FINITE-TIME BLOW-UP OF L^∞-WEAK SOLUTIONS OF AN AGGREGATION EQUATION *

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Abstract. We consider the aggregation equation $u_t + \nabla\cdot[(\nabla K)*u)u] = 0$ with nonnegative initial data in $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for $n\geq 2$. We assume that K is rotationally invariant, nonnegative, decaying at infinity, with at worst a Lipschitz point at the origin. We prove existence, uniqueness, and continuation of solutions. Finite time blow-up (in the L^∞ norm) of solutions is proved when the kernel has precisely a Lipschitz point at the origin.

Key words.
subject classifications.

1. Introduction

In this paper we consider the evolution equation

$$u_t + \nabla\cdot[(\nabla K)*u)u] = 0 \tag{1.1}$$

with nonnegative initial data in $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for $n\geq 2$. We prove existence, uniqueness, and continuation of solutions. Finite time blow-up (in the L^∞ norm) of solutions is proved when the kernel has precisely a Lipschitz point at the origin. Equation (1.1) arises in the study of animal aggregations [11, 20, 22, 23] and also certain problems in materials science [13, 14]. In the case of animal aggregations, u represents population density, while in materials applications u typically represents a particle density. The case $K(x) = e^{-|x|}$, which we focus on here, arises in both the biological and materials literature [3, 13, 14, 20, 23].

By differentiation, (1.1) can be written as

$$\frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u = -(\Delta K * u)u \quad \text{and} \quad \vec{v} = \nabla K * u. \tag{1.2}$$

This demonstrates that (1.1) is an advection-reaction equation; the solution u is amplified along characteristics by the nonlocal operator $(-\Delta K * u)u$. Problems such as (1.1), in which a quantity is transported by a vector field obtained by applying a nonlocal operator to that quantity, are known as active scalar problems [8]. Active scalar problems are common in fluid dynamics and have been used as model problems for vortex stretching in the 3-D Euler equations. One source of interest in vortex stretching is its intimate connection with the regularity of solutions of the incompressible Euler equations; it was proven in [2] that smooth solutions to the Euler equations develop singularities only if the vorticity becomes infinite in a certain sense. According to the Euler equations, the vorticity $\vec{\omega}$ satisfies

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{v}\cdot\nabla\vec{\omega} = \vec{\omega}\cdot\nabla\vec{v} \quad \text{and} \ \vec{v} = \vec{K_3} * \vec{\omega} \tag{1.3}$$

where K_3 is the 3-D Biot-Savart kernel, which is homogeneous of degree -2. Vortex stretching is the result of the non-local amplification term $\vec{\omega}\cdot\nabla\vec{v}$ in (1.3). Two examples of active scalar problems bearing similarity to (1.3) are the 2-D quasi-geostrophic

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equations [10] and the Constantin-Lax-Majda one-dimensional problem $u_t = H(u)u$, where H denotes the Hilbert transform [9]. In [9], an explicit formula for the solution to the Constant-Lax-Majda one dimensional model problem was derived and used to provide conditions under which finite time blow-up occurs. In [10], an analogue of the Beale-Kato-Majda theorem [2] was proven for the quasigeostrophic active scalar. A third active scalar problem, which we discuss in more detail below, is the 2-D vorticity equation

$$\omega_t + \vec{v}\cdot\nabla\omega = 0 \quad \text{where} \quad \vec{v} = \vec{K}*\omega \quad \text{and} \quad \vec{K}(x) = \frac{1}{|x|^2}(-x_2, x_1). \tag{1.4}$$

Singularities do not occur in (1.4) due to the absence of an amplification term. Note that (1.1) involves advection by a gradient field, a distinguishing feature from the active scalar problems in fluid dynamics involving transport by a divergence free flow.

In two space dimensions and higher, the questions of short-time existence and uniqueness of solutions of (1.1) were addressed in [3, 16] for initial data belonging to $H^s(\mathbb{R}^n)$, $s \geq 2$. Global-in-time solutions of (1.1), with the addition of a density-dependent diffusion used to model repulsion, were studied in [6] for initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ also satisfying $u_0^2 \in H^1(\mathbb{R}^n)$. Uniqueness of an entropy solution was proved for this problem. In one space dimension, short-time existence and uniqueness of (1.1) were established in [4] for C^1 initial data. In [7], equation (1.1), with the addition of nonlinear diffusion and an external potential, was studied in one dimension. Existence and uniqueness of weak solutions for initial data belonging to $L^1(\mathbb{R}^n)$ was proven by viewing solutions as gradient flows in the Wasserstein metric and long-time behavior of solutions was also studied.

