T). \tag{23}$$

The boundedness of the left-hand side in L^2 furthermore implies that along a subsequence

$$\rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \rightharpoonup \rho(\nabla K * \rho) \qquad \text{in } L^2(\Omega_T).$$

Therefore we can take the limit as $j \to \infty$ in the weak formulation of the equation (E2): For $\phi \in C_0^\infty(\Omega \times (0,T))$

$$\int_0^T \int_\Omega \rho_{\varepsilon_j}\phi_t - \nabla A(\rho_{\varepsilon_j}) \cdot \nabla\phi + \rho_{\varepsilon_j}(\nabla K * \rho_{\varepsilon_j}) \cdot \nabla \phi \, dxdt = 0.$$

to obtain that (2) holds. Note also that uniform L^∞ bound on ρ_ε and the L^1 convergence of ρ_{ε_j} yield that $\rho \in L^\infty(\Omega_T)$. The proof that $\rho \in C(0,T,\tilde{H}^{-1}(\Omega))$ and $\rho(t) \to \rho_0$ in $\tilde{H}^{-1}$ as $t \to 0$ is the same as before.

It follows that $\rho(t) : [0,T] \to L^2(\Omega)$ is continuous with respect to weak $L^2(\Omega)$ topology. In particular, it suffices to establish that $\int_\Omega \rho(x,s)\psi(x)dx \to \int_\Omega \rho(x,t)\psi(x)dx$ as $s \to t$ for all $\psi \in L^2(\Omega)$. By a density argument it is enough to consider smooth ψ. Finally for smooth ψ the claim holds since $\rho \in C(0,T,\tilde{H}^{-1}(\Omega))$.

Since Ω is bounded, $\rho(t)$ is also continuous with respect to weak L^1 topology. Let $F(z) := \int_0^z A(s)ds$. Lemma 6 and Lemma 10 then imply that $t \mapsto \int_\Omega F(\rho(t))$ is continuous. Lemma 9 then implies that $\rho(t)$ is continuous with respect to $L^2(\Omega)$ topology. Using the boundedness of domain, and interpolating with L^∞ bound on ρ implies that $\rho \in C(0,T,L^p(\Omega))$ for all $p \in [1,\infty)$. □

3. Solution on $\mathbb{R}^N$

We now consider the Cauchy problem on $\mathbb{R}^N$:

$$\rho_t - \Delta A(\rho) + \nabla \cdot [(\rho \nabla K * \rho)] = 0 \qquad \text{on } \mathbb{R}^N \times [0,T] \tag{24}$$
$$\rho(\,\cdot\,,0) = \rho_0 \qquad \text{on } \mathbb{R}^N$$

The notion of the weak solution is as in Definition 1, only that it is also assumed that $\rho_0 \in L^1(\mathbb{R}^N)$ and $\rho \in L^\infty(0,T,L^1(\mathbb{R}^N))$.

Theorem 14. *Assume that $\rho_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\rho_0 \geq 0$. Then there exists a unique weak solution of (24). Furthermore the solution preserves the integral $\int_{\mathbb{R}^N} \rho(x,t)dt$.*

Proof. As there are no new essential estimates needed, we only sketch out the proof. Let $\Omega_n := B(0,n)$. Let ρ_n be the unique weak solutions of (E1) on Ω_n with initial data the restriction of ρ_0 to Ω_n. Note that the bounds of Lemma 5 and of Lemma 10 are independent of Ω_n. These are sufficient to extract a convergent subsequence, via a diagonal argument: There exist $\rho \in L^\infty(\mathbb{R}^N \times [0,T]) \cap L^2(0,T,L^2(\mathbb{R}^N))$ and $w \in L^2(0,T,H^1(\mathbb{R}^N))$ such that

$$\rho_n \rightharpoonup \rho \text{ in } L^2(U \times [0,T]) \quad \text{and} \quad A(\rho_n) \rightharpoonup w \text{ in } L^2(0,T,H^1(U))$$

