Communications in Mathematical Physics manuscript No.
(will be inserted by the editor)

Well-Posedness Theory for Aggregation Sheets

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Abstract: In this paper, we consider distribution solutions to the aggregation equation $\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,\ \mathbf{u} = -\nabla V * \rho$ in $\mathbb{R}^d$ where the density ρ concentrates on a co-dimension one manifold. We show that an evolution equation for the manifold itself completely determines the dynamics of such solutions. We refer to such solutions aggregation sheets. When the equation for the sheet is linearly well-posed, we show that the fully non-linear evolution is also well-posed locally in time for the class of bi-Lipschitz surfaces. Moreover, we show that if the initial sheet is C^1 then the solution itself remains C^1 as long as it remains Lipschitz. Lastly, we provide conditions on the kernel $g(s) = -\frac{\mathrm{d}V}{\mathrm{d}s}$ that guarantee the solution remains a bi-Lipschitz surface globally in time, and construct explicit solutions that either collapse or blow up in finite time when these conditions fail.

1. Background

Systems with a large number of pairwise interacting particles pervade many disciplines, ranging from models of self-assembly processes in physics and chemistry [21,22,23,29] to models for biological swarming [1,8,16,28] to algorithms for the cooperative control of autonomous vehicles [32]. A simple example of these models employs a first order system of ordinary differential equations for the positions $\mathbf{x}_i(t) \in \mathbb{R}^d$ of N particles,

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \sum_{j \neq i} g\left(\frac{1}{2}|\mathbf{x}_i - \mathbf{x}_j|^2\right)(\mathbf{x}_i - \mathbf{x}_j), \quad 1 \leq i \leq N. \tag{1}$$

The interaction kernel $g(s)$ describes the manner in which particles interact with one another, and therefore depends on the particular application for the model. The formal continuum limit of this system then yields the well-known

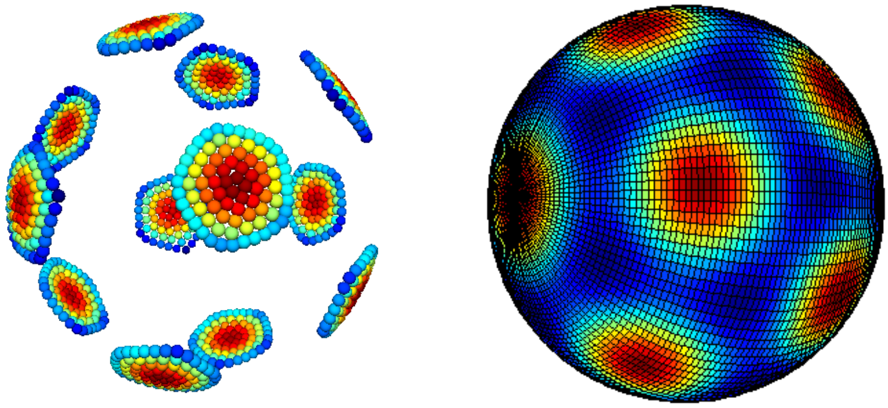

Fig. 1. Left: a "soccer ball" steady-state to the ODE model (1). Right: approximation of the steady state using the co-dimension one continuum model (8), i.e. an approximately spherical surface with color indicating particle density along the manifold.

aggregation equation

$$\frac{\partial \rho}{\partial t}(\mathbf{y},t) + \operatorname{div}(\rho(\mathbf{y},t)\mathbf{u}(\mathbf{y},t)) = 0, \quad \mathbf{y} \in \mathbb{R}^d, \quad t \geq 0,$$
$$\mathbf{u}(\mathbf{y},t) = \int_{\mathbb{R}^d} g\left(\frac{1}{2}|\mathbf{y}-\mathbf{z}|^2\right)(\mathbf{y}-\mathbf{z})\,\rho(\mathbf{z},t)\,\mathrm{d}\mathbf{z}, \tag{2}$$

for the density ρ of particles.

This equation has received significant attention in recent years, and the majority of the analysis largely falls into two categories. More classical treatments focus on densities ρ that are absolutely continuous with respect to Lebesgue measure, such as those lying in an $L^p(\mathbb{R}^d)$ space [6,2,3,4,10,9,14,5]. For densities that merely define a Borel measure on $\mathbb{R}^d$, such as point masses, ideas from optimal transport have proven fruitful for demonstrating the well-posedness of (2) for some classes of interaction kernels [7,19,13,12,20]. However, several recent studies [30,26,15] have found that rings, spheres and more complicated surface-like states naturally occur in the ODE systems (1) and the full PDE models. This suggests that a co-dimension one description of (2) might prove useful for studying such particle distributions (see figure 1). In this context, i.e. when the density must have support of co-dimension one, even the most basic well-posedness results do not yet exist. We therefore provide them in this paper.

Specifically, we analyze distribution solutions to (2) that have support homeomorphic to the $(d-1)$ sphere $\mathcal{S}^{d-1} \subset \mathbb{R}^d$, and so take the form

$$\rho(\mathbf{y},t) := \int_{\mathcal{S}^{d-1}} \delta\left(\mathbf{y} - \Phi(\mathbf{x},t)\right) f(\mathbf{x},t)\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{x}). \tag{3}$$

The map $\Phi(\cdot,t) : \mathcal{S}^{d-1} \to \mathbb{R}^d$ parametrizes the manifold. The function $f(\cdot,t) : \mathcal{S}^{d-1} \to \mathbb{R}$ is such that $f(\mathbf{x},t)\mathrm{d}\mathcal{S}^{d-1}(\mathbf{x}) = \rho_\Phi(\mathbf{x},t)\mathrm{d}\mathcal{H}_\Phi(\mathbf{x})$, where $\rho_\Phi(\mathbf{x},t)$ describes the density of particles along the manifold and $\mathrm{d}\mathcal{H}_\Phi(\mathbf{x})$ denotes the surface measure on the manifold. By (3), we mean that ρ acts as a distribution on $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ as $\rho[\psi] = \int_0^\infty \int_{\mathcal{S}^{d-1}} \psi(\Phi(\mathbf{x},t),t) f(\mathbf{x},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x})\mathrm{d}t$. In the usual manner, we then require that

$$\int_0^\infty \int_{\mathcal{S}^{d-1}} \left(\psi_t + \langle \mathbf{u}, \nabla\psi\rangle\right)(\Phi(\mathbf{x},t),t) f(\mathbf{x},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x})\mathrm{d}t \ = \ 0, \tag{4}$$

$$\mathbf{u}(\mathbf{y}) \ = \ \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w},t)|^2\right)(\mathbf{y} - \Phi(\mathbf{w},t))\, f(\mathbf{w},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}), \tag{5}$$

hold for all $\psi \in C_0^\infty$ in order for (3) to define a formal distribution solution to (2). As $\Phi(\mathbf{x},t)$ gives a Lagrangian parametrization of the manifold, it evolves according to

$$\frac{\partial \Phi}{\partial t}(\mathbf{x},t) \ = \ \mathbf{u}(\Phi(\mathbf{x},t),t) \ = \tag{6}$$

$$\int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\Phi(\mathbf{x},t) - \Phi(\mathbf{w},t)|^2\right)(\Phi(\mathbf{x},t) - \Phi(\mathbf{w},t))\, f(\mathbf{w},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

Combining (4) and (6) with the fact that

$$\frac{\partial}{\partial t}\left\{\psi(\Phi(\mathbf{x},t),t)\right\} = \left(\psi_t + \left\langle \frac{\partial \Phi}{\partial t}, \nabla\psi \right\rangle\right)(\Phi(\mathbf{x},t),t),$$

we discover f must satisfy

$$0 = \int_0^\infty \int_{\mathcal{S}^{d-1}} \frac{\partial}{\partial t}\left\{\psi(\Phi(\mathbf{x},t),t)\right\} f(\mathbf{x},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x})\mathrm{d}t =$$
$$- \int_0^\infty \int_{\mathcal{S}^{d-1}} \psi(\Phi(\mathbf{x},t),t) \frac{\partial f}{\partial t}(\mathbf{x},t)\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x})\mathrm{d}t$$

for all ψ, whence

$$f(\mathbf{x},t) \equiv f(\mathbf{x},0). \tag{7}$$

Therefore, given an initial density

$$\rho(\mathbf{y},0) = \rho_0(\mathbf{y}) = \int_{\mathcal{S}^{d-1}} \delta\left(\mathbf{y} - \Phi_0(\mathbf{w})\right) f_0(\mathbf{w})\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}),$$

we formally obtain a distribution solution to (2) by evolving the surface according to

$$\frac{\partial \Phi}{\partial t} = \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\Phi(\mathbf{x},t) - \Phi(\mathbf{w},t)|^2\right)(\Phi(\mathbf{x},t) - \Phi(\mathbf{w},t))\, f_0(\mathbf{w})\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}), \tag{8}$$

if $\mathbf{x} \in \mathcal{S}^{d-1}$ and $t > 0$, together with the initial condition $\Phi(\mathbf{x},0) = \Phi_0(\mathbf{x})$. Conversely, provided the integro-differential equation (IDE) (8) has a solution that results in a sufficiently regular velocity field (6), we can justify the preceding computations to obtain distribution solutions to the original equation.

Variants of the IDE (8) appear in numerous contexts. The classical Birkhoff-Rott equation in two dimensions results from taking $g(s) = -(\pi s)^{-1}$, then rotating the resulting velocity field to make it incompressible. Similarly, in [26] the authors derived a generalization of the two dimensional Birkhoff-Rott equation directly from the principle of mass conservation. This results in velocity fields of mixed type that contain both an incompressible contribution and a gradient contribution. The IDE (8), then, extends their generalized equation to arbitrary dimensions $d \geq 2$ in the case when the incompressible contribution vanishes. Although we do not consider the fully general case, local well-posedness for two dimensional mixed kernels does follow from our arguments as well.

Our primary concern insted lies in developing a well-posedness theory for (8). To this end, we first demonstrate that solutions to (8) exist locally in time when the initial data $\Phi_0(\mathbf{x})$ defines a Lipschitz homeomorphism. Specifically, if the IDE (8) is linearly well-posed we prove that the fully non-linear problem is also well-posed across the full range of linearly well-posed kernels. We also show that if $\Phi_0 \in C^1$ then the solution itself remains C^1 as long as it remains Lipschitz. We then address issues regarding continuation and global existence of solutions. We prove that a unique continutation exists provided Φ and its inverse remain Lipschitz, and by explicit construction we show that finite time singularities of each type of may occur. For kernels with an attractive singularity at the origin, we generalize the results for $L^\infty(\mathbb{R}^d)$ [3] and general $L^p(\mathbb{R}^d)$ solutions [4] to (2) that show finite time singularity occurs if and only if the kernel is Osgood. Finally, for a subclass of the natural potentials studied in [7] [4] [3] we show that the solution exists globally when the kernel has a repulsive singularity at the origin.

To make our hypotheses on the interaction kernel $g(s)$ for these results precise, we recall that the linear theory from [30,15] shows the solution $\Phi(\mathbf{x}) \equiv R\mathbf{x}$ is linearly well-posed only if

$$g\left(R^2(1-s)\right)(1-s^2)^{\frac{d-3}{2}} \in L^1([-1,1]). \tag{9}$$

For simplicity, we assume the kernel behaves as a power law, $g(s) = \mathcal{O}(s^p)$, near the origin, although our arguments apply in a more general context. The linear well-posedness condition then enforces

$$p > \frac{1-d}{2}. \tag{10}$$

This suggests the following assumptions on the interaction kernel:

Definition 1. *Let $g(s) : \mathbb{R}^+ \to \mathbb{R}$. Then $g(s)$ defines an* ***admissible interaction kernel*** *if $g \in C^1(\mathbb{R}^+ \setminus \{0\})$, and there exist constants $C > 0, \delta > 0, p > \frac{1-d}{2}$ such that*

$$\max\{|g(s)|, |sg'(s)|\} \leq Cs^p \quad \forall\, s \in (0, \delta). \tag{11}$$

These hypotheses suffice to establish local well-posedness, and are sufficiently mild to still include many of the kernels that prove relevant for applications.