For pointy potentials such as $K(x) = e^{-|x|}$ which have a discontinuity in their first derivative at the origin, blow-up of (1.1) has been shown to occur, under appropriate initial conditions, in all space dimensions [3, 4]. However, currently known conditions under which blow-up will occur in space dimensions two or more are much more restrictive than those for the one dimensional case. In one space dimension, L^∞ finite-time blow-up of (1.1) when $K(x) = e^{-|x|}$ was proven for a large class of initial data by integrating (1.2) along characteristics [4]. Since in one space dimension the most singular part of $\Delta(e^{-|x|})$ is the δ distribution, we see from the right hand side of (1.2) that (1.1) exhibits quadratic amplification along characteristics. By establishing estimates controlling the behavior of particle paths, the authors of [4] showed that in one space dimension solutions of (1.1) satisfy $\frac{D}{Dt}u \geq Cu^2 - Du$. Applying this differential inequality, along with an appropriate continuation theorem on solutions of (1.1), it follows that whenever the initial data u_0 satisfies $u_0(x) \geq \frac{D}{C}$ at some point x, the solution exhibits finite time blow up in the L^∞ norm. Analytic and computational aspects of blow-up of (1.1) in one dimension were also studied in [17]. As the space dimension increases, $\Delta(e^{-|x|})$ becomes progressively less singular; in dimensions two and higher $\Delta(e^{-|x|})$ does not contain a Dirac delta function, its singular part is of the form $\frac{1}{|x|}$. As a result the convolution on the right hand side of (1.2) becomes a smoothing operator which is less localized than in one dimension; as a result, its effect is less severe. It has been shown that if the kernel K has bounded second derivatives, then solutions exist for all time [16, 22]; again, we see that the degree of singularity of K controls blow-up. Finite-time blow-up of (1.1) in the case $K(x) = e^{-|x|}$ was proven for radially symmetric initial data supported inside a small enough ball in [3]. In order to show this, the authors introduce an energy $E(u)$ such that for solutions u of (1.1), $E(u) \leq C_1 \|u_0\|^2_{L^1(\mathbb{R}^n)}$, where u_0 is the initial data, and $\frac{dE(u)}{dt} \geq C_2 > 0$. Combining these bounds on $E(u)$, we find that solutions cease to exist beyond some time. It then follows from the continuation theorem in [3] that solutions exhibit blow-up in all L^p such that $p>2$ in two space dimensions and $p\geq 2$ in three dimensions or more. We extend this blow-up result to more general initial data in section 6 of this paper.

A natural choice of initial data is pointwise finite density (L^∞) and finite total mass (L^1); this is the physical motivation for the choice of problem considered here. Since (1.1) is a transport equation, there is no gain in regularity and solutions remain in $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ at later times. We derive an explicit representation formula for smooth solutions of (1.1) using the method of characteristics which demonstrates this. In sections 2-4, we give a definition of weak solution of (1.1) for nonnegative initial data belonging to $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and prove the existence of such solutions. Because (1.1) is a transport equation, the Sobolev space arguments used in [3] to prove local-in-time existence for $H^s(\mathbb{R}^n)$ initial data do not naturally extend to $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ initial data. This paper presents a different method, set in a Lagrangian reference frame, to establish local-in-time existence for such initial data. In section 5, we prove uniqueness of weak solutions which obey a mild decay hypothesis at infinity. Our proof uses a technique from [1] to establish an H^{-1} energy estimate for the difference of two solutions, followed by an application of Grönwall's inequality. We prove finite-time blow-up in section 6 for initial data that is radially symmetric and supported inside a small enough ball. Blow-up is shown to occur in all L^p such that $p>2$ in two space dimensions and $p\geq 2$ in three or more space dimensions following the energy argument in [3].

The bulk of our work is devoted to proving existence of weak solutions. In order to do this, we introduce ideas from incompressible flow. In [25], Yudovich established existence and uniqueness of solutions for all time of the vorticity stream formulation of the Euler equations (1.4) for initial vorticity belonging to $L^1(\Omega)\cap L^\infty(\Omega)$ where Ω is a bounded domain. Although (1.1) and (1.4) are both transport equations with non-local velocity, a direct application of the methods used in studying (1.4) to (1.1) is not possible since (1.1) and (1.4) describe different phenomenon. Equation (1.4) describes convection by a divergence-free flow, while (1.1) corresponds to convection by a gradient flow with additional amplification of the density along characteristics. Nevertheless, some of the Lagrangian constructions arising from the study of (1.4) are useful for our problem; in particular, we borrow ideas from chapter eight of [19]. We also introduce new ideas to control the amplification along characteristics of (1.1).

2. Weak solutions Our first step is to define a notion of weak solution for (1.1) given initial data

$$0\leq u(\cdot,t=0)=u_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n). \tag{2.1}$$

We follow the approach taken in [1].

Definition 2.1. *A function u is a weak solution of (1.1) on [0,T] for initial data (2.1) if it satisfies*

1. $u\in L^\infty([0,T];L^1(\mathbb{R}^n))$ *and* ess sup $_{0\leq s\leq T}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}<\infty$.
2. *(Weak differentiability in time) Solutions must satisfy* $u_t\in L^\infty([0,T];V^*)$ *and*

$$\int_0^T \langle u_t,\phi\rangle\ dt+\int_0^T\int_{\mathbb{R}^n}(u-u_0)\phi_t\ dx\ dt=0 \tag{2.2}$$

for every test function $\phi\in W^{1,1}([0,T];V)$ such that $\phi(T)=0$. Here, $V=\{f\in L^\infty(\mathbb{R}^n):\nabla f\in L^2(\mathbb{R}^n)\}$ endowed with the norm $\|f\|_V=\|f\|_{L^\infty(\mathbb{R}^n)}+\|\nabla f\|_{L^2(\mathbb{R}^n)}$.