for any compact set U. The estimate in Lemma 11 also does not depend on Ω, however obtaining compactness in L^1, (22), relies on estimates that are domain-size dependent. Thus we only have $\rho_n \to \rho$ in $L^1_{loc}(\mathbb{R}^N \times [0,T])$. That is, nevertheless, sufficient to establish that $w = A(\rho)$. Furthermore $\|\rho\|_{L^\infty(0,T,L^1(B(0,n)))} \leq \|\rho_0\|_{L^1(\mathbb{R}^N)}$ for every n. Therefore, by the monotone convergence theorem, $\rho \in L^\infty(0,T,L^1(\mathbb{R}^N))$. Combining the L^∞ estimates and the fact that $\nabla K \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is enough to establish that $\rho_n(\nabla K * \rho_n) \to \rho(\nabla K * \rho)$ in $L^1_{loc}(\mathbb{R}^N \times [0,T])$. Since, as before, $\rho_n(\nabla K * \rho_n)$ is bounded in $L^2(\mathbb{R}^N \times [0,T])$ we can extract a weakly convergent subsequence in L^2 and identify the limit as $\rho(\nabla K * \rho)$. This is now enough to establish that ρ is a weak solution and that $\rho_t \in L^2(0,T,H^{-1}(\mathbb{R}^N))$.

Uniqueness arguments given in Theorem 4 carry over to $\mathbb{R}^N$ with minor modifications.

To show the conservation of $\int_{R^N} \rho(x,t)dx$ consider in the definition of a weak solution (1) test functions $\phi_n \in C^\infty(\mathbb{R}^N,[0,1])$ supported on $B(0,n+1)$ and equal to 1 on $B(0,n)$ and such that their gradient and laplacian are bounded in L^∞ uniformly in n. We use the fact that $A(\rho)$ is in $L^1(\mathbb{R}^N \times [0,T])$ which follows from $A \in C^1([0,\infty))$ and $\rho \in L^\infty(\mathbb{R}^N) \cap L^\infty(0,T,L^1(\mathbb{R}^N))$. From (1) follows, via integrating in time and integrating by parts in space, that for $0 \leq s < t \leq T$

$$\left| \int_{\mathbb{R}^N} \rho(t)\phi_n dx - \int_{\mathbb{R}^N} \rho(s)\phi_n dx \right| = \left| \int_s^t \int_{\mathbb{R}^N} -A(\rho(\tau))\Delta\phi_n - \rho(\tau)(\nabla K * \rho(\tau))\nabla\phi_n dx d\tau \right|$$
$$\leq C \int_s^t \int_{B(0,n+1)\setminus B(0,n)} A(\rho(\tau)) + \rho(\tau) dx d\tau.$$

Taking $n \to \infty$ and using monotone convergence theorem on the LHS and the fact that $A(\rho) + \rho \in L^1(\Omega_T)$ on the RHS completes the proof. □

4. Energy.

We now rewrite the equation in a slightly different form:

$$\rho_t = \nabla \cdot (\rho \nabla (g(\rho) - K * \rho)) \tag{25}$$

where g is smooth on $(0, \infty)$, and $a \,(= A')$ and g are related by

$$a(z) = z g'(z) \tag{26}$$

Let $G(z) := \int_0^z g(s) ds$. Integration by parts gives $A(z) = z g(z) - G(z)$.

An important quantity associated to the equation (25) is the energy:

$$E(\rho) := \int_\Omega G(\rho) - \frac{1}{2} \rho K * \rho dx. \tag{27}$$

The variational derivative of E in the direction $v \in L^2$, for which $\int_\Omega v = 0$

$$DE(\rho)[v] = \left\langle \frac{\delta E}{\delta \rho}, v \right\rangle_{L^2(\Omega)} = \int_\Omega (g(\rho) - K * \rho) v dx$$

Let $p := -\frac{\delta E}{\delta \rho}$ and flux $J = \rho \nabla p$. Then the equation can be written as

$$\rho_t = -\nabla \cdot J = -\nabla \cdot (\rho \nabla p) = \nabla \cdot \left(\rho \nabla \frac{\delta E}{\delta \rho} \right)$$

If the solution is smooth a simple calculation shows that the energy (27) is dissipated and

$$\frac{dE}{dt} = -\int_\Omega \rho |\nabla p|^2 dx = -\int_\Omega \frac{1}{\rho} |J|^2 dx.$$

For weak solutions we claim the following:

Lemma 15 (**Energy dissipation**). *Assume (A1) and (K1)-(K2). Let ρ be a weak solution of* (E1) *on $\Omega \times [0, T]$. Then for almost all $\tau \in (0, T)$*

$$E(\rho(0)) - E(\rho(\tau)) \geq \int_0^\tau \int_\Omega \frac{1}{\rho} |J|^2 dx \tag{28}$$

where $J = \nabla A(\rho) - \rho \nabla K * \rho$.