We shall demonstrate well-posedness of the IDE in the space $C^{0,1}(\mathcal{S}^{d-1})$ of Lipschitz functions over the sphere $\mathcal{S}^{d-1}$, where $C^{0,1}(\mathcal{S}^{d-1})$ has the usual norm

$$||\Phi||_{C^{0,1}} := \max_{\mathcal{S}^{d-1}} |\Phi(\mathbf{x})| + \mathrm{Lip}[\Phi], \quad \mathrm{Lip}[\Phi] := \sup_{\mathbf{x} \neq \mathbf{w}} \frac{|\Phi(\mathbf{x}) - \Phi(\mathbf{w})|}{|\mathbf{x} - \mathbf{w}|}. \tag{12}$$

To allow for the singularity in $g(s)$ at zero, we restrict attention to initial data $\Phi_0(\mathbf{x})$ lying in the subset $\mathcal{O}_M \subset C^{0,1}(\mathcal{S}^{d-1})$ of all functions where both $\mathrm{Lip}[\Phi] \leq M$ and $\mathrm{Lip}[\Phi^{-1}] \leq M$,

$$\mathcal{O}_M := \left\{ \Phi \in C^{0,1}(\mathcal{S}^{d-1}) : \frac{1}{M} \leq \inf_{\mathbf{x} \neq \mathbf{w}} \frac{|\Phi(\mathbf{x}) - \Phi(\mathbf{w})|}{|\mathbf{x} - \mathbf{w}|} \leq \mathrm{Lip}[\Phi] \leq M \right\}. \tag{13}$$

This class of initial data proves less restrictive than the requirements on initial data that appear in related problems. As we enforce regularity in the kernel $g(s)$ this allows us to relax the regularity requirements on the initial sheet itself, and this makes our task somewhat easier. In particular, we need not assume any regularity in addition to boundedness of derivatives along the sheet. Similar results for vortex patches [17, 18] require Hölder regularity in derivatives, and results for the Birkhoff-Rott equation typically require analyticity [25] or other additional regularity hypotheses [31]. Proving an existence result for more singular kernels, such as the Newtonian potential, would therefore require a different approach than we adopt here, so we make no effort in this direction. Also in contrast to many studies on the Birkhoff-Rott equation, we consider compact sheets instead of sheets homeomorphic to the real line. This also causes our approach to demonstrating existence to differ to a large extent.

The remainder of the paper proceeds as follows: in section 2 we first establish the necessary estimates on the nonlocal term in the IDE, and this allows us derive local existence in section 3 using a modified version of simple Picard iteration; subsection 3.1 addresses issues regarding differentiability of solutions and the final section addresses questions regarding the long term behavior of solutions; we finish with some concluding remarks.

2. Elementary Properties and A-Priori Estimates

Like its co-dimension zero counterpart (2), solutions to the IDE (8) exhibit several conserved quantities. Foremost, it formally expresses conservation of mass in that

$$M_\rho := \int_{\mathbb{R}^d} \rho(\mathbf{x}, t) \equiv \int_{\mathcal{S}^{d-1}} f_0(\mathbf{x}) \, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x}) \tag{14}$$

for all time. Moreover, we have conservation of center of mass

$$\int_{\mathbb{R}^d} \mathbf{x}\rho(\mathbf{x}, t) = \int_{\mathcal{S}^{d-1}} \Phi(\mathbf{x}, t) \equiv \int_{\mathcal{S}^{d-1}} \Phi_0(\mathbf{x}), \tag{15}$$

which we assume equals zero throughout the remainder of the paper. Potential energy also dissipates along solutions. Indeed, let $V(s)$ denote a potential for the evolution, i.e. that $\frac{\mathrm{d}V}{\mathrm{d}s} = -g(s)$, and define

$$E_\Phi(t) := \frac{1}{2} \int_{\mathcal{S}^{d-1} \times \mathcal{S}^{d-1}} V\left(\frac{1}{2} |\Phi(\mathbf{x}, t) - \Phi(\mathbf{z}, t)|^2 \right) f_0(\mathbf{x}) f_0(\mathbf{z}) \, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{x}) \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})$$

A simple calculation then formally yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\Phi(t) = - \int_{\mathcal{S}^{d-1}} \left| \frac{\partial \Phi}{\partial t} \right|^2. \tag{16}$$

These statements can be readily justified using the arguments that follow.

We begin by recalling a standard theorem that will prove useful on several occasions, i.e. the Funk-Hecke formula for spherical harmonics [24]. Our desire to satisfying the integrability hypothesis for the formula further motivates for the growth rate (10) on $g(s)$ near the origin.

Theorem 1. *(Funk-Hecke Theorem)*

Let $h(s)(1-s^2)^{\frac{d-3}{2}} \in L^1([-1,1])$. Then for any $\mathbf{x} \in \mathcal{S}^{d-1}$ and any spherical harmonic $S^l(\mathbf{x})$ of degree l,

$$\int_{\mathcal{S}^{d-1}} h(\langle \mathbf{x},\mathbf{w}\rangle) S^l(\mathbf{w}) \, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) = \mathrm{vol}(\mathcal{S}^{d-2}) \left(\int_{-1}^{1} h(s)(1-s^2)^{\frac{d-3}{2}} P_{l,d}(s) \, \mathrm{d}s \right) S^l(\mathbf{x}),$$

where $P_{l,d}(s)$ denotes the Gegenbauer polynomial $P_l^{(\frac{d}{2}-1)}(s)$ from [27] normalized to $P_{l,d}(1)=1$.

Before turning our attention to estimating the nonlocality in (8), we first illustrate utilize theorem 1 to construct the simple but important class of exact spherical solutions to (8). These solutions will later prove useful in determing how solutions to (8) behave for large times.

Example 1. (Spherical Solutions) Let $\Phi(\mathbf{x},t) = R(t)\mathbf{x}$ for $R(t) > 0$ and $f_0(\mathbf{w}) \equiv 1$. Substituting this expression into (8) yields

$$\frac{\mathrm{d}R}{\mathrm{d}t}\mathbf{x} = R(t) \int_{\mathcal{S}^{d-1}} g\left(\frac{R(t)^2}{2}|\mathbf{x}-\mathbf{w}|^2\right)(\mathbf{x}-\mathbf{w}) \, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

The facts that $\mathbf{w}$ is a spherical harmonic of degree one and that $P_{l,d}(s) = s$ combine with the Funk-Hecke theorem for $l = 0, 1$ to show

$$\frac{\mathrm{d}R}{\mathrm{d}t}\mathbf{x} = \mathrm{vol}(\mathcal{S}^{d-2}) R(t) \left[\int_{-1}^{1} g\left(R(t)^2(1-s)\right)(1-s)(1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s \right] \mathbf{x}.$$

Therefore $\Phi(\mathbf{x},t) = R(t)\mathbf{x}$ defines a solution to (8) if $R(t)$ solves the ordinary differential equation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \mathrm{vol}(\mathcal{S}^{d-2}) R(t) \int_{-1}^{1} g\left(R(t)^2(1-s)\right)(1-s)(1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s. \qquad (17)$$

The case $l = 0$ of theorem 1 also proves useful in establishing the following two technical lemmas. Their proof constitutes the majority of the effort needed to establish theorem 2, as they suffice to show the right hand side of (8) is locally Lipschitz in $C^0(\mathcal{S}^{d-1})$. A combination of Picard iteration and a-posteriori estimates then yields the theorem. The first lemma estimates expressions of the form

$$H(\mathbf{y}) := \int_{\mathcal{S}^{d-1}} h\left(\frac{1}{2}|\mathbf{y}-\Phi(\mathbf{w})|^2\right) f_0(\mathbf{w}) \, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) \qquad (18)$$

for all $\mathbf{y} \in \mathbb{R}^d$, where we envision $h(s) = g(s)$ or $h(s) = sg'(s)$ so that h satisfies a hypothesis similar to (11).

Lemma 1. *Let $h(s) : \mathbb{R}^+ \to \mathbb{R}$ be locally bounded away from zero, and suppose $\exists K > 0, \sigma > 0, q > \frac{1-d}{2}$ with $|h(s)| \le Ks^q$ for all $s \in (0, \sigma)$. If $\Phi(\mathbf{x}) \in \mathcal{O}_M$ then*

$$|H(\mathbf{y})| \le C(h, d_{\mathbf{y}}, M)||f_0||_{L^\infty}.$$

The constant C depends only on h, M and $d_{\mathbf{y}} := \min_{\mathbf{x}\in\mathcal{S}^{d-1}} |\mathbf{y} - \Phi(\mathbf{x})|$, and increases with both $d_{\mathbf{y}}$ and M.

Proof. Fix $\mathbf{y} \in \mathbb{R}^d$ and decompose

$$H(\mathbf{y}) = \int_{|\mathbf{y}-\Phi(\mathbf{w})|\ge\sqrt{2\sigma}} + \int_{|\mathbf{y}-\Phi(\mathbf{w})|<\sqrt{2\sigma}} := \mathrm{I} + \mathrm{II}.$$

Let $\mathbf{x}_0$ denote a minimizer of $|\mathbf{y} - \Phi(\mathbf{x})|$ over $\mathbf{x} \in \mathcal{S}^{d-1}$, so that $d_{\mathbf{y}} = |\mathbf{y} - \Phi(\mathbf{x}_0)|$. Due to the boundedness of h away from zero,

$$|\mathrm{I}| \le \mathrm{vol}(\mathcal{S}^{d-1})||h||_{L^\infty([\sigma,2M^2+d_{\mathbf{y}}^2])}||f_0||_{L^\infty(\mathcal{S}^{d-1})}.$$

As for the second integral, the growth hypothesis on h near zero implies that

$$|\mathrm{II}| \le K2^{-q}||f_0||_{L^\infty(\mathcal{S}^{d-1})} \int_{\mathcal{S}^{d-1}} |\mathbf{y} - \Phi(\mathbf{w})|^{2q} \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

If $q \ge 0$ then $|\mathrm{II}| \le K2^{-q}||f_0||_{L^\infty(\mathcal{S}^{d-1})}(d_{\mathbf{y}} + 2M)^{2q}\mathrm{vol}(\mathcal{S}^{d-1})$, so assume that $q < 0$. The facts that $|\mathbf{y} - \Phi(\mathbf{w})| \ge \frac{1}{2}|\Phi(\mathbf{x}_0) - \Phi(\mathbf{w})|$ and that Φ^{-1} is Lipschitz with constant M suffice to show

$$|\mathrm{II}| \le K2^{-2q}||f_0||_{L^\infty(\mathcal{S}^{d-1})}M^{-2q} \int_{\mathcal{S}^{d-1}} (1 - \langle \mathbf{x}_0, \mathbf{w}\rangle)^q \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

As $q + \frac{d-3}{2} > -1$, the case $l = 0$ of theorem 1 allows us to compute the last term,

$$\int_{\mathcal{S}^{d-1}} (1 - \langle \mathbf{x}_0, \mathbf{w}\rangle)^q \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) = \mathrm{vol}(\mathcal{S}^{d-2}) \int_{-1}^{1} (1 - s)^q(1 - s^2)^{\frac{d-3}{2}} \,\mathrm{d}s < \infty.$$

□

The second lemma allows us to differentiate expressions of the form

$$\mathbf{v}^i_\Phi(\mathbf{y}) := \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right) (\mathbf{y}^i - \Phi^i(\mathbf{w}))f_0(\mathbf{w}) \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}), \tag{19}$$

for any $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^d)^t \in \mathbb{R}^d$, where the subscript notation $\mathbf{v}^i_\Phi(\mathbf{y})$ indicates the possibly changing dependence on $\Phi(\mathbf{x})$. A combination of both lemmas then establishes the required properties of the right hand side of (8) as corollaries.

Lemma 2. *Suppose $g(s)$ defines an admissible kernel and $f_0 \in L^\infty(\mathcal{S}^{d-1})$. If $\Phi(\mathbf{x}) \in \mathcal{O}_M$ and $\Psi(\mathbf{x}) \in C^{0,1}(\mathcal{S}^{d-1})$, then for fixed $\mathbf{y} \in \mathbb{R}^d$*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})|^2\right) (\mathbf{y}^i - \Phi^i(\mathbf{w}) - \epsilon\Psi^i(\mathbf{w}))f_0(\mathbf{w}) \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) \,\Big|_{\epsilon=0}$$

$$= - \int_{\mathcal{S}^{d-1}} \Bigg[g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right) \Psi^i(\mathbf{w}) + g'\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right) \langle \mathbf{y} - \Phi(\mathbf{w}), \Psi(\mathbf{w})\rangle (\mathbf{y}^i - \Phi^i(\mathbf{w}))\Bigg] f_0(\mathbf{w}) \,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

Proof. For fixed $1 \le i \le d$ and $\mathbf{y} \in \mathbb{R}^d$ consider the quantity

$$\frac{\mathbf{v}^i_{\Phi+\epsilon\Psi}(\mathbf{y}) - \mathbf{v}^i_{\Phi}(\mathbf{y})}{\epsilon} =$$
$$\frac{1}{\epsilon}\int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})|^2\right)(\mathbf{y}^i - \Phi^i(\mathbf{w}) - \epsilon\Psi^i(\mathbf{w}))f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) -$$
$$\frac{1}{\epsilon}\int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right)(\mathbf{y}^i - \Phi^i(\mathbf{w}))f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}).$$

Let g_ϵ denote the integrand. As g is differentiable away from zero and Φ is one-to-one it follows that

$$g_\epsilon \to -\left[g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right)\Psi^i(\mathbf{w}) + \right.$$
$$\left. g'\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right)\langle \mathbf{y} - \Phi(\mathbf{w}), \Psi(\mathbf{w})\rangle (\mathbf{y}^i - \Phi^i(\mathbf{w}))\right] f_0(\mathbf{w})$$

for almost every $\mathbf{w} \in \mathcal{S}^{d-1}$. The aim thus becomes to conclude that in fact

$$\int_{\mathcal{S}^{d-1}} g_\epsilon \to -\int_{\mathcal{S}^{d-1}}\left[g\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right)\Psi^i(\mathbf{w}) + \right.$$
$$\left. g'\left(\frac{1}{2}|\mathbf{y} - \Phi(\mathbf{w})|^2\right)\langle \mathbf{y} - \Phi(\mathbf{w}), \Psi(\mathbf{w})\rangle (\mathbf{y}^i - \Phi^i(\mathbf{w}))\right] f_0(\mathbf{w}). \quad (20)$$

If $d_{\mathbf{y}} = \min_{\mathcal{S}^{d-1}} |\mathbf{y} - \Phi(\mathbf{w})| > 0$, this immediately follows as $g \in C^1(\mathbb{R}^+ \setminus \{0\})$ and the dominated convergence theorem. The difficulty comes when $\mathbf{y} = \Phi(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathcal{S}^{d-1}$. In this case, it suffices show that the g_ϵ are uniformly integrable: for any $\gamma > 0$ there exists $N > 0$ so that

$$\sup_\epsilon \int_{\mathcal{S}^{d-1}} |g_\epsilon|\mathbf{1}_{\{|g^\epsilon| > N\}} < \gamma.$$

The Vitali convergence theorem then yields the desired result.