3. (Weak solution of evolution equation) Solutions satisfy a weak form of (1.1):

$$\int_0^T \langle u_t,\phi\rangle \; dt - \int_0^T \int_{\mathbb{R}^n} \langle (\nabla K * u)u, \nabla\phi\rangle \; dx \; dt = 0 \tag{2.3}$$

for all $\phi \in L^1([0,T];V)$.

Note that classical solutions to the aggregation equation are also weak solutions and weak solutions to (1.1) for initial data (2.1) with sufficient regularity are classical solutions. It follows from our definition of weak solution that mass is conserved.

LEMMA 2.1. *(Mass conservation) Let* u *be a weak solution of (1.1) on* $[0,T]$ *for initial data* u_0*. Then* u *conserves mass:*

$$\int_{\mathbb{R}^n} u(x,t) \; dx = \int_{\mathbb{R}^n} u_0(x) \; dx \quad \textit{for a.e. } t \in [0,T].$$

Proof.

Let $\psi \in C([0,T])$ *and set* $\phi(x,t) = \int_0^t \psi(s) \; ds - \int_0^T \psi(s) \; ds$*. From (2.3),*

$$\int_0^T \langle u_t,\phi\rangle \; dt + \int_0^T \int_{\mathbb{R}^n} (u-u_0)\psi \; dx \; dt = 0.$$

Since ϕ *has no dependence on the spatial variable, it follows from(2.3) that* $\int_0^T \langle u_t,\phi\rangle \; dt = 0$*. Hence, we have*

$$\int_0^T \psi(s) \int_{\mathbb{R}^n} (u-u_0) \; dx \; dt = 0.$$

Since ψ *was arbitrary, the lemma is proved.* $\square$

Definition of admissible kernel. Because we use results from [16], we assume throughout that $\nabla K \in L^2(\mathbb{R}^n)$ and $\Delta K \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \in [p^\star, 2]$ where $\frac{1}{p^\star} = \frac{1}{2} + \frac{1}{n}$.

3. Short time existence of weak solutions

In this section, we prove short time existence of solutions to (1.1) for initial data (2.1). Short time existence of classical solutions to (1.1) for initial data belonging to $H^s(\mathbb{R}^n), s \geq 2$ was demonstrated using energy methods in an Eulerian reference frame [16]. Because we are interested in solutions to (1.1) belonging to $L^\infty([0,T]; L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$, it is more convenient to work in a Lagrangian reference frame. We will prove

THEOREM 3.1. *(Local existence) Let* $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then we can find* $T > 0$ *such that there exists a weak solution of (1.1) for initial data (2.1) on the time interval* $[0,T]$*.* T *is a function of* $\|u_0\|_{L^1(\mathbb{R}^n)}$ *and* $\|u_0\|_{L^\infty(\mathbb{R}^n)}$*.*

Our approach is based on a reformulation of (1.1) in terms of particle paths. For smooth solutions of (1.1), recall that $v = \nabla K * u$ and introduce the particle paths $X(\alpha,t)$ defined by the family of ODEs

$$\frac{dX}{dt}(\alpha,t) = v(X(\alpha,t),t), \quad X(\alpha,t=0) = \alpha.$$

Let $X^{-t}(\alpha)$ denote the inverse particle path: $X^{-t}(X(\alpha,t))=\alpha$. Note that equation (1.2) is equivalent to $\frac{d}{dt}u(X(\alpha,t))=-\text{div }v(X(\alpha,t),t)u$ which implies

$$u(x,t)=u_0(X^{-t}(x))e^{\int_0^t -\text{div }v(X^{s-t}(x),s)\ ds}. \tag{3.1}$$

In the above, we use the shorthand X^{s-t} for $X^s\circ X^{-t}$. Equation (3.1) provides a simple representation of solutions of (1.1) in terms of the initial data, particle paths, and div v. The proof of existence is constructive and utilizes (3.1); it relies on the following five steps.

Outline of existence argument

1. We construct a family of approximating solutions $\{u_\epsilon\}$ which we show exist on a common time interval. Each u_ϵ solves (1.1) with initial data $u_0 * J_\epsilon$, where J_ϵ is a mollifier.
2. We establish uniform Lipschitz estimates for the $\{v_\epsilon\}$ and particle maps in space and time. From these estimates, we show that there exists a sequence $\{\epsilon_j\}$, $\lim_{j\to\infty}\epsilon_j=0$, such that the $\{X_{\epsilon_j}\}$ converge to a particle map X.
3. We show that the $\{\text{div } v_\epsilon\}$ are uniformly quasi-Lipschitz in space in the two dimensional case, and uniformly Lipschitz in space in higher dimensions.
4. We use steps 2 and 3 to demonstrate that the $a_\epsilon(x,t)=\int_0^t \text{div } v_\epsilon(X_\epsilon^{s-t}(x),s)\ ds$ form an equicontinuous and uniformly bounded family of functions in space and time. This implies that a subsequence, which we refer to as $\{a_{\epsilon_j}\}$, converges to a function $a(x,t)$.
5. Recalling our representation (3.1), define the weak solution

$$u(x,t)=u_0(X^{-t}(x))a(x,t) \quad \text{and} \quad v(x,t)=\nabla K * u(x,t). \tag{3.2}$$

The proof of Theorem 3.1 follows from steps 1-5 and Proposition 3.1, which we state below.