Proof. Let us regularize the equation as before by considering smooth a_ε such that $a + \varepsilon \leq a_\varepsilon \leq a + 2\varepsilon$. Define g and g_ε using (26) and setting $g(1) = g_\varepsilon(1) = 0$ Then for $z > 0$

$$g'(z) \leq g'_\varepsilon(z) \leq g'(z) + \frac{2\varepsilon}{z}$$

$$\text{for } z \leq 1 \quad g(z) \geq g_\varepsilon(z) \geq g(z) + 2\varepsilon \ln z$$

$$\text{for } z \geq 1 \quad g(z) \leq g_\varepsilon(z) \leq g(z) + 2\varepsilon \ln z$$

integrating from 0 to z gives

$$G(z) - 2\varepsilon \leq G_\varepsilon(z) \leq G(z) + 2\varepsilon(z \ln z - z)_+$$

Let ρ_ε be the (smooth) solutions of the regularized equation. Using the smoothness of ρ_ε one verifies via direct computation:

$$E_\varepsilon(\rho_\varepsilon(0)) - E_\varepsilon(\rho_\varepsilon(\tau)) = \int_0^\tau \int_\Omega \frac{1}{\rho_\varepsilon} |J_\varepsilon|^2 dx \tag{29}$$

We claim that for almost all $0 < \tau < T$

$$\lim_{\varepsilon \to 0} E_\varepsilon(\rho_\varepsilon(\tau)) \to E(\rho(\tau)) \tag{30}$$

From (23) follows that for almost all $\tau \in (0,T)$, along a subsequence as $\varepsilon \to 0$

$$\rho_\varepsilon(\tau) \nabla K * \rho_\varepsilon(\tau) \to \rho(\tau) \nabla K * \rho(\tau) \text{ in } L^1(\Omega).$$

Thus for almost all $\tau \in (0,T)$

$$\int_\Omega \rho_\varepsilon(\tau) \nabla K * \rho_\varepsilon(\tau) dx \to \int_\Omega \rho(\tau) \nabla K * \rho(\tau) dx. \tag{31}$$

along subsequence as $\varepsilon \to 0$. Let us show that

$$\int_\Omega G_\varepsilon(\rho_\varepsilon(\tau)) dx \to \int_\Omega G(\rho(\tau)) dx \tag{32}$$

for almost all τ. Using the uniform L^∞ bound on ρ_ε

$$\left| \int_\Omega G_\varepsilon(\rho_\varepsilon(\tau)) - G(\rho_\varepsilon(\tau))) dx \right| \leq 2\varepsilon \int_\Omega 1 + (\rho_\varepsilon(\tau) \ln \rho_\varepsilon(\tau))_+ dx \leq C|\Omega|\varepsilon.$$

$$\int_\Omega |G(\rho_\varepsilon(\tau)) - G(\rho(\tau))| dx \leq \|G\|_{C^1([0, \max_\varepsilon \|\rho_\varepsilon\|_{L^\infty}])} \|\rho_\varepsilon(\tau) - \rho(\tau)\|_{L^1(\Omega)}$$

which, for almost all τ converges to 0 along a further subsequence in ε. Thus (32) holds, and combined with (31) implies (30).

Regarding the right hand side of (28), we use the following weak lower-semicontinuity property, proven in Otto [19][pg. 165-166]: Assume that $\sigma_\varepsilon \geq 0$ are in $L^1(\Omega_\tau)$ and f_ε are L^1 vector fields on Ω_τ such that

$$\int_{\Omega_\tau} \sigma_\varepsilon \phi dx dt \to \int_{\Omega_\tau} \sigma \phi dx dt \quad \text{and}$$
$$\int_{\Omega_\tau} f_\varepsilon \cdot \xi dx dt \to \int_{\Omega_\tau} f \cdot \xi dx$$

for all $\phi \in C_0^\infty(\overline{\Omega}_\tau)$ and all $\xi \in C_0^\infty(\overline{\Omega}_\tau, \mathbb{R}^N)$. Then

$$\int_{\Omega_\tau} \frac{1}{\sigma} |f|^2 dx dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\tau} \frac{1}{\sigma_\varepsilon} |f_\varepsilon|^2 dx dt. \tag{33}$$

the proof is simple and relies on observation that

$$\int_{\Omega_\tau} \frac{1}{\sigma}\,|f|^2 dxdt = \sup_{\xi \in C_0^\infty(\overline{\Omega}_\tau, \mathbb{R}^N)} \int_{\Omega_\tau} 2f\cdot\xi - \sigma|\xi|^2 dxdt.$$

The bounds on ρ_ε and J_ε that stated in the proof of Theorem 13 imply that along a subsequence as $\varepsilon \to 0$

$$\rho_\varepsilon \rightharpoonup \rho \text{ in } L^2(\Omega_\tau) \quad \text{and} \quad J_\varepsilon \rightharpoonup J \text{ in } L^2(\Omega_\tau, \mathbb{R}^N).$$