To show uniform integrability, let $\mathbf{z}^\epsilon := \Phi(\mathbf{x}_0) - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})$ and $\mathbf{z} := \Phi(\mathbf{x}_0) - \Phi(\mathbf{w})$. For fixed ϵ let $A := \{||\epsilon\Psi||_\infty \le \frac{|\mathbf{z}|}{2}\}$ and write

$$g_\epsilon = g_\epsilon \mathbf{1}_A + g_\epsilon \mathbf{1}_{\mathcal{S}^{d-1}\setminus A} := g^1_\epsilon + g^2_\epsilon$$
$$g_\epsilon = \left[-g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right)\Psi^i(\mathbf{w}) + \frac{g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right) - g\left(\frac{1}{2}|\mathbf{z}|^2\right)}{\epsilon}\mathbf{z}^i\right] f_0(\mathbf{w}).$$

If $\mathbf{w} \in A$ then $|\mathbf{z}| \le 2|\mathbf{z}^\epsilon|$. To estimate g^1_ϵ, the mean value theorem furnishes $s_0 \in (\frac{|\mathbf{z}|^2}{8}, \frac{9|\mathbf{z}|^2}{8})$ with $g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right) - g\left(\frac{1}{2}|\mathbf{z}|^2\right) = \frac{1}{2}g'(s_0)(|\mathbf{z}^\epsilon|^2 - |\mathbf{z}|^2)$. Therefore

$$|g^1_\epsilon| \le ||f_0||_{L^\infty(\mathcal{S}^{d-1})}||\Psi||_{L^\infty(\mathcal{S}^{d-1})}\left(\left|g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right)\right| + |g'(s_0)|\frac{5|\mathbf{z}|^2}{4}\right).$$

To estimate g_ϵ^2, since $\mathbf{w} \notin A$ then $\frac{|\mathbf{z}|}{|\epsilon|} \leq 2||\Psi||_\infty$, so that

$$
\begin{aligned}
|g_\epsilon^2| &\leq ||f_0||_{L^\infty(\mathcal{S}^{d-1})} \left(\left| g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right) \right| ||\Psi||_\infty + \left| g\left(\frac{1}{2}|\mathbf{z}^h|^2\right) \right| \frac{|\mathbf{z}|}{|\epsilon|} + \left| g\left(\frac{1}{2}|\mathbf{z}|^2\right) \right| \frac{|\mathbf{z}|}{|\epsilon|} \right) \\
&\leq 3||f_0||_{L^\infty(\mathcal{S}^{d-1})}||\Psi||_{L^\infty(\mathcal{S}^{d-1})} \left(\left| g\left(\frac{1}{2}|\mathbf{z}^h|^2\right) \right| + \left| g\left(\frac{1}{2}|\mathbf{z}|^2\right) \right| \right).
\end{aligned}
$$

Combining these estimates yields

$$
|g_\epsilon| \leq C_1 \left(\left| g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right) \right| + \left| g\left(\frac{1}{2}|\mathbf{z}|^2\right) \right| + |g'(s_0)||\mathbf{z}|^2 \right) := \mathrm{I} + \mathrm{II} + \mathrm{III}.
$$

for some absolute constant C_1 that depends only on $||f_0||_{L^\infty(\mathcal{S}^{d-1})}, ||\Psi||_{L^\infty(\mathcal{S}^{d-1})}$. As a linear combination of uniformly integrable functions is uniformly integrable, it suffices to show the uniform of $\mathrm{I} - \mathrm{III}$ individually.

To show the uniform integrability of I, as in the proof of lemma 1 let $\mathbf{x}_0^\epsilon$ denote a minimizer of $|\Phi(\mathbf{x}_0) - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})|$ over $\mathbf{w} \in \mathcal{S}^{d-1}$, and decompose

$$
\int_{\{\mathrm{I}>N\}} \mathrm{I}\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) \leq \int_{\{\mathrm{I}>N\}\cap\{|\mathbf{z}^\epsilon|\geq\sqrt{2\delta}\}} + \int_{\{\mathrm{I}>N\}\cap\{|\mathbf{z}^\epsilon|\leq\sqrt{2\delta}\}}
$$

If $|\mathbf{z}^\epsilon| \geq \sqrt{2\delta}$ then $\mathrm{I} \leq ||g||_{L^\infty([\delta,(2M+||\Psi||_\infty)^2])} := K(M, ||\Psi||_\infty)$ whenever $|\epsilon| \leq 1$. When $|\mathbf{z}^\epsilon| < \sqrt{2\delta}$, the growth rate of $g(s)$ near zero demonstrates

$$
\mathrm{I} \leq C2^{-p}|\mathbf{z}^\epsilon|^{2p}.
$$

If $p \geq 0$ the dominated convergence theorem gives the desired result. If $p < 0$, the fact that $|\mathbf{z}^\epsilon| \geq \frac{1}{2}|\Phi(\mathbf{x}_0^\epsilon) + \epsilon\Psi(\mathbf{x}_0^\epsilon) - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})|$ yields

$$
\mathrm{I} \leq C2^{-3p}|\Phi(\mathbf{x}_0^\epsilon) + \epsilon\Psi(\mathbf{x}_0^\epsilon) - \Phi(\mathbf{w}) - \epsilon\Psi(\mathbf{w})|^{2p}.
$$

Now, as $\Psi \in C^{0,1}(\mathcal{S}^{d-1})$ and $\Phi(\mathbf{x}) \in \mathcal{O}_M$, for all ϵ sufficiently small $\Phi(\mathbf{x}) + \epsilon\Psi(\mathbf{x}) \in \mathcal{O}_{2M}$ as well. Therefore, for $|\mathbf{z}^\epsilon| < \sqrt{2\delta}$

$$
\mathrm{I} \leq C2^{-3p}(2M)^{-2p}|\mathbf{x}_0^\epsilon - \mathbf{w}|^{2p} = C2^{-4p}M^{-2p}(1 - \langle \mathbf{x}_0^\epsilon, \mathbf{w}\rangle)^p := f_M(\langle \mathbf{x}_0^\epsilon, \mathbf{w}\rangle).
$$

Summarizing the preceding, when $N > 0$

$$
\begin{aligned}
\int_{\{\mathrm{I}>N\}} \mathrm{I} &\leq \int_{\{\mathrm{I}>N\}\cap\{|\mathbf{z}^\epsilon|\geq\sqrt{2\delta}\}} + \int_{\{\mathrm{I}>N\}\cap\{|\mathbf{z}^\epsilon|\leq\sqrt{2\delta}\}} \\
&\leq K(M, ||\Psi||_\infty) \int_{\{K>N\}} + \int_{\{f_M>N\}} f_M.
\end{aligned}
$$

By the case $l = 0$ of theorem 1,

$$
\int_{\{f_M>N\}} f_M = C2^{-4p}M^{-2p}\mathrm{vol}(\mathcal{S}^{d-2}) \int_{-1}^{1} (1-s)^p(1-s^2)^{\frac{d-3}{2}} \mathbf{1}_{\{(1-s)^p>N\}}\, \mathrm{d}s.
$$

Taking N sufficiently large, independently of ϵ, shows that

$$
\sup_\epsilon \int_{\mathcal{S}^{d-1}} \left| g\left(\frac{1}{2}|\mathbf{z}^\epsilon|^2\right) \right| \mathbf{1}_{\{\mathrm{I}>N\}} < \gamma
$$

as desired. For II, as $\Phi(\mathbf{x}) \in \mathcal{O}_M$, by lemma 1 $\int_{\mathcal{S}^{d-1}} \mathrm{II} \leq C(g,0,M) < \infty$. By the dominated convergence theorem,

$$\int_{\mathcal{S}^{d-1}} \mathrm{II}\mathbf{1}_{\{\mathrm{II}>N\}} \to 0$$

uniformly in ϵ as well. For III, again decompose

$$\int_{\{\mathrm{III}>N\}} \mathrm{III}\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) \leq \int_{\{\mathrm{III}>N\}\cap\{|s_0|\geq\sqrt{2\delta}\}} + \int_{\{\mathrm{III}>N\}\cap\{|s_0|\leq\sqrt{2\delta}\}},$$

and recall that $s_0 \in (\frac{|\mathbf{z}|^2}{8}, \frac{9|\mathbf{z}|^2}{8})$. As in lemma 1, each term can be dominated by an integrable function that does not depend on ϵ, so III is uniformly integrable as well.

□

Now, let $\mathbf{v}_\Phi(\mathbf{y}) : \mathbb{R}^d \to \mathbb{R}^d$ denote the right hand side of (8) evaluated at an arbitrary point $\mathbf{y} \in \mathbb{R}^d$,

$$\mathbf{v}_\Phi(\mathbf{y}) := (\mathbf{v}_\Phi^1(\mathbf{y}), \ldots, \mathbf{v}_\Phi^d(\mathbf{y}))^t$$

with $\mathbf{v}_\Phi^i(\mathbf{y})$ given by (19). By taking $\Psi(\mathbf{x}) \equiv -\mathbf{e}_j$ for $1 \leq j \leq d$ in lemma 2, we conclude

$$\begin{aligned}\left[\nabla \mathbf{v}_\Phi\right](\mathbf{y}) = \int_{\mathcal{S}^{d-1}} \Bigg[g\left(\frac{1}{2}|\mathbf{y}-\Phi(\mathbf{w})|^2\right) \mathrm{Id} + \\ g'\left(\frac{1}{2}|\mathbf{y}-\Phi(\mathbf{w})|^2\right)(\mathbf{y}-\Phi(\mathbf{w}))(\mathbf{y}-\Phi(\mathbf{w}))^t\Bigg] f_0(\mathbf{w})\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}), \quad (21)\end{aligned}$$

where Id denotes the $d \times d$ identity matrix. Applying lemma 1 then shows the matrix norm $||\nabla \mathbf{v}_\Phi||_2(\mathbf{y}) \leq C(g, g', d_{\mathbf{y}}, M)||f_0||_\infty$, for some constant C that increases with $d_{\mathbf{y}}$. The mean value theorem then yields

Corollary 1. *Let $\Phi(\mathbf{x}) \in \mathcal{O}_M$. Then for any two points $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d$*

$$|\mathbf{v}_\Phi(\mathbf{y}_1) - \mathbf{v}_\Phi(\mathbf{y}_2)| \leq C(g, g', \max\{d_{\mathbf{y}_1}, d_{\mathbf{y}_2}\}, M)||f_0||_\infty|\mathbf{y}_1 - \mathbf{y}_2|. \quad (22)$$

Similarly, fix $\mathbf{y} \in \mathbb{R}^d, \Phi, \Psi \in C^{0,1}(\mathcal{S}^{d-1})$ and suppose that for $0 \leq \epsilon \leq 1$ the line $L_\epsilon := \epsilon\Psi + (1-\epsilon)\Phi \in \mathcal{O}_M$. We can then use lemma 2 to deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbf{v}_{L_\epsilon}^i(\mathbf{y}) = -\int_{\mathcal{S}^{d-1}} \Bigg[g\left(\frac{1}{2}|\mathbf{y}-L_\epsilon(\mathbf{w})|^2\right)(\Psi^i(\mathbf{w}) - \Phi^i(\mathbf{w})) + \\ g'\left(\frac{1}{2}|\mathbf{y}-L_\epsilon(\mathbf{w})|^2\right)\langle \mathbf{y}-L_\epsilon(\mathbf{w}), \Psi(\mathbf{w})-\Phi(\mathbf{w})\rangle\,(\mathbf{y}^i - L_\epsilon^i(\mathbf{w}))\Bigg] f_0(\mathbf{w})\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w})\end{aligned}$$