PROPOSITION 3.1. *(Uniform bounds) Let u_0 satisfy (2.1) and let (u_ϵ,v_ϵ) be defined as above. Then the (u_ϵ,v_ϵ) exist on a common time interval $[0,T]$ and possess the following properties.*

1. *The u_ϵ and v_ϵ are uniformly bounded on $[0,T]$ and there exist a constant $c>0$ depending on $\|u_0\|_{L^1(\mathbb{R}^n)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^n)}$ such that*

$$\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)}+\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\le c\Big(\|u_0\|_{L^1(\mathbb{R}^n)},\|u_0\|_{L^\infty(\mathbb{R}^n)}\Big) \text{ for } t\in[0,T].$$

2. *If we define u and v according to formula (3.2) above, we have that there exists a sequence $\{\epsilon_k\}$, $\lim_{k\to\infty}\epsilon_k=0$, such that*

$$\lim_{k\to\infty}\sup_{0\le s\le T}\|u_{\epsilon_k}(\cdot,s)-u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}=0 \tag{3.3}$$

$$\lim_{k\to\infty} v_{\epsilon_k}(\cdot,t)=v(\cdot,t) \quad \textit{pointwise a.e.} \tag{3.4}$$

Proposition 3.1 is proven in the next section. We now show how the proof of Theorem 3.1 follows directly from steps 1-5 and Proposition 3.1.

Proof.

We will show that u_t exists and satisfies

$$\int_0^T \langle u_t,\phi\rangle\ dt=\int_0^T\int_{\mathbb{R}^n}\langle(\nabla K * u)u,\nabla\phi\rangle\ dx\ dt$$

for all $\phi \in L^2([0,T];V)$. Let $\phi \in H^1([0,T];V)$ be such that $\phi(T)=0$; we have by Proposition 3.1 that

$$\int_0^T \int_{\mathbb{R}^n} (u-u_0)\phi_t \; dx \; dt = \lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^n} (u_\epsilon - u_0 * J_\epsilon)\phi_t \; dx \; dt$$

$$= \lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^n} \langle (\nabla K * u_\epsilon)u_\epsilon, \nabla\phi\rangle \; dx \; dt = \int_0^T \int_{\mathbb{R}^n} \langle (\nabla K * u)u, \nabla\phi\rangle \; dx \; dt.$$

By density of $\{\phi \in H^1([0,T];V) : \phi(T)=0\}$ in $L^2([0,T];V)$, it follows that u_t exists and satisfies $\int_0^T \langle u_t, \phi\rangle \; dt = \int_0^T \int_{\mathbb{R}^n} \langle (\nabla K * u)u, \nabla\phi\rangle \; dx$. ∎

4. Uniform bounds

In this section we present the details of steps 1-5 and the proof of Proposition 3.1. At the end of the section, we discuss two corollaries regarding the continuity in time of the constructed solution; these results play an important role in our discussion of blow-up in section 6. We begin by establishing step 1), the existence of a common time interval for the $\{u_\epsilon\}$. We assume that $J_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$ for some $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$.

Lemma 4.1. *Let $0 \le u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and let u_ϵ denote the solution to 1.1 with initial data $u_0 * J_\epsilon$, where J_ϵ is a mollifier. Then the $\{u_\epsilon\}$ exist on a common time interval $[0,T]$ and there exists a constant c depending on $\|u_0\|_{L^1(\mathbb{R}^n)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^n)}$ such that*

$$\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} + \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c\Big(\|u_0\|_{L^1(\mathbb{R}^n)}, \|u_0\|_{L^\infty(\mathbb{R}^n)}\Big) \text{ for } t \in [0,T].$$

Proof. Note that $\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)}$ is constant in time by mass conservation [16]. We work in a Lagrangian reference frame and use equation (1.1) to derive uniform bounds on the growth rate of $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$. Setting $\frac{D}{Dt} = u_t + v\cdot\nabla u$, equation (1.1) implies that

$$|\frac{D}{Dt}u_\epsilon| = |-(\Delta K * u_\epsilon)u_\epsilon| \le \|\Delta K\|_{L^p(\mathbb{R}^n)} \|u_\epsilon\|_{L^{p'}(\mathbb{R}^n)} \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le$$

$$\le \|\Delta K\|_{L^p(\mathbb{R}^n)} \|u_\epsilon\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p'}} \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1+\frac{p'-1}{p'}}.$$

Since $\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|u_\epsilon(\cdot,0)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$ we have from above that

$$|\frac{D}{Dt}u_\epsilon| \le \|\Delta K\|_{L^p(\mathbb{R}^n)} \|u_0\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p'}} \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1+\frac{p'-1}{p'}} = c\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{\alpha}$$

where

$$c = \|\Delta K\|_{L^p(\mathbb{R}^n)} \|u_0\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p'}} \quad \text{and} \quad \alpha = 1 + \frac{p'-1}{p'}.$$