Therefore the claim above implies

$$\int_{\Omega_\tau} \frac{1}{\rho}\,|J|^2 dxdt \le \liminf_{\varepsilon\to 0} \int_{\Omega_\tau} \frac{1}{\rho_\varepsilon}\,|J_\varepsilon|^2 dxdt. \tag{34}$$

Finally, claims (30) and (34), and observing that (30) holds when $\tau = 0$, imply (28). $\square$

Corollary 16. *The claim of Lemma 15 also holds when $\Omega = \mathbb{R}^N$.*

Proof. Let, as in the proof of the existence of weak solutions on $\mathbb{R}^N$, ρ_n be the solutions of the problem (E1) on $\Omega_n = B(0,n)$. The available bounds imply that

$$\nabla A(\rho_n) \rightharpoonup \nabla A(\rho) \text{ and } \rho_n K * \rho_n \rightharpoonup \rho K * \rho \text{ in } L^2(\mathbb{R}^N \times [0,\tau])$$

along a subsequence, which for simplicity we assume to be the whole sequence. In the above claim the quantities defined on Ω_n have been extended by zero to $\mathbb{R}^N$. By the monotone convergence theorem

$$E(\rho_n(0)) \to E(\rho(0)) \quad \text{as } n \to \infty. \tag{35}$$

As in the proof of existence, we have $\rho_n(K*\rho_n) \to \rho(K*\rho)$ in $L^1_{loc}(\mathbb{R}^N \times [0,T])$. Using a diagonal procedure, for almost all $\tau \in [0,T]$, there exist a subsequence n_j such that $\rho_{n_j}(\tau) \to \rho(\tau)$ a.e. and $\rho_{n_j}(\tau)(K*\rho_{n_j}(\tau)) \to \rho(\tau)(K*\rho(\tau))$ in $L^1(B(0,k))$ for each integer $k > 0$. Thus

$$\lim_{k\to\infty}\lim_{j\to\infty} \int_{B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) = \int_{\mathbb{R}^N} \rho(\tau) K * \rho(\tau)$$

We claim that the two limits on the left-hand side commute. The proof is elementary and relies on the fact that the integrals are monotone in k, and the following uniform integrability: For every $\varepsilon > 0$ there exist k_0, j_0 such that for all $k > k_0$ and $j > j_0$

$$\int_{R^N \setminus B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) < \varepsilon \tag{36}$$

To show this note that using "mass" conservation

$$\int_{R^N \setminus B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) \le C \int_{R^N \setminus B(0,k)} \rho_{n_j}(\tau) \le C \left(\int_{R^N} \rho(\tau) - \int_{B(0,k)} \rho_{n_j}(\tau) \right)$$

Now pick k_0 large enough so that $\int_{R^N \setminus B(0,k_0)} \rho(\tau) < \frac{\varepsilon}{2C}$ and j_0 so that $\int_{B(0,k_0)} |\rho_{n_j}(\tau) - \rho(\tau)| dx < \frac{\varepsilon}{2C}$ for all $j > j_0$, which we can do thanks to L^1_{loc} convergence of $\rho_{n_j}(\tau)$. This implies (36). Consequently

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) = \lim_{k\to\infty} \lim_{j\to\infty} \int_{B(0,k)} \rho_{n_j}(\tau) K * \rho_{n_j}(\tau) = \int_{\mathbb{R}^N} \rho(\tau) K * \rho(\tau).$$

Note also that since ρ_n are bounded in $L^\infty(0,T,L^2(\mathbb{R}^N))$ for almost all $\tau > 0$ $\rho_{n_j}(\tau) \rightharpoonup \rho(\tau)$ along a subsequence in $L^2(\mathbb{R}^N)$. Since G is convex the mapping $u \mapsto \int_{R^N} G(u)dx$ is weakly lower-semicontinuous with respect to $L^2(\mathbb{R}^N)$ topology. Combining the two claims we conclude

$$\liminf_{n\to\infty} E(\rho_n(\tau)) \geq E(\rho(\tau)).$$

By Lemma 15

$$E(\rho_n(0)) - E(\rho_n(\tau)) \geq \int_0^\tau \int_{\Omega_n} \frac{1}{\rho_n} |\nabla A(\rho_n) - \rho_n \nabla K * \rho_n|^2 dx.$$

The claims we have proven, along with the lower-semicontinuity claim (33) are sufficient to pass to limit $n \to \infty$. $\square$

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