An application of lemma 1 then shows

$$\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbf{v}_{L_\epsilon}^i\right| \leq C(g, g', d_{\mathbf{y}}, M)||f_0||_\infty \max_{\mathcal{S}^{d-1}}|\Psi(\mathbf{x}) - \Phi(\mathbf{x})|,$$

where $d_{\mathbf{y}} = \min_{\mathcal{S}^{d-1}} |\mathbf{y} - L_\epsilon|$ and the constant C depends only on L_ϵ through M. For $\mathbf{y} \in \mathbb{R}^d$ fixed, the fundamental theorem of calculus then shows

$$|\mathbf{v}^i_\Psi(\mathbf{y}) - \mathbf{v}^i_\Phi(\mathbf{y})| = \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathbf{v}^i_{L_\epsilon}(y)\, \mathrm{d}\epsilon \right| \leq C(g, g', d_{\mathbf{y}}, M)||f_0||_\infty \max_{\mathcal{S}^{d-1}} |\Psi(\mathbf{x}) - \Phi(\mathbf{x})|.$$

We therefore get the following corollary

Corollary 2. *Let $\Phi, \Psi \in C^{0,1}$ be such that the line $L_\epsilon := \epsilon\Psi + (1-\epsilon)\Phi \in \mathcal{O}_M$ for all $0 \leq \epsilon \leq 1$. Then for any $\mathbf{y} \in \mathbb{R}^d$,*

$$|\mathbf{v}_\Psi(\mathbf{y}) - \mathbf{v}_\Phi(\mathbf{y})| \leq C(g, g', d_{\mathbf{y}}, M)||f_0||_\infty \max_{\mathcal{S}^{d-1}} |\Psi(\mathbf{x}) - \Phi(\mathbf{x})|. \qquad (23)$$

The arguments in the proof of 2 also establish the following lemma that demonstrates continuity of the gradient $[\nabla \mathbf{v}_\Phi](\mathbf{y})$ of the Eulerian velocity field. To avoid redundancy, we leave the proof as an exercise for the reader.

Lemma 3. *Suppose $g(s)$ defines an admissible kernel and $f_0 \in L^\infty(\mathcal{S}^{d-1})$. If $\Phi(\mathbf{x}) \in \mathcal{O}_M$, then the matrix $[\nabla \mathbf{v}_\Phi](\mathbf{y})$ given by (21) is continuous as a function on $\mathbb{R}^d$.*

3. Local Well-Posedness

We may now proceed to demonstrate our main result, i.e. local existence for the IDE (8)—

Theorem 2. *(Local Well-Posedness for the IDE) Let $g(s)$ define an admissible kernel, $f_0 \in L^\infty$ and $\Phi_0(\mathbf{x}) \in \mathcal{O}_{M/2}$. Then there exists $T = T(g, g', M, ||f_0||_\infty)$ such that the IDE (8) has a solution*

$$\Phi(\mathbf{x}, t) \in C^1([-T, T]; C^0(\mathcal{S}^{d-1})) \cap C([-T, T]; C^{0,1}(\mathcal{S}^{d-1}) \cap \mathcal{O}_M).$$

If $\Psi(\mathbf{x}, t) \in C([-T', T']; C^{0,1}(\mathcal{S}^{d-1}))$ denotes another solution for any $T' \leq T$, then $\Phi(\mathbf{x}, t) \equiv \Psi(\mathbf{x}, t)$ on $[-T', T']$.

Fix an initial datum $\Phi_0(\mathbf{x}) \in \mathcal{O}_{M/2}$ and let $\Phi(\mathbf{x}, t) \in C([0, T]; C^{0,1}(\mathcal{S}^{d-1}))$. In the usual manner, define a mapping $A[\Phi]$ by

$$\Phi(\mathbf{x}, t) \to A[\Phi](\mathbf{x}, t) := \Phi_0(\mathbf{x}) + \\ \int_0^t \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\Phi(\mathbf{x}, s) - \Phi(\mathbf{w}, s)|^2\right) (\Phi(\mathbf{x}, s) - \Phi(\mathbf{w}, s)) f_0(\mathbf{w})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w})\mathrm{d}s,$$

so that it suffices to show this mapping has a fixed point. To this end, we need to prove the following three propositions regarding the mapping, and may then proceed to apply straightforward Picard iteration.

Proposition 1. *Let $\Phi_0(\mathbf{x}) \in \mathcal{O}_{M/2}$ and $\mathrm{Lip}[\Phi - \Phi_0](t) \leq \min\left\{\frac{M}{2}, \frac{1}{M}\right\}$ for all $t \in [0, T]$. Then $\Phi(\mathbf{x}, t) \in \mathcal{O}_M$ for all $t \in [0, T]$.*

Proof. By the triangle inequality, $|\Phi(\mathbf{x},t)-\Phi(\mathbf{w},t)| = |\Phi(\mathbf{x},t)-\Phi_0(\mathbf{x})-(\Phi(\mathbf{w},t)-\Phi_0(\mathbf{w}))+\Phi_0(\mathbf{x})-\Phi_0(\mathbf{w})| \leq \mathrm{Lip}[\Phi-\Phi_0](t)|\mathbf{x}-\mathbf{w}|+|\Phi_0(\mathbf{x})-\Phi_0(\mathbf{w})| \leq M|\mathbf{x}-\mathbf{w}|$. By the reverse triangle inequality, $|\Phi(\mathbf{x},t)-\Phi(\mathbf{w},t)| \geq |\Phi_0(\mathbf{x})-\Phi_0(\mathbf{w})||\mathbf{x}-\mathbf{w}| - \mathrm{Lip}[\Phi-\Phi_0](t)|\mathbf{x}-\mathbf{w}| \geq (\frac{2}{M}-\mathrm{Lip}[\Phi-\Phi_0](t))|\mathbf{x}-\mathbf{w}| \geq \frac{1}{M}|\mathbf{x}-\mathbf{w}|$.

□

Proposition 2. *Let $\Phi(\mathbf{x},t) \in \mathcal{O}_M$ for all $t \in [0,T]$. If $T = T(g,g',M,||f_0||_\infty)$ is sufficiently small, then*

$$||A[\Phi]-\Phi_0||_{C^{0,1}}(t) < \min\left\{\frac{M}{2},\frac{1}{M}\right\}.$$

for all $t \in [0,T]$.

Proof. Set $h(\mathbf{x},t) := A[\Phi](\mathbf{x},t)-\Phi_0(\mathbf{x}) = \int_0^t \mathbf{v}_\Phi(\Phi(\mathbf{x},s))\ \mathrm{d}s$. By lemma 1, $|\mathbf{v}_\Phi(\Phi(\mathbf{x},s))| \leq C(g,0,M)||f_0||_\infty$, so that $||h||_\infty(t) \leq TC(g,0,M)||f_0||_\infty$. By corollary 1,

$$\begin{aligned}
|h(\mathbf{x},t)-h(\mathbf{w},t)| &\leq \int_0^t |\mathbf{v}_\Phi(\Phi(\mathbf{x},s))-\mathbf{v}_\Phi(\Phi(\mathbf{w},s))|\ \mathrm{d}s \\
&\leq C(g,g',2M,M)||f_0||_\infty \int_0^t |\Phi(\mathbf{x},s)-\Phi(\mathbf{w},s)|\ \mathrm{d}s \\
&\leq C(g,g',2M,M)||f_0||_\infty MT|\mathbf{x}-\mathbf{w}|.
\end{aligned}$$

Taking $T = T(g,g',M,||f_0||_\infty)$ sufficiently small yields the desired bound for both $\max_{\mathcal{S}^{d-1}}|h(\mathbf{x},t)|$ and $\mathrm{Lip}[h](t)$ for all $t \in [0,T]$.

□

Proposition 3. *Suppose that both $||\Phi-\Phi_0||_{C^{0,1}} < \min\left\{\frac{M}{2},\frac{1}{M}\right\}$ and $||\Psi-\Phi_0||_{C^{0,1}} < \min\left\{\frac{M}{2},\frac{1}{M}\right\}$. Then for T sufficiently small depending only on M,*

$$\sup_{t\in[0,T]}\max_{\mathcal{S}^{d-1}}|A[\Psi](\mathbf{x},t)-A[\Phi](\mathbf{x},t)| \leq K \sup_{t\in[0,T]}\max_{\mathcal{S}^{d-1}}|\Psi(\mathbf{x},t)-\Phi(\mathbf{x},t)|$$

for some $K < 1$.

Proof. We have

$$\begin{aligned}
&|A[\Psi](\mathbf{x},t)-A[\Phi](\mathbf{x},t)| = \\
&\left|\int_0^t \mathbf{v}_\Psi(\Psi(\mathbf{x},s))-\mathbf{v}_\Psi(\Phi(\mathbf{x},s))+\mathbf{v}_\Psi(\Phi(\mathbf{x},s))-\mathbf{v}_\Phi(\Phi(\mathbf{x},s))\ \mathrm{d}s\right| \leq \\
&\int_0^t |\mathbf{v}_\Psi(\Psi(\mathbf{x},s))-\mathbf{v}_\Psi(\Phi(\mathbf{x},s))|\ \mathrm{d}s + \int_0^t |\mathbf{v}_\Psi(\Phi(\mathbf{x},s))-\mathbf{v}_\Phi(\Phi(\mathbf{x},s))|\ \mathrm{d}s.
\end{aligned}$$

As $\mathrm{Lip}[\epsilon\Psi+(1-\epsilon)\Phi-\Phi_0](t) \leq \min\left\{\frac{M}{2},\frac{1}{M}\right\}$ for all $0 \leq \epsilon \leq 1$, proposition 1 shows that the line $L_\epsilon := \epsilon\Psi(\cdot,t)+(1-\epsilon)\Phi \in \mathcal{O}_M$. Corollary 1 provides a sufficient estimate for the first term,

$$\int_0^t |\mathbf{v}_\Psi(\Psi(\mathbf{x},s))-\mathbf{v}_\Psi(\Phi(\mathbf{x},s))|\ \mathrm{d}s \leq$$

$$C(g,g',2M,M)||f_0||_\infty \int_0^t |\Psi(\mathbf{x},s)-\Phi(\mathbf{x},s)|\ \mathrm{d}s,$$

whereas corollary 2 provides a sufficient estimate for the second term,

$$\int_0^t |\mathbf{v}_\Psi(\Phi(\mathbf{x},s)) - \mathbf{v}_\Phi(\Phi(\mathbf{x},s))|\ \mathrm{d}s \le$$
$$C(g,g',3M,M)||f_0||_\infty \int_0^t \max_{\mathcal{S}^{d-1}} |\Psi(\mathbf{x},s) - \Phi(\mathbf{x},s)|\ \mathrm{d}s.$$

□

Straightforward Picard iteration now does the work. Given $\Phi_0(\mathbf{x}) \in \mathcal{O}_{M/2}$, take $T = T(g,g',M,||f_0||_\infty)$ sufficiently small as in propositions 2 and 3, and begin by defining $\Phi^0(\mathbf{x},t) \equiv \Phi_0(\mathbf{x})$ for all $t \in [0,T]$. Then, iteratively set

$$\Phi^n(\mathbf{x},t) := A[\Phi^{n-1}](\mathbf{x},t).$$

Inductively, assume that $||\Phi^{n-1} - \Phi_0||_{C^{0,1}}(t) \le \min\{\frac{M}{2}, \frac{1}{M}\}$ and $\Phi^{n-1}(\mathbf{x},t) \in \mathcal{O}_M$ for all $t \in [0,T]$. By proposition 2, $||\Phi^n - \Phi_0||_{C^{0,1}}(t) \le \min\{\frac{M}{2}, \frac{1}{M}\}$ for all $t \in [0,T]$, so that $\Phi^n(\mathbf{x},t) \in \mathcal{O}_M$ as well from proposition 1. Therefore,

$$\sup_{[0,T]} ||\Phi^n - \Phi^{n-1}||_\infty(t) \le K \sup_{[0,T]} ||\Phi^{n-1} - \Phi^{n-2}||_\infty(t)$$

for some $K < 1$ by proposition 3, yielding a contraction in $C([0,T];C^0(\mathcal{S}^{d-1}))$. We therefore have a limit function $\Phi(\mathbf{x},t) \in C([0,T];C^0(\mathcal{S}^{d-1}))$ with $||\Phi^n - \Phi||_{C([0,T];C^0(\mathcal{S}^{d-1}))} \to 0$. However, we may note that

$$\sup_{[0,T]} ||\Phi^n - \Phi_0||_{C^{0,1}}(t) \le \min\left\{\frac{M}{2}, \frac{1}{M}\right\},$$

i.e. that each Φ^n lies in a fixed ball in $C^{0,1}(\mathcal{S}^{d-1})$ with center $\Phi_0(\mathbf{x})$. As they converge uniformly to $\Phi(\mathbf{x},t)$, we conclude

$$\sup_{[0,T]} ||\Phi - \Phi_0||_{C^{0,1}}(t) \le \min\left\{\frac{M}{2}, \frac{1}{M}\right\}$$

as well. Proposition 3 then demonstrates

$$\left|\int_0^t \mathbf{v}_{\Phi^n}(\Phi^n) - \int_0^t \mathbf{v}_\Phi(\Phi)\right| \le K \sup_{[0,T]} ||\Phi^n - \Phi||_\infty(t) \to 0,$$

so that

$$\Phi(\mathbf{x},t) = \Phi_0(\mathbf{x}) +$$
$$\int_0^t \int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\Phi(\mathbf{x},s) - \Phi(\mathbf{w},s)|^2\right)(\Phi(\mathbf{x},s) - \Phi(\mathbf{w},s))f_0(\mathbf{w})\ \mathrm{d}\mathcal{S}^{d-1}(\mathbf{w})\mathrm{d}s$$

as desired.