Written in integral form, this is

$$\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c\int_0^t \|u_\epsilon(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}^{\alpha} \; ds + \|u_\epsilon(\cdot,0)\|_{L^\infty(\mathbb{R}^n)}. \tag{4.1}$$

We now apply (4.1) to finish the proof. First, recall that the continuation theorem from [3] states that if a classical solution u of (1.1) cannot be continued past a time T, then $\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ blows up as t approaches T. We require that $q>2$ if $n=2$ and $q\geq 2$ if $n>2$. Next, note that from the existence theorem in [3] and the Sobolev embedding theorem that $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is a continuous function of time. It then follows from (4.1) that

$$\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq (\|u_0\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)} + c(1-\alpha)t)^{\frac{1}{1-\alpha}}.$$

Since $\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|u_\epsilon(\cdot,0)\|_{L^1(\mathbb{R}^n)}$ by mass conservation, and $\|u_\epsilon(\cdot,0)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$ using properties of the mollifier, we have that $\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$. Combined with the above estimate for $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n}$, this implies that we cannot have blow-up of $\|u_\epsilon(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ on $[0,T]$, for $T < \frac{\|u_0\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)}}{c(\alpha-1)}$. ∎

Next, we move to step 2. We first establish uniform Lipschitz estimates on the velocities $\{v_\epsilon\}$ and then use these to obtain uniform Lipschitz estimates for the particle paths.

LEMMA 4.2. *(Uniform Lipschitz estimates for velocities) Let* $v_\epsilon = \nabla K * u_\epsilon$. *The family* $\{v_\epsilon\}$ *is uniformly Lipschitz in space and time.*

Proof. First, we show that the v_ϵ are uniformly Lipschitz in time:

$$|v_\epsilon(x,t_1) - v_\epsilon(x,t_1)| \leq c|t_1 - t_2| \quad \text{for some } c \text{ and all } (x,t) \in \mathbb{R}^n \times [0,T].$$

Recalling that $v_\epsilon = \nabla K * u_\epsilon$, and assuming $t_1 \leq t_2$ we have that

$$|v_\epsilon(x,t_1) - v_\epsilon(x,t_1)| \leq |\int_{\mathbb{R}^n} \nabla K(x-y)(u_\epsilon(y,t_1) - u_\epsilon(y,t_2)\ dy| = |\int_{\mathbb{R}^n} \nabla K(x-y) \int_{t_1}^{t_2} (u_\epsilon)_t(y,s)\ ds\ dy| =$$

$$= |\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \nabla K(x-y)[-\nabla \cdot (\nabla K * u_\epsilon\ u_\epsilon)]\ dy\ ds| \text{ (by Fubini and the fact that the } u_\epsilon \text{ solve (1.1))} \leq$$

$$\leq \int_{t_1}^{t_2} |\int_{\mathbb{R}^n} -D^2K(x-y)[\nabla K * u_\epsilon](y,s) u_\epsilon(y,s)\ dy|\ ds \quad \text{by integrating by parts.}$$

The result then follows from our assumption that $|D^2K| \in L^r(\mathbb{R}^n)$ for some $r \geq 1$ and our uniform bounds on $\|\nabla K * u_\epsilon u_\epsilon(\cdot,t)\|_{L^{r'}(\mathbb{R}^n)}$.

Now we show that v_ϵ are Lipschitz continuous in space:

$$|v_\epsilon(x_1,t) - v_\epsilon(x_2,t)| \leq c|x_1 - x_2| \quad \text{for some } c \text{ and all } (x,t) \in \mathbb{R}^n \times [0,T].$$

This follows from

$$|Dv_\epsilon| = |\int_{\mathbb{R}^n} D^2K(x-y)u_\epsilon(y)\ dy| \leq \|\ |D^2K|\ \|_{L^r(\mathbb{R}^n)} \||u_\epsilon|\|_{L^{r'}(\mathbb{R}^n)}.$$

∎

We now apply the uniform Lipschitz estimates on the $\{v_\epsilon\}$ to deduce equicontinuity of the particle maps $\{X_\epsilon\}$ and their inverses.

LEMMA 4.3. *(Uniform Lipschitz estimates for particle maps and their inverses) If* $t \leq T$, *then there exists* c *such that*

$$|X_\epsilon^{-t}(x_1) - X_\epsilon^{-t}(x_2)| \leq c|x_1 - x_2|$$

$$|X_\epsilon(\alpha_1,t)-X_\epsilon(\alpha_2,t)|\le c|\alpha_1-\alpha_2|.$$

If $t_1,t_2\le T$, we have that

$$|X_\epsilon^{-t_1}(x)-X_\epsilon^{-t_2}(x)|\le c|t_1-t_2|$$

$$|X_\epsilon(\alpha,t_1)-X_\epsilon(\alpha,t_2)|\le c|t_1-t_2|.$$

Proof. We first prove the second inequality. Setting

$$\rho(t)=|X_\epsilon(\alpha_1,t)-X_\epsilon(\alpha_2,t)|$$

we have that

$$\frac{d\rho}{dt}\le|v_\epsilon(X_\epsilon(\alpha_1,t),t)-v_\epsilon(X_\epsilon(\alpha_2,t),t)|\le c\rho(t)\Rightarrow$$

$$\rho(t)\le\rho(0)e^{ct}.$$

The first inequality follows similarly. Introduce the backward particle paths $Y_\epsilon(x,t;\tau)$ defined by

$$\frac{dY_\epsilon}{d\tau}(x,t;\tau)=-v_\epsilon(Y_\epsilon(x,t;\tau),t-\tau)\quad Y_\epsilon(x,t;0)=x.$$