This yields a solution $\Phi(\mathbf{x},t) \in C([0,T];C^0(\mathcal{S}^{d-1}))$ that lies in $\mathcal{O}_M$ for each $t \in [0,T]$. However, for $t_1 > t_0$ writing

$$\Phi(\mathbf{x},t_1) = \Phi(\mathbf{x},t_0) + \int_{t_0}^{t_1} \mathbf{v}_\Phi(\Phi(\mathbf{x},s))\ \mathrm{d}s$$

and paralleling the proof of proposition 2 demonstrates that in fact $\Phi(\mathbf{x},t) \in C([0,T]; C^{0,1}(\mathcal{S}^{d-1}) \cap \mathcal{O}_M)$. The relation

$$\frac{\partial \Phi(\mathbf{x},t)}{\partial t} = \mathbf{v}_\Phi(\Phi(\mathbf{x},t))$$

and the fact that $\Phi(\mathbf{x},t) \in \mathcal{O}_M$ combine to show that $\frac{\partial \Phi}{\partial t}$ is Lipschitz, by corollary 1. The contraction furnished by proposition 3 shows that $\Phi(\mathbf{x},t)$ is the unique solution that lies in $C([0,T]; C^{0,1}(\mathcal{S}^{d-1}))$. Finally, each of the preceding arguments work equally well backward in time. All together, this yields theorem 2.

3.1. Differentiability Properties of Solutions. Fix an arbitrary bi-Lipschitz solution $\Phi(\mathbf{x},t)$ to (8) on $[0,T]$, and choose $M = M(T)$ so that $\Phi(\mathbf{x},t) \in \mathcal{O}_M$ for all $t \in [0,T]$. For any such solution, we aim in this subsection to prove

Theorem 3. *Let $\Phi(\mathbf{x},t)$ denote a solution to* (8) *on $[0,T]$ that lies in $\mathcal{O}_M$ for all $t \in [0,T]$. If $D_j\Phi_0(\mathbf{x})$ exists at $\mathbf{x} \in \mathcal{S}^{d-1}$, then $D_j\Phi(\mathbf{x},t) \in C([0,T])$ also exists at $\mathbf{x}$ for all $t \in [0,T]$, and it satisfies the linear ordinary differential equation*

$$\frac{\mathrm{d}D_j\Phi}{\mathrm{d}t}(\mathbf{x},t) = [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x},t))D_j\Phi(\mathbf{x},t), \quad D_j\Phi(\mathbf{x},0) = D_j\Phi_0(\mathbf{x}). \tag{24}$$

In particular, when $\Phi_0(\mathbf{x}) \in C^1(\mathcal{S}^{d-1})$ it follows that if $\Phi(\mathbf{x},t)$ bi-Lipschitz then actually $\Phi(\mathbf{x},t) \in C^1(\mathcal{S}^{d-1})$.

Let $\mathbf{x} \in \mathcal{S}^{d-1}$ denote an arbitrary but fixed point on the sphere, and write $\mathbf{x} = \mathbf{x}(\eta_1,\ldots,\eta_{d-1})$ where $(\eta_1,\ldots,\eta_{d-1}) \in \mathbb{R}^{d-1}$ denote spherical coordinates. For fixed $1 \le j \le d-1$ and any $|h| > 0$ define $\mathbf{x}^h_j = \mathbf{x}(\eta_1,\ldots,\eta_j + h,\ldots,\eta_{d-1}) \in \mathcal{S}^{d-1}$, and for an arbitrary function $\Psi(\mathbf{x}) : \mathcal{S}^{d-1} \to \mathbb{R}^d$ define the difference quotient

$$(D^h_j\Psi)(\mathbf{x}) := \frac{\Psi(\mathbf{x}^h_j) - \Psi(\mathbf{x})}{h}.$$

If the limit of the difference quotient exists as $h \to 0$ then the j^{th} partial derivative D_j of Ψ exists at $\mathbf{x}$, and we write

$$D_j\Psi(\mathbf{x}) := \lim_{h\to 0}(D^h_j\Psi)(\mathbf{x}).$$

As $\Phi(\mathbf{x},t)$ satifies (8) for $t \in [0,T]$, we can take difference quotients in the integral form of the equation to find that

$$(D^h_j\Phi)(\mathbf{x},t) = (D^h_j\Phi_0)(\mathbf{x}) + \frac{1}{h}\int_0^t \big(\mathbf{v}_\Phi(\Phi(\mathbf{x}^h_j,s)) - \mathbf{v}_\Phi(\Phi(\mathbf{x},s))\big)\ \mathrm{d}s$$

holds for all $t \in [0,T]$. The fundamental theorem of calculus then shows that

$$(D^h_j\Phi)(\mathbf{x},t) = (D^h_j\Phi_0)(\mathbf{x}) + \int_0^t\int_0^1 [\nabla\mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}^h_j,s) + (1-\epsilon)\Phi(\mathbf{x},s))(D^h_j\Phi)(\mathbf{x},s)\ \mathrm{d}\epsilon\mathrm{d}s.$$

As $\Phi(\mathbf{x},t) \in \mathcal{O}_M$ for all $t \in [0,T]$, we have that the bound $|\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s) - \Phi(\mathbf{w})| \leq 2M$ holds independent of the values that s and h assume. By lemma 1, then,

$$||[\nabla \mathbf{v}_\Phi]||_2(\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s)) \leq C(g,g',M)||f_0||_\infty \tag{25}$$

for C some universal constant.

Now, for an arbitrary $\Psi(t) \in C([0,T];\mathbb{R}^d)$ define a linear operator B_h as

$$B_h[\Psi](t) = \int_0^t \int_0^1 [\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s))\Psi(s)\, \mathrm{d}\epsilon \mathrm{d}s. \tag{26}$$

Due to (25), we conclude that for any $t_1, t_2 \in [0,T]$

$$|B_h[\Psi](t_2) - B_h[\Psi](t_1)| \leq ||\Psi||_{C([0,T])} C(g,g',M)||f_0||_\infty |t_2 - t_1|.$$

The operator B_h therefore maps $C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$. Moreover, by taking $t_1 = 0$ we see that if $T' \leq T$ is sufficiently small, depending only on C, then the operator B_h maps $C([0,T'];\mathbb{R}^d) \to C([0,T'];\mathbb{R}^d)$ with operator norm $||B_h||_{\mathrm{op}} \leq 1/2$. In particular, $\mathrm{Id} - B_h$ is invertible. We therefore have that

$$(D_j^h \Phi)(\mathbf{x},t) = (\mathrm{Id} - B_h)^{-1}[(D_j^h \Phi_0)(\mathbf{x})](t)$$

for all $t \in [0,T']$. Analogously, define the linear operator $B : C([0,T'];\mathbb{R}^d) \to C([0,T'];\mathbb{R}^d)$ as

$$B[\Psi](t) = \int_0^t [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x},s))\Psi(s)\, \mathrm{d}s. \tag{27}$$

Note that $||B||_{\mathrm{op}} \leq 1/2$ for the same value of T' as well. For these operators, we then have the following lemma:

Lemma 4. *Let $B_h, B : C([0,T'];\mathbb{R}^d) \to C([0,T'];\mathbb{R}^d)$ denote the linear operators in* (26) *and* (27)*, respectively. If $g \in C^1(\mathbb{R}^+ \setminus \{0\})$, satisfies* (11) *and $\Phi(\mathbf{x},t) \in \mathcal{O}_M$ for all $t \in [0,T']$, then $B_h \to B$ as $h \to 0$ in operator norm.*

Proof. Let $\Psi(t) \in C([0,T'];\mathbb{R}^d)$ with $||\Psi||_{C([0,T'])} \leq 1$. Then from the definitions of the operators B_h and B,

$$||(B_h - B)[\Psi]||_{C([0,T'])} \leq \\ C \int_0^{T'} \int_0^1 ||[\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s)) - [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x},s))||_2\, \mathrm{d}\epsilon \mathrm{d}s.$$

By definition of the operator norm, then,

$$\lim_{h\to 0} ||B_h - B||_{\mathrm{op}} \leq \\ C \lim_{h\to 0} \int_0^{T'} \int_0^1 ||[\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s)) - [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x},s))||_2\, \mathrm{d}\epsilon \mathrm{d}s.$$

The matrix $[\nabla \mathbf{v}_\Phi](\mathbf{y})$ is continuous by lemma 3. This fact combines with the continuity of Φ itself and the fact that $\mathbf{x}_j^h \to \mathbf{x}$ to yield

$$||[\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}_j^h,s) + (1-\epsilon)\Phi(\mathbf{x},s)) - [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x},s))||_2 \to 0$$

for all $s \in [0,T']$ and $\epsilon \in [0,1]$. The estimate (25) and the dominated convergence theorem then show

$$\lim_{h\to 0} ||B_h - B|| \leq$$
$$C\int_0^{T'}\int_0^1 \lim_{h\to 0} ||[\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x}^h_j, s) + (1-\epsilon)\Phi(\mathbf{x}, s)) - [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x}, s))||_2 \, \mathrm{d}\epsilon \mathrm{d}s = 0$$

as desired.

□

Returning to the task at hand, we have that the uniform estimates $||B_h||_{\mathrm{op}} \leq \frac{1}{2}$ and $||B||_{\mathrm{op}} \leq \frac{1}{2}$ guarantee that both $(\mathrm{Id} - B_h)^{-1}$ and $(\mathrm{Id} - B)^{-1}$ exist. Moreover, by using the power series representations of the inverse operators, the uniform operator norm estimates and the fact that $B_h \to B$ in operator norm we see that $||(\mathrm{Id} - B_h)^{-1} - (\mathrm{Id} - B)^{-1}||_{\mathrm{op}} \to 0$ as well. If $D_j\Phi_0(\mathbf{x})$ exists, we may define the constant functions $\Psi_h, \Psi \in C([0,T'];\mathbb{R}^d)$ by $\Psi_h(t) \equiv (D^h_j\Phi_0)(\mathbf{x})$ and $\Psi(t) \equiv D_j\Phi_0(\mathbf{x})$. Lemma 4 then shows

$$\begin{aligned}
&|(D^h_j\Phi)(\mathbf{x}, t) - (\mathrm{Id} - B)^{-1}[\Psi](t)| = |(\mathrm{Id} - B_h)^{-1}[\Psi_h](t) - (\mathrm{Id} - B)^{-1}[\Psi](t)| \\
&\leq 2||\Psi_h - \Psi||_{C([0,T'])} + ||(\mathrm{Id} - B_h)^{-1} - (\mathrm{Id} - B)^{-1}||_{\mathrm{op}}||\Psi||_{C([0,T'])} \\
&= 2|(D^h_j\Phi_0)(\mathbf{x}) - D_j\Phi_0(\mathbf{x})| + ||(\mathrm{Id} - B_h)^{-1} - (\mathrm{Id} - B)^{-1}||_{\mathrm{op}}|D_j\Phi_0(\mathbf{x})| \to 0
\end{aligned}$$

as $h \to 0$. In other words, $D_j\Phi(\mathbf{x}, t)$ exists at $\mathbf{x}$ as well, and we have the representation

$$D_j\Phi(\mathbf{x}, t) = (\mathrm{Id} - B)^{-1}[D_j\Phi_0(\mathbf{x})](t) \tag{28}$$

Moreover, $D_j\Phi(\mathbf{x}, t)$ is a continuous function in t for all $t \in [0,T']$. Pre-multiplying by $(\mathrm{Id} - B)$ in (28) and using the definition (27) of B then shows that $D_j\Phi(\mathbf{x}, t)$ satisfies the integral equation

$$D_j\Phi(\mathbf{x}, t) = D_j\Phi_0(\mathbf{x}) + \int_0^t [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x}, s))D_j\Phi(\mathbf{x}, s) \, \mathrm{d}s \tag{29}$$

on $[0,T']$. Taking $\Phi(\mathbf{x}, T')$ as inititial data and applying the same argument then shows that

$$D_j\Phi(\mathbf{x}, t) = D_j\Phi(\mathbf{x}, T') + \int_{T'}^t [\nabla \mathbf{v}_\Phi](\Phi(\mathbf{x}, s))D_j\Phi(\mathbf{x}, s) \, \mathrm{d}s$$

for $t \in [T', 2T']$, so that (29) actually holds on $[0, 2T']$. Applying the argument a finite number of times then shows that $D_j\Phi(\mathbf{x}, t) \in C([0,T])$ and satisfies (29) on $[0,T]$. By the fundamental theorem of calculus, then, (24) holds.