With this definition, we have that $Y_\epsilon(x,t;t)=X_\epsilon^{-t}(x)$. Setting

$$\rho(\tau)=|Y_\epsilon(x_1,t;\tau)-Y_\epsilon(x_2,t;\tau)|$$

we have that $\rho(0)=|x_1-x_2|$ and $\rho(t)=|X_\epsilon^{-t}(x_1)-X_\epsilon^{-t}(x_2)|$. The uniform Lipschitz estimates on $\{v_\epsilon\}$ from Lemma 4.2 imply that

$$\frac{d}{d\tau}\rho(\tau)\le c\rho(\tau)\quad\Rightarrow$$

$$|X_\epsilon^{-t}(x_1)-X_\epsilon^{-t}(x_2)|=\rho(t)\le e^{ct}|x_1-x_2|.$$

The fourth inequality follows from

$$|X_\epsilon(\alpha,t_1)-X_\epsilon(\alpha,t_2)|=|\int_{t_1}^{t_2}v(X(\alpha,s),s)\ ds|\le\Big(\sup_{\mathbb{R}^n\times[0,T^*]}|v|\Big)|t_1-t_2|.$$

To prove the third inequality, we proceed as in [19], page 319. Assuming $t_1\le t_2$, we have that $X_\epsilon^{-t_2}(x)=X_\epsilon^{-t_1}(\alpha^*)$, where $\alpha^*=Y_\epsilon(x,t_2;t_2-t_1)$. It follows that

$$|X_\epsilon^{-t_1}(x)-X_\epsilon^{-t_2}(x)|=|X_\epsilon^{-t_1}(x)-X_\epsilon^{-t_1}(\alpha^*)|\le c|x-\alpha^*|$$

$$=c|\int_{t_1}^{t_2}v_\epsilon(X_\epsilon(\alpha^*,\tau),\tau)\ d\tau\le c|t_2-t_1|.$$

In the above, we used

$$|X_\epsilon(\alpha_1,t)-X_\epsilon(\alpha_2,t)|\le c|\alpha_1-\alpha_2|$$

and the fact that we have a uniform bound on $\sup_{\mathbb{R}^n\times[0,T^*]}|v_\epsilon(x,t)|$. □

Due to Lemma 4.3, we may now apply the Arzela-Ascoli theorem to the families $\{X_{\frac{1}{n}}^{-t}(x)\}_{n\in\mathbb{N}}$, $\{X_{\frac{1}{n}}(\alpha,t)\}_{n\in\mathbb{N}}$, and $\{v_{\frac{1}{n}}(x,t)\}_{n\in\mathbb{N}}$ to conclude that there exists a sequence $\{\epsilon_j\}$, $\lim_{j\to\infty}\epsilon_j=0$, such that $X_{\epsilon_j}^{-t}(x)\to X^{-t}(x)$, $X_{\epsilon_j}(\alpha,t)\to X(\alpha,t)$ and $v_{\epsilon_j}(x,t)\to v(x,t)$ uniformly on compact sets of $\mathbb{R}^n\times[0,T]$ as $j\to\infty$. We then have

$$\frac{dX}{dt}(\alpha,t)=v(X(\alpha,t),t).$$

This follows by taking the limit as $j\to\infty$ in the identity

$$X_{\epsilon_j}(\alpha,t)=X_{\epsilon_j}(\alpha,0)+\int_0^t v_\epsilon(X_{\epsilon_j}(\alpha,t),t)\ ds$$

to arrive at

$$X(\alpha,t)=X(\alpha,0)+\int_0^t v(X(\alpha,t),t)\ ds$$

using the the uniform convergence of the X_{ϵ_j} and v_{ϵ_j} to X and v, respectively.

Before we can use the representation formula (3.2), we must establish steps 3 and 4. We begin by establishing step 3; in two dimensions this amounts to proving the existence of uniform quasi-Lipschitz estimates on the $\{\text{div } v_\epsilon\}=\{\Delta K*u_\epsilon\}$ for the kernel $K(x)=e^{-|x|}$, while in three dimensions or more it amounts to proving uniform Lipschitz bounds. We first discuss the higher dimensional case.