For the last statement in theorem 3, by lemma 3 the equation (24) defines a linear ODE with coefficients that depend continuously on the parameter $\mathbf{x} \in \mathcal{S}^{d-1}$. Its solutions therefore depend continuously on both the parameter $\mathbf{x} \in \mathcal{S}^{d-1}$ and on the initial data. As $\Phi_0(\mathbf{x}) \in C^1(\mathcal{S}^{d-1})$ the initial data also depends continuously on $\mathbf{x} \in \mathcal{S}^{d-1}$, so that the solution $D_j\Phi(\mathbf{x}, t) \in C(\mathcal{S}^{d-1})$ as desired.

4. Blowup, Collapse, and Global Existence

In the previous section, we demonstrated that if $\Phi_0(\mathbf{x}) \in \mathcal{O}_{\frac{M}{2}}$ then there exists $T = T(g, g', M, ||f_0||_\infty)$ such that integral equation (8) has a unique solution $\Phi(\mathbf{x}, t)$ on $t \in [0, T]$. The solution lies in $\mathcal{O}_M$ for all $t \in [0, T]$ as well. Clearly, we can take $\Phi(\mathbf{x}, T) \in \mathcal{O}_M$ as initial data and then repeat the argument. This yields a unique solution on some larger time interval $[0, T_1]$ with $T_1 > T$, and this process can continue as long as $\mathrm{Lip}[\Phi](t)$ and $\mathrm{Lip}[\Phi^{-1}](t)$ remain finite. Summarizing, we have the following continuation result:

Theorem 4. *Let g and Φ_0 satsify the assumptions of theorem 2 and $\Phi(\mathbf{x}, t)$ denote the corresponding solution to the IDE (8). If $[0, T_f)$ denotes the largest time interval on which $\Phi(\mathbf{x}, t)$ exists as a bi-Lipschitz solution, then at least one of*

$$\text{(i)} \limsup_{t \nearrow T_f} \mathrm{Lip}[\Phi](t) = \infty \quad \text{(ii)} \limsup_{t \nearrow T_f} \mathrm{Lip}[\Phi^{-1}](t) = \infty \quad \text{(iii)}\ T_f = \infty \tag{30}$$

must hold.

By recalling the class of solutions $\Phi(\mathbf{x}, t) = R(t)\mathbf{x}$ from example 1, we find simple examples that demonstrate each of (i), (ii) and (iii) can happen in isolation. Indeed, if $g(s) = s^p$ for $p > 0$ the ODE (17) reduces to $R' = C_p R^{1+2p}$; the constant

$$C_p = \mathrm{vol}(\mathcal{S}^{d-2}) \int_{-1}^{1} (1 - s)^{1+p}(1 - s^2)^{\frac{d-3}{2}}\, \mathrm{d}s$$

is positive. We readily compute the explicit solution and maximal interval of existence $[0, T_f)$ as

$$R(t) = \left(\frac{1}{R(0)^{-2p} - 2pC_p t} \right)^{\frac{1}{2p}}, \quad T_f = \frac{1}{2pC_p R(0)^{2p}}, \tag{31}$$

so that (i) occurs as $t \nearrow T_f$ while (ii) remains finite. Conversely, suppose $g(s) = -s^{-p}$ for $0 < p < \frac{d-1}{2}$. Then

$$C_p = \mathrm{vol}(\mathcal{S}^{d-2}) \int_{-1}^{1} (1 - s)^{1-p}(1 - s^2)^{\frac{d-3}{2}}\, \mathrm{d}s > 0$$

$$R(t) = \left(R(0)^{2p} - 2pC_p t\right)^{\frac{1}{2p}}, \quad T_f = \frac{R(0)^{2p}}{2pC_p} \tag{32}$$

and the solution can collapse to zero in finite time. That is, (ii) occurs at T_f while (i) remains finite.

As these examples indicate, we must prevent both blowup and collapse in order to guarantee the solution exists as a bi-Lipschitz surface for all time. It comes as no surprise that this amounts to having control over the gradient matrix $[\nabla \mathbf{v}_\Phi](\mathbf{y})$ generated by the Eulerian velocity field $\mathbf{v}_\Phi(\mathbf{y})$, as similar criteria abound for related active scalar problems. Specifically, it proves both necessary and sufficient to have

$$\int_0^T ||\nabla \mathbf{v}_\Phi||_{L^\infty(|\mathbf{y}| \leq ||\Phi||_\infty(t))} \mathrm{d}t < \infty \tag{33}$$

Precisely analogous conditions guarantee existence for related problems, such as solutions to the Euler equations ([18], Chapter 5) and for the boundary of a vortex patch written in contour dynamics form ([18], Chapter 8).

Theorem 5. *Suppose $g(s)$ defines an admissible kernel and $f_0 \in L^\infty$. Then the solution $\Phi(\mathbf{x},t) \in C([0,T];C^{0,1}(\mathcal{S}^{d-1}))$ to* (8) *exists as a bi-Lipschitz surface past time T if and only if both $||\Phi||_\infty(T) < \infty$ and* (33) *hold.*

Proof. Clearly, if $\Phi(\mathbf{x},t)$ is bi-Lipschitz on $[0,T']$ for $T' > T$ then $||\Phi||_\infty(T) < \infty$ and $M := \sup_{[0,T]} \mathrm{Lip}[\Phi^{-1}](t) < \infty$ as well. Recalling from (21) that

$$[\nabla \mathbf{v}_\Phi](\mathbf{y}) = \int_{\mathcal{S}^{d-1}} \left[g\left(\frac{1}{2}|\mathbf{y}-\Phi(\mathbf{w})|^2\right)\mathrm{Id} + \right.$$
$$\left. g'\left(\frac{1}{2}|\mathbf{y}-\Phi(\mathbf{w})|^2\right)(\mathbf{y}-\Phi(\mathbf{w}))(\mathbf{y}-\Phi(\mathbf{w}))^t \right] f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}),$$

the proof of lemma 1 shows that $||\nabla \mathbf{v}_\Phi||_\infty(\mathbf{y},t) \leq C(M, D_{\mathbf{y}})$. The constant C increases with M and $D_{\mathbf{y}} := \max_{\mathcal{S}^{d-1}} |\mathbf{y} - \Phi(\mathbf{w},t)|$ and remains finite provided M and $D_{\mathbf{y}}$ stay bounded. Of course $D_{\mathbf{y}} \leq 2||\Phi||_\infty(t) \leq 2\sup_{[0,T]} ||\Phi||_\infty(t) < \infty$ provided $|\mathbf{y}| \leq ||\Phi||_\infty(t)$, so that

$$||\nabla \mathbf{v}_\Phi||_{L^\infty(|\mathbf{y}|\leq||\Phi||_\infty(t))} \leq C\left(M, 2\sup_{[0,T]}||\Phi||_\infty(t)\right) < \infty$$

and (33) holds.

For the converse, it suffices to show that both $\mathrm{Lip}[\Phi](T)$ and $\mathrm{Lip}[\Phi^{-1}](T)$ remain bounded. To this end, for $\mathbf{x},\mathbf{z} \in \mathcal{S}^{d-1}$ let $\Delta(\mathbf{x},\mathbf{z},t) := \Phi(\mathbf{x},t) - \Phi(\mathbf{z},t)$. The fundamental theorem of calculus then yields

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{z},t)|^2 =$$
$$\int_0^1 \langle \Delta(\mathbf{x},\mathbf{z},t), [\nabla \mathbf{v}_\Phi](\epsilon\Phi(\mathbf{x},t) + (1-\epsilon)\Phi(\mathbf{z},t))\Delta(\mathbf{x},\mathbf{z},t)\rangle\;\mathrm{d}\epsilon. \tag{34}$$

As $|\epsilon\Phi(\mathbf{x},t) + (1-\epsilon)\Phi(\mathbf{z},t)| \leq ||\Phi||_\infty(t)$, the relation (34) implies

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{z},t)|^2 \geq -K||\nabla \mathbf{v}_\Phi||_{L^\infty(|\mathbf{y}|\leq||\Phi||_\infty(t))}|\Delta(\mathbf{x},\mathbf{z},t)|^2$$

for some absolute constant K that depends only on the size of the matrix. By Gronwall's inequality

$$|\Delta(\mathbf{x},\mathbf{z},0)|\mathrm{e}^{-K\int_0^t ||\nabla \mathbf{v}_\Phi||_{L^\infty(|\mathbf{y}|\leq||\Phi||_\infty(s))}\mathrm{d}s} \leq |\Delta(\mathbf{x},\mathbf{z},t)|.$$

Dividing through by $|\mathbf{x}-\mathbf{w}|$ and taking an infimum gives the estimate

$$\mathrm{Lip}[\Phi^{-1}](T) \leq \mathrm{Lip}[\Phi_0^{-1}]\mathrm{e}^{K\int_0^T ||\nabla \mathbf{v}_\Phi||_{L^\infty(|\mathbf{y}|\leq||\Phi||_\infty(s))}\mathrm{d}s} < \infty$$

due to (33). Analogously, the fundamental theorem of calculus and the proof of lemma 1 combine to show

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{z},t)|^2 \leq C(\mathrm{Lip}[\Phi^{-1}](t), 2||\Phi||_\infty||(t)).$$

Applying Gronwall's inequality, then dividing by $|\mathbf{x}-\mathbf{w}|$ and taking a supremum yields

$$\mathrm{Lip}[\Phi](T) \leq \mathrm{Lip}[\Phi_0]\mathrm{e}^{\int_0^T C(\mathrm{Lip}[\Phi^{-1}](s),2||\Phi||_\infty(s))\,\mathrm{d}s} < \infty.$$

The last inequality holds since $\mathrm{Lip}[\Phi^{-1}](t)$ remains finite on $[0,T]$ due to the previous estimate, and since $||\Phi||_\infty(t)$ remains bounded for all $t \in [0,T]$ by hypothesis.

□

Remark 1. From the proof of the previous theorem, we can rephrase the result to say that the solution $\Phi(\mathbf{x},t) \in C([0,T];C^{0,1}(\mathcal{S}^{d-1}))$ exists as a bi-Lipschitz surface past time T if and only if both $\mathrm{Lip}[\Phi^{-1}](T)$ and $||\Phi||_\infty(T)$ remain finite. This rephrasing generally proves more useful than the statement in theorem 5.

4.1. The Osgood Condition for Locally Attractive Kernels. We first focus our attention on the case when $g(s)$ has an attractive (i.e., negative) singularity at the origin, such as $g(s) = -s^{-p}$. From (32) we know collapse can occur in finite time, so we wish to characterize precisely when this happens. Earlier studies on the aggregation equation (2) have shown that the Osgood condition on the kernel $g(s)$ provides a precise characterization. Indeed, for initial data $\rho_0 \in L^\infty(\mathbb{R}^d)$ the Osgood condition proves both necessary and sufficient for ρ to remain in L^∞ for all positive times [3]. For initial data in $\rho_0 \in L^p(\mathbb{R}^d)$ with $p > \frac{d}{d-1}$, the Osgood condition proves necessary and sufficient for global existence as well [4]. For our co-dimension one distribution solutions, we show that this characterization holds for the surface equation (8) in this section.

Following [3], we say that the kernel $g(s)$ is Osgood if

$$\lim_{\epsilon\downarrow 0} \int_\epsilon^1 \frac{1}{sg(s)}\,\mathrm{d}s = -\infty. \tag{35}$$

Adapting the arguments from [3] to our setting easily yields the necessity of (35) for global existence, as we demonstrate in the lemma that follows.

Lemma 5. *Suppose $g(s)$ is non-positive and non-decreasing in some neighborhood $(0,\delta]$ of the origin and that $f_0(\mathbf{w}) \geq 0$. If (35) fails, then all solutions with $||\Phi_0||_\infty^2 < \delta/2$ collapse to the origin in finite time.*

Proof. The proof follows exactly as in [3]. As long as $\Phi(\mathbf{x},t)$ exists, by continuity there exists $\mathbf{x} \in \mathcal{S}^{d-1}$ with $|\Phi(\mathbf{x},t)| = ||\Phi||_\infty(t)$. From the hypotheses on g, f_0 and the fact that $\langle \Phi(\mathbf{x},t), \Phi(\mathbf{x},t) - \Phi(\mathbf{w},t)\rangle \geq 0$ for all $\mathbf{w} \in \mathcal{S}^{d-1}$ it then follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}|\Phi(\mathbf{x},t)|^2 = \\
&2\int_{\mathcal{S}^{d-1}} g\left(\frac{1}{2}|\Phi(\mathbf{x},t)-\Phi(\mathbf{w},t)|^2\right)\langle \Phi(\mathbf{x},t), \Phi(\mathbf{x},t)-\Phi(\mathbf{w},t)\rangle f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) \\
&\leq 2g\left(2|\Phi(\mathbf{x},t)|^2\right)\left[\int_{\mathcal{S}^{d-1}} |\Phi(\mathbf{x},t)|^2 f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w}) - \right. \\
&\qquad\qquad \left.\left\langle \Phi(\mathbf{x},t), \int_{\mathcal{S}^{d-1}} \Phi(\mathbf{w},t) f_0(\mathbf{w})\,\mathrm{d}\mathcal{S}^{d-1}(\mathbf{w})\right\rangle\right] \\
&= 2M_\rho|\Phi(\mathbf{x},t)|^2 g\left(2|\Phi(\mathbf{x},t)|^2\right) \leq 0.
\end{aligned}$$

The last line results from (14),(15) and our assumption that $\Phi_0(\mathbf{x})$ has zero center of mass. If (35) fails, the solution to the ODE

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2M_\rho r g(2r) \quad r(0) = ||\Phi_0||_\infty^2 \tag{36}$$

reaches zero in finite time, whence $||\Phi||_\infty(t)$ must reach zero in finite time as well.

□

As a consequence, in general (35) must hold in order to guarantee that solutions to (8) do not collapse in finite time. We therefore assume (35), and turn our attention toward demonstrating the sufficiency of the Osgood condition for global existence. For this it will prove useful to rewrite $g(s)$ in the form

$$g(s) = \frac{h\left((2s)^p\right)}{(2s)^p}, \; 0 < p \le 1/2, \tag{37}$$

so that the Osgood condition then reads

$$\lim_{\epsilon \downarrow 0} \int_\epsilon^1 \frac{1}{h(u)} \, \mathrm{d}u = -\infty. \tag{38}$$

Following [4], we shall say $h(r)$ defines a natural kernel provided it satisfies the following regularity, boundedness and monotonicity conditions:

Definition 2. *Let $g(s)$ satisfy (37) for some $0 < p \le 1/2$ if $d > 2$ and $0 < p < 1/2$ if $d = 2$. We then say $h(r)$ defines a **natural kernel** if*

(H1) $h(r) \in C^1\left(\mathbb{R}^+ \setminus \{0\}\right)$
(H2) $h(r) \in L^\infty(\mathbb{R}^+)$
(H3) $h'(r)$ *is monotonic (either increasing or decreasing) near zero*

Remark 2. The additional restriction $0 < p < 1/2$ if $d = 2$ arises due to the integrability constraint (11).

Using the arguments from [4], we establish

Lemma 6. *Let $h(r)$ define a natural kernel with $h(0) = 0$. Then either*

(a) $\min\left\{\frac{h(r)}{r}, h'(r)\right\} \ge C_0$ *for some $C_0 > -\infty$ and all $r \in [0,1]$, or both*

(b1) $\frac{h(r)}{r} \to -\infty$ *and* $h'(r) \to -\infty$ *as* $r \to 0^+$, *and*

(b2) $\exists \delta > 0$ *such that* $\forall r \in (0, \delta]$ $h'(r) \ge \frac{h(r)}{r}$, $\frac{h(r)}{r}$ *increases,* $h(r)$ *decreases, and if* $\delta_1 \le \delta$ *then* $\inf_{r \ge \delta_1} \frac{h(r)}{r} = \frac{h(\delta_1)}{\delta_1}$ *and* $\inf_{r \ge \delta_1} h'(r) = h'(\delta_1)$.

Proof. Suppose first that there exists $C_1 > -\infty$ such that

$$\liminf_{r\to 0^+} \frac{h(r)}{r} > C_1.$$

As $h(0) = 0$, given any r sufficiently small there exists $s < r$ with

$$h'(s) = \frac{h(r)}{r} > C_1.$$

It then follows from (H3) that $\lim_{r\to 0^+} h'(r) \geq C_0$. Thus $h'(r)$ is bounded from below in a neighborhood of the origin as well, so (a) holds. Otherwise, there exists sequences $r_n \to 0^+$ and $s_n < r_n$ with

$$\lim_{n\to\infty} \frac{h(r_n)}{r_n} = h'(s_n) = -\infty. \tag{39}$$

When combined with (H3), this gives both that $h'(r) \to -\infty$ and that $h'(r)$ is increasing on some neighborhood $(0, \sigma]$ of zero. Clearly h decreases in this neighborhood as $h' < 0$. Moreover, for any $r \in (0, \sigma]$ there exists $s < r \leq \sigma$ with

$$\frac{h(r)}{r} = h'(s) \leq h'(r)$$

as desired. This also gives that $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{h(r)}{r}\right) = \frac{1}{r}\left(h'(r) - \frac{h(r)}{r}\right) \geq 0$, so that $\frac{h(r)}{r}$ increases. Coupled with (39) this shows $\frac{h(r)}{r} \to -\infty$, completing the proof of (b1). Finally, from these statements it follows that $\frac{h(r)}{r}$ and $h'(r)$ are monotonic in $(0, \sigma]$ and tend to $-\infty$ as $r \to 0^+$, so the remainder of (b2) follows provided $\delta \leq \sigma$ is sufficiently small.

□

Note that if $g(s)$ is Osgood, it follows from (38) that necessarily $h(0) = 0$. We can therefore apply lemma 6 to such kernels, and this allows us to provide a lower bound for the time of collapse of $1/\mathrm{Lip}[\Phi^{-1}](t)$ to zero in terms of the solution to an ODE. When part (a) of the lemma holds, a crude estimate suffices to demonstrate global existence from this ODE. When (b1) and (b2) hold the ODE proves more complicated. However, as $g(s)$ is Osgood, the solution to this ODE still remains positive for all time, and this yields global existence in the second case.

Lemma 7. *Let $h(r)$ define a natural kernel $g(s)$ that is Osgood. Suppose further that $f_0(\mathbf{z}) \geq 0$. If (a) in lemma 6 holds then the solution $\Phi(\mathbf{x}, t)$ exists globally in time.*

Proof. By the remark following theorem 5, this follows from a straightforward upper bound for $\mathrm{Lip}[\Phi^{-1}](t)$ and $||\Phi||_\infty(T)$. For $\mathbf{x}, \mathbf{w}, \mathbf{z} \in \mathcal{S}^{d-1}$ and $\epsilon \in \mathbb{R}$ let $\Delta(\mathbf{x}, \mathbf{w}, t) := \Phi(\mathbf{x}, t) - \Phi(\mathbf{w}, t)$ and

$$L_\epsilon(\mathbf{x}, \mathbf{w}, \mathbf{z}) = \epsilon\Phi(\mathbf{x}, t) + (1 - \epsilon)\Phi(\mathbf{w}, t) - \Phi(\mathbf{z}, t). \tag{40}$$

Using the fundamental theorem of calculus as before shows

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{w},t)|^2 = |\Delta|^2 \int_0^1 \int_{\mathcal{S}^{d-1}} \left[\frac{h(|L_\epsilon|^{2p})}{|L_\epsilon|^{2p}} \left(1 - 2p\cos^2(\theta_\epsilon)\right) + h'(|L_\epsilon|^{2p}) 2p\cos^2(\theta_\epsilon) \right] f_0(\mathbf{z})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z}) \mathrm{d}\epsilon, \tag{41}$$

where θ_ϵ denotes the angle between L_ϵ and Δ. Let

$$C_0(t) = C_0(||\Phi||_\infty(t)) = \inf_{r \in \left[0, 2^{2p}||\Phi||_\infty^{2p}(t)\right]} \min\left\{ \frac{h(r)}{r}, h'(r) \right\}$$

When (a) holds, it follows from (H1) that $C_0(t) > -\infty$ provided $||\Phi||_\infty(t)$ remains finite. Therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{w},t)|^2 \geq C_0(t) M_\rho |\Delta(\mathbf{x},\mathbf{w},t)|^2.$$

Gronwall's inequality then yields

$$|\Delta(\mathbf{x},\mathbf{w},t)| \geq |\Delta(\mathbf{x},\mathbf{w},0)| \mathrm{e}^{M_\rho \int_0^t C_0(s)\mathrm{d}s}.$$

Dividing through by $|\mathbf{x}-\mathbf{w}|$ and taking an infimum yields

$$\mathrm{Lip}[\Phi^{-1}](t) \leq \mathrm{Lip}[\Phi_0^{-1}] \mathrm{e}^{-M_\rho \int_0^t C_0(s)\mathrm{d}s},$$

so that $\mathrm{Lip}[\Phi^{-1}](t)$ remains bounded for all finite times provided $||\Phi||_\infty(t)$ does. As $h(r)$ defines a natural kernel, the hypotheses (H2) shows that

$$\frac{\partial}{\partial t}||\Phi||_\infty(t) \leq K ||\Phi||_\infty^{1-2p}(t),$$

for some absolute constant K, so that $||\Phi||_\infty(t)$ does indeed remain bounded for all finite time as desired.

□

Now let us turn to the second case, i.e. that (b1) and (b2) from lemma 6 hold. For use in the following lemma, let us define the quantity we wish to estimate, $r(t) := 1/\mathrm{Lip}[\Phi^{-1}](t)$, and the integral

$$\mathcal{I}(r^{2p}(t)) = \int_{-1}^{1} \frac{h\left(r^{2p}(t) 2^{-p}(1-s)^p\right)}{2^{-p}(1-s)^p} (1-s^2)^{\frac{d-3}{2}}\, \mathrm{d}s. \tag{42}$$

With these definitions, and taking δ as in lemma 6 part (b2) we can demonstrate

Lemma 8. *Let $h(r)$ define a natural kernel $g(s)$ that is Osgood. Suppose further that $f_0(\mathbf{z}) \geq 0$ and $0 < r(t_0) < \delta$ for some $t_0 \geq 0$. If* (b1) *and* (b2) *in lemma 6 holds, then $r^{2p}(t)$ remains bounded below by the solution $q(t)$ to the ODE*

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 2p\left[\mathrm{vol}(\mathcal{S}^{d-2})||f_0||_\infty \mathcal{I}(q(t)) + M_\rho h(q(t))\right], \quad q(t_0) = r^{2p}(t_0) \tag{43}$$

for all $t \geq t_0$.

Proof. Use the fundamental theorem of calculus as in the first case, define L_ϵ as in (40) and let $f(\epsilon, \mathbf{z})$ denote the integrand. Then split the resulting integral (41) into two terms to find

$$\int_0^1 \int_{\mathcal{S}^{d-1}} f(\epsilon, \mathbf{z})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})\mathrm{d}\epsilon = \int_0^1 \int_{\mathcal{S}^{d-1}\cap\{|L_\epsilon|^{2p}\leq\delta_1\}} + \int_0^1 \int_{\mathcal{S}^{d-1}\cap\{|L_\epsilon|^{2p}\geq\delta_1\}}$$
$$:=\mathrm{I}+\mathrm{II}.$$

For any $\delta_1 \leq \delta$ with δ as in lemma 6, as $h'(|L_\epsilon|^{2p}) \geq \frac{h(|L_\epsilon|^{2p})}{|L_\epsilon|^{2p}}$ and $h \leq 0$ it follows that

$$\mathrm{I} \geq ||f_0||_\infty \int_0^1 \int_{\mathcal{S}^{d-1}\cap\{|L_\epsilon|^{2p}\leq\delta_1\}} \frac{h(|L_\epsilon|^{2p})}{|L_\epsilon|^{2p}}\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z}).$$

Let $\mathbf{x}_0 = \mathbf{x}_0(\epsilon)$ denote a minimizer of $|\epsilon\Phi(\mathbf{x},t) + (1-\epsilon)\Phi(\mathbf{w},t) - \Phi(\mathbf{z},t)|$ over $\mathbf{z} \in \mathcal{S}^{d-1}$, so that

$$|L_\epsilon| \geq \frac{1}{2}|\Phi(\mathbf{x}_0,t) - \Phi(\mathbf{z},t)| \geq \frac{r(t)}{2}|\mathbf{x}_0 - \mathbf{z}|.$$

Combining this with the facts that $\frac{h(r)}{r}$ is non-decreasing and that $h \leq 0$ then shows

$$\begin{aligned}\mathrm{I} \geq & 2^{2p}||f_0||_\infty \int_0^1 \int_{\{|L_\epsilon|^{2p}\leq\delta_1\}} \frac{h\left(r^{2p}(t)2^{-2p}|\mathbf{x}_0-\mathbf{z}|^{2p}\right)}{r^{2p}(t)|\mathbf{x}_0-\mathbf{z}|^{2p}}\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})\mathrm{d}\epsilon \geq \\ & 2^{2p}||f_0||_\infty \int_0^1 \int_{\{r^{2p}(t)2^{-2p}|\mathbf{x}_0-\mathbf{z}|^{2p}\leq\delta_1\}} \frac{h\left(r^{2p}(t)2^{-2p}|\mathbf{x}_0-\mathbf{z}|^{2p}\right)}{r^{2p}(t)|\mathbf{x}_0-\mathbf{z}|^{2p}}\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})\mathrm{d}\epsilon.\end{aligned}$$