Lemma 4.4. *(Lipschitz estimates for div v in $\mathbb{R}^n$, $n\ge 3$) Let $n\ge 3$ and set $K(x)=e^{-|x|}+g(x)$, where $\Delta g=h\in C^{0,1}(\mathbb{R}^2)$. Also, let $u\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Then $\Delta K*u$ is Lipschitz continuous:*

$$|\Delta K*u(x_1)-\Delta K*u(x_2)|\le c(\|u\|_{L^1(\mathbb{R}^n)}+\|u\|_{L^\infty(\mathbb{R}^n)}+\|\nabla h\|_{L^\infty(\mathbb{R}^n)})|x_1-x_2|.$$

Proof. It suffices to show that $\Delta K*u$ has a bounded first derivative; to do this, we differentiate $\Delta K*u(x)=\int_{\mathbb{R}^n}\Delta K(x-y)u(y)\ dy$ under the integral sign. Since $\Delta K=(\frac{1}{|x|}+1)e^{-|x|}+g(x)$, this is permitted when $n\ge 3$. □

In the two dimensional case, we cannot differentiate under the integral sign, since $\nabla\frac{1}{|x|}e^{-|x|}$ is not locally integrable at the origin. Computing the weak derivative of $\frac{1}{|x|}e^{-|x|}$ in two dimensions, we find that

$$\frac{\partial}{\partial x_j}\frac{1}{|x|}e^{-|x|}=\frac{-x_j}{|x|^2}e^{-|x|}+\ \text{PV}\ \frac{-x_j}{|x|^3}e^{-|x|}+c\delta_{x=0}$$

where PV u is the distribution defined by

$$\text{PV}\ u\ (\phi)=\lim_{\epsilon\to 0}\int_{|x|>\epsilon}u(y)\phi(y)\ dy\qquad\text{for } \phi\in C_c^1(\mathbb{R}^n).$$

This highlights the difference between the two and three dimensional cases. One consequence of this is that the $\{\text{div } v_\epsilon\}$ are only uniformly quasi-Lipschitz in $\mathbb{R}^2$.

Lemma 4.5. *(Quasi-Lipschitz estimates for div v in $\mathbb{R}^2$) Let $K(x)=e^{-|x|}+g(x)$ where $\Delta g=h\in C^{0,1}(\mathbb{R}^n)$. Also, let $u\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then $\Delta K*u$ is quasi-Lipschitz with constants that depend only on $\|u\|_{L^1(\mathbb{R}^2)}$ and $\|u\|_{L^\infty(\mathbb{R}^2)}$:*

$$|\Delta K*u(x_1)-\Delta K*u(x_2)|\leq c(\|u\|_{L^1(\mathbb{R}^2)}+\|u\|_{L^\infty(\mathbb{R}^2)})|x_1-x_2|(1-ln^-|x_1-x_2|)+\|\nabla h\|_{L^\infty(\mathbb{R}^n)}|x_1-x_2|.$$

Here, c is independent of u and $ln^- x=ln\ x$ for $0<x<1$ and $ln^- x=0$ for $x\geq 1$.

Proof. See appendix. □

We now apply Lemmas 4.4 and 4.5 to establish step 4. We begin with a definition.

Definition 4.1. *Set $a_\epsilon=\int_0^t div\ v_\epsilon(X_\epsilon^{s-t}(x),s)\ ds$.*

Lemma 4.6. *The $\{a_\epsilon\}$ form an equicontinuous and uniformly bounded family of functions on $\mathbb{R}^n\times[0,T]$.*

Proof. Uniform boundedness of the family comes from our assumptions on the kernel K and Lemma 4.1. Equicontinuity in space comes from Lemmas 4.3-4.5. Carrying out the calculation in the case of two dimensions, we have that

$$\Big|\int_0^t \text{div } v_\epsilon(X_\epsilon^{s-t}(x),s)\ ds-\int_0^t \text{div } v_\epsilon(X_\epsilon^{s-t}(y),s)\ ds\Big|\leq\int_0^t\Big|\text{div } v_\epsilon(X_\epsilon^{s-t}(x),s)-\text{div } v_\epsilon(X_\epsilon^{s-t}(y),s)\Big|\ ds$$

$$\leq\int_0^t C\Big|X_\epsilon^{s-t}(x)-X_\epsilon^{s-t}(y)\Big|\Big(1-\ln^-\Big|X_\epsilon^{s-t}(x)-X_\epsilon^{s-t}(y)\Big|\Big)\ ds \text{ by Lemma 4.5.}$$

Applying Lemma 4.3 to the above gives equicontinuity in space of the $\{\int_0^t \text{div } v_\epsilon(X_\epsilon^{s-t}(x),s)\ ds\}$. Equicontinuity in time can be shown similarly. First note that, assuming $t_1\leq t_2$,

$$\int_0^{t_1}\text{div } v_\epsilon(X_\epsilon^{s-t_1}(x),s)\ ds-\int_0^{t_2}\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\ ds=$$

$$\int_0^{t_1}\Big(\text{div } v_\epsilon(X_\epsilon^{s-t_1}(x),s)-\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\Big)\ ds-\int_{t_1}^{t_2}\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\ ds.$$

Since $\Delta K\in L^1(\mathbb{R}^n)$, we have that the div $v_\epsilon=\Delta K*u_\epsilon$ are uniformly bounded. This implies that

$$\Big|\int_{t_1}^{t_2}\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\ ds\Big|\leq C|t_1-t_2|.$$