The case $l = 0$ of theorem 1 then implies

$$\mathrm{I} \geq \mathrm{vol}(\mathcal{S}^{d-2})||f_0||_\infty \int_{\{(r^2(1-s)/2)^p\leq\delta_1\}} \frac{h\left(r^{2p}(t)2^{-p}(1-s)^p\right)}{r^{2p}(t)2^{-p}(1-s)^p}(1-s^2)^{\frac{d-3}{2}}\, \mathrm{d}s.$$

For II, using the last part of (b2) it follows that $\frac{h(r)}{r} \geq \frac{h(\delta_1)}{\delta_1}$ for all $r \geq \delta_1$ and similarly that $h'(r) \geq h'(\delta_1) \geq \frac{h(\delta_1)}{\delta_1}$ provided $\delta_1 \leq \delta$. Therefore

$$\mathrm{II} \geq \frac{h(\delta_1)}{\delta_1}\int_0^1 \int_{\mathcal{S}^{d-1}} f_0(\mathbf{z})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})\mathrm{d}\epsilon = M_\rho \frac{h(\delta_1)}{\delta_1}.$$

For any time when $r^{2p}(t) < \delta$, the choice $\delta_1 = r^{2p}(t)$ yields

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta|^2 \geq |\Delta|^2 \left[\mathrm{vol}(\mathcal{S}^{d-2})||f_0||_\infty \mathcal{I}(r^{2p}(t)) + M_\rho h(r^{2p}(t))\right] r^{-2p}(t). \tag{44}$$

An application of Gronwall's inequality then shows

$$\begin{aligned}|\Delta(\mathbf{x},\mathbf{w},t)|^{2p} \geq |\Delta(\mathbf{x},\mathbf{w},0)|^{2p} \exp\Bigg(2p\int_{t_0}^t & \left[\mathrm{vol}(\mathcal{S}^{d-2})||f_0||_\infty \mathcal{I}(r^{2p}(s)) + \right.\\ & \left. M_\rho h(r^{2p}(s))\right] r^{-2p}(s)\, \mathrm{d}s\Bigg).\end{aligned}$$

Dividing through by $|\mathbf{x}-\mathbf{w}|$ and taking infimums yields the estimate

$$r^{2p}(t)/r^{2p}(t_0) \geq \exp\left(2p\int_{t_0}^{t}\left[\mathrm{vol}(\mathcal{S}^{d-2})||f_0||_\infty \mathcal{I}(r^{2p}(s)) + M_\rho h(r^{2p}(s))\right] r^{-2p}(s)\,\mathrm{d}s\right) \tag{45}$$

which holds for all $t \geq t_0$ such that $r^{2p}(t) < \delta$ on $[t_0, t]$. Using (45) and a standard bootstrap argument shows that $r^{2p}(t) \geq q(t)$ for all such $t \geq t_0$. Of course, $q(t) < \delta$ for all $t \geq t_0$ as $h \leq 0$ on $(0, \delta]$, so that in fact $r^{2p}(t) \geq q(t)$ for all $t \geq t_0$.

□

The last ingredient we need demonstrates that in the second case, the solution to (43) remains positive for all time when $h(r)$ defines a natural, Osgood kernel.

Lemma 9. *Let $h(r)$ define a natural, Osgood kernel satisfying* (b1) *and* (b2)*, and take $\delta > 0$ as in lemma 6. Then the solution $\Phi(\mathbf{x}, t)$ with initial data $\Phi_0(\mathbf{x})$ exists globally in time.*

Proof. It suffices to show that $\mathcal{I}(q(t)) \geq Ch(q(t))$ where C denotes some finite, positive constant. Indeed, as $h(r)$ defines an Osgood kernel the solution to (43) then remains positive for all time, whence $\mathrm{Lip}[\Phi^{-1}](t)$ remains finite for all time by lemma 8. From (H2) it follows that $||\Phi||_\infty(t)$ also remains bounded for all time, and the claim then follows.

To see that $\mathcal{I}(q(t)) \geq Ch(q(t))$ holds, recall from lemma 6 part (b2) that $h(r)$ decreases on $(0, \delta]$. As $q(t)2^{-p}(1-s)^p \leq q \leq \delta$ for $s \in [-1, 1]$, it then follows that

$$\begin{aligned}\mathcal{I}(q(t)) &:= \int_{-1}^{1} \frac{h\left(q(t)2^{-p}(1-s)^p\right)}{2^{-p}(1-s)^p}(1-s^2)^{\frac{d-3}{2}}\,\mathrm{d}s \\ &\geq 2^p h(q(t)) \int_{-1}^{1} (1-s)^{\frac{d-3}{2}-p}(1+s)^{\frac{d-3}{2}}\,\mathrm{d}s.\end{aligned}$$

As $p < \frac{d-1}{2}$ by hypothesis, the last integral is finite, which gives $\mathcal{I}(q(t)) \geq Ch(q(t))$ as desired.

□

We may now encapsulate the previous lemmas into the main result of this section, i.e. the following theorem demonstrating the equivalence between the Osgood condition (38) and the global existence of all solutions to the IDE (8) for the class of natural kernels.

Theorem 6. *(Necessary and Sufficient Condition for Global Existence) Let $g(s)$ satisfy* (37)*, where $h(r)$ defines a natural kernel and $h(r) \leq 0$ in a neighborhood of the origin. Then all solutions to* (8) *exist globally in time if and only if* (38) *holds.*

Proof. Suppose first that (38) fails. Then either $h(0) < 0$ or $h(0) = 0$. In the first case, there exists $\epsilon > 0$ so that

$$g(s) < \frac{h(0)}{2(2s)^p},$$

for some $p > 0$ and all $s \in [0, \epsilon]$. The proof of lemma 5 then shows that all solutions with $||\Phi_0||_\infty^2 \le \epsilon/2$ collapse to the origin in finite time. In the second case, either (a) or (b1,b2) in lemma 6 holds. If (a) holds then $\exists C_0 > 0$ so that

$$h(r) \ge -C_0 r$$

for all r in a neighborhood of the origin. This contradicts the assumption that (38) fails, so both (b1) and (b2) must hold. As a consequence, $g(s)$ is non-negative and non-decreasing in a neighborhood of the origin. Lemma 5 then applies, so that all solutions with $||\Phi_0||_\infty$ sufficiently small must collapse in finite time.

Conversely, if (38) holds then necessarily $h(0) = 0$. Thus either lemma 7 or lemma 9 applies, yielding global existence of all solutions in either case.

□

4.2. Locally Repulsive Kernels. Lastly, we provide a global existence result for locally repulsive kernels, i.e. when $g(s)$ has a positive singularity near the origin. As before, we assume

$$g(s) = \frac{h\left((2s)^p\right)}{(2s)^p},$$

for some $0 < p \le 1/2$ and $p < 1/2$ if $d = 2$. We modify the assumtions on $h(r)$ slightly, in that we replace the monotonicity condition (H3) with a boundedness condition (H4). We therefore assume

$$\begin{aligned}
&\text{(H1) } h(r) \in C^1\left(\mathbb{R}^+ \setminus \{0\}\right) \\
&\text{(H2) } h(r) \in L^\infty(\mathbb{R}^+) \\
&\text{(H4) } \inf_{(0,1)} h'(r) > -\infty.
\end{aligned} \tag{46}$$

These hypotheses include many kernels that appear in applications, including the power laws $g(s) = s^{-p}$ for $p \le \frac{1}{2}$ as well as the ubiquitous Morse potential [16,11]

$$g(s) = \frac{\mathrm{e}^{-\sqrt{2s}} - F\mathrm{e}^{-L\sqrt{2s}}}{\sqrt{2s}}.$$

Under these assumptions, we have the following global existence result:

Theorem 7. *Let $g(s) = h((2s)^p)(2s)^{-p}$ for some $0 < p \le 1/2$ if $d \ge 3$ and $p < 1/2$ if $d = 2$. Let $h(r)$ satisfy (46) and $f_0(\mathbf{w}) \ge 0$. If there exists a neighborhood $(0, \delta]$ of the origin on which $h(r) \ge 0$, then the solution $\Phi(\mathbf{x}, t)$ given by theorem 2 exists globally in time.*

Proof. Again using the remark following theorem 5, this follows from a straightforward upper bound for $\mathrm{Lip}[\Phi^{-1}](t)$ and $||\Phi||_\infty(T)$. For $\mathbf{x}, \mathbf{w}, \mathbf{z} \in \mathcal{S}^{d-1}$ and $\epsilon \in \mathbb{R}$ let $\Delta(\mathbf{x}, \mathbf{w}, t) := \Phi(\mathbf{x}, t) - \Phi(\mathbf{w}, t)$ and

$$L_\epsilon(\mathbf{x}, \mathbf{w}, \mathbf{z}) = \epsilon\Phi(\mathbf{x}, t) + (1-\epsilon)\Phi(\mathbf{w}, t) - \Phi(\mathbf{z}, t).$$

Then as before it follows that

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x}, \mathbf{w}, t)|^2 = &|\Delta|^2 \int_0^1 \int_{\mathcal{S}^{d-1}} \left[\frac{h(|L_\epsilon|^{2p})}{|L_\epsilon|^{2p}}\left(1 - 2p\cos^2(\theta_\epsilon)\right) + \right. \\
&\left. h'(|L_\epsilon|^{2p}) 2p\cos^2(\theta_\epsilon)\right] f_0(\mathbf{z})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z})\mathrm{d}\epsilon,
\end{aligned}$$

where θ_ϵ denotes the angle between L_ϵ and Δ. As $h \geq 0$ when $|L_\epsilon|^{2p} < \delta$ and h' is bounded below it follows that

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta(\mathbf{x},\mathbf{w},t)|^2 \geq |\Delta|^2 \int_0^1 \int_{\mathcal{S}^{d-1}} \left[\frac{h(|L_\epsilon|^{2p})}{|L_\epsilon|^{2p}} \left(1 - 2p\cos^2(\theta_\epsilon)\right) \mathbf{1}_{\{|L_\epsilon|^{2p} \geq \delta\}} + \right.$$
$$\left. h'(|L_\epsilon|^{2p}) 2p \cos^2(\theta_\epsilon) \right] f_0(\mathbf{z})\, \mathrm{d}\mathcal{S}^{d-1}(\mathbf{z}) \mathrm{d}\epsilon \geq$$
$$|\Delta|^2 \mathrm{vol}(\mathcal{S}^{d-1}) ||f_0||_\infty \left(\min\left\{ \inf_{r \in (0, 2^{2p}||\Phi||_\infty^{2p}(t)]} h'(r), 0 \right\} - \frac{||h||_\infty}{\delta} \right).$$

Using Gronwall's inequality as before shows that $\mathrm{Lip}[\Phi^{-1}](t)$ remains finite for all time provided $||\Phi||_\infty(t)$ does. However, as in lemma 7 the hypothesis (H2) shows that

$$\frac{\partial}{\partial t}||\Phi||_\infty(t) \leq K||\Phi||_\infty^{1-2p}(t),$$

for some absolute constant K, so that $||\Phi||_\infty(t)$ does remains bounded for all finite time as desired.

□

5. Concluding Remarks

This paper provides the basic local in time well-posedness theory for an aggregation sheet, i.e. a solution to the aggregation equation that concentrates on a co-dimension one manifold. We focused our efforts on the case when the evolution equation (8) is linearly well-posed, and used the linear well-posedness condition to demonstrate that nonlinear well-posedness also holds. This condition enforces regularity in the kernel, and we therefore assumed only a modest amount regularity for the sheet itself. This contrasts to similar problems in the linearly ill-posed regime, most notably the Birkhoff-Rott equation, where local existence results have been known for some time for analytic sheets in two and three dimensions [25], and for chord-arc initial data [31] in two dimensions. Demonstrating local existence of sheet solutions to the aggregation equation (2) in the ill-posed regime proves an interesting open problem.

Regarding global existence, we showed that for attractive kernels the Osgood condition (35) determines whether or not solutions collapse in finite time. This makes a nice connection to the existing literature on the co-dimension zero aggregation equation, where similar results exist [3,4]. For a class of kernels with a repulsive singularity near the origin we provided a simple global existence result. While this class includes many kernels that appear in applications, such as the Morse potential, it fails to capture reasonable examples such as the power laws $g(s) = s^{-p}$ for $p > 1/2$. Our current methods for demonstrating global existence do not apply to such kernels, so we leave the problem of proving global existence for a broader class of repulsive kernels for future research.

6. Acknowledgements

We would like to thank David Uminsky for his editorial comments regarding the manuscript. This work was supported by NSF Grants DMS-0907931 and EFRI-1024765.

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