Finally, we can show that, for each $\delta>0$, we can choose $\delta'>0$ such that

$$|t_1-t_2|<\delta'\Rightarrow\int_0^{t_1}\Big|(\text{div } v_\epsilon(X_\epsilon^{s-t_1}(x),s)-\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\Big|\ ds<\delta.$$

To do this, first note that Lemma 4.5 implies that

$$|\int_0^{t_1}(\text{div } v_\epsilon(X_\epsilon^{s-t_1}(x),s)-\text{div } v_\epsilon(X_\epsilon^{s-t_2}(x),s)\ ds|$$

$$\leq c(\|u\|_{L^1(\mathbb{R}^2)}+\|u\|_{L^\infty(\mathbb{R}^2)})\int_0^{t_1}|X_\epsilon^{s-t_1}(x)-X_\epsilon^{s-t_2}(x)|(1-\ln^-|X_\epsilon^{s-t_1}(x)-X_\epsilon^{s-t_2}(x)|)\ ds.$$

An application of Lemma 4.3 finishes the argument. ▯

We now apply the Arzelà-Ascoli theorem to conclude that there exists a subsequence of the $\{a_{\epsilon_j}\}$, which we refer to as $\{a_{\epsilon_k}\}$, which converges uniformly on compact subsets of $\mathbb{R}^n \times [0,T]$ to a function $a(x,t)$:

$$a_{\epsilon_k} \to a \text{ uniformly on compact sets of } \mathbb{R}^n \times [0,T] \text{ as } k \to \infty.$$

We follow step 5 and set

$$u(x,t) = u_0(X^{-t}(x))a(x,t) \quad \text{and} \quad v = \nabla K * u.$$

All that remains is proving (3.3) and (3.4). We first prove a preliminary proposition.

Lemma 4.7. *(Bounds on volume distortion) There exists a constant C such that for all $f \in L^1(\mathbb{R}^n)$ and $t \in [0,T]$*

$$\frac{1}{C}\|f\|_{L^1(\mathbb{R}^n)} \leq \|f(X^{-t}(x))\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)} \tag{4.2}$$

and

$$\frac{1}{C}\|f\|_{L^1(\mathbb{R}^n)} \leq \|f(X^{t}(x))\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}. \tag{4.3}$$

In particular, denoting Lebesgue measure by m, all measurable subsets $E \subset \mathbb{R}^n$ satisfy

$$\frac{1}{C}m(E) \leq m(X^t(E)) \leq Cm(E) \tag{4.4}$$

and

$$\frac{1}{C}m(E) \leq m(X^{-t}(E)) \leq Cm(E). \tag{4.5}$$

Proof. Since X^t and X^{-t} are Lipschitz continuous by Lemma 4.3, it suffices, by the change of variables formula for Lipschitz maps [12], to prove that there exists $C > 0$ such that

$$\frac{1}{C} \leq \; J(X^t(x)),\; J(X^{-t}(x)) \; \leq C \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^n.$$

Here, $J(X^t(x))$ refers to the Jacobian of X^t evaluated at x. In order to prove this, first note that by Lemma 4.3 there exists C such that

$$J(X^t(x)),\; J(X^{-t}(x)) \; \leq C \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^n.$$

Since Lipschitz functions map sets of measure zero to sets of measure zero and $X^t \circ X^{-t} = X^{-t} \circ X^t = I$, the above upper bound implies the lower bound

$$\frac{1}{C} \leq \; J(X^t(x)),\; J(X^{-t}(x)).$$

▯

We now apply Lemma 4.7 to finish our proof of Proposition 3.1.

Proof. All that remains is proving (3.3) and (3.4). Let $u_0^\epsilon = u_0 * J_\epsilon$. First, note that for any $\delta > 0$, we can find a compact set $K \subset \mathbb{R}^n$ such that $\int_{\mathbb{R}^n \setminus K} u_0(\alpha)\; d\alpha < \delta$

and $\int_{\mathbb{R}^n\setminus K} u_0^\epsilon(\alpha)\ d\alpha < \delta$ for small $\epsilon > 0$. Given such a K, by Proposition 4.3 we can find a compact set K' such that

$$x \in (K')^c \Rightarrow X_\epsilon^{-t}(x), X^{-t}(x) \in K^c \text{ for all } \epsilon > 0.$$

Since the $\{a_\epsilon\}$ are uniformly bounded, by Lemma 4.6, we have that

$$\int_{(K')^c} |u_{\epsilon_k}(x,t) - u(x,t)|\ dx < b\delta$$

for some constant b. We are left with showing that for all k large enough,

$$\int_{K'} |u_{\epsilon_k}(x,t) - u(x,t)|\ dx < \delta.$$

In order to do this, we break the integral into three pieces:

$$I = \int_{K'} |u_0^{\epsilon_k}(X_{\epsilon_k}^{-t}(x)) a_{\epsilon_k}(x,t) - u_0(X_{\epsilon_k}^{-t}(x)) a_{\epsilon_k}(x,t)|\ dx,$$

$$II = \int_{K'} |u_0(X_{\epsilon_k}^{-t}(x)) a_{\epsilon_k}(x,t) - u_0(X^{-t}(x)) a_{\epsilon_k}(x,t)|\ dx, \quad \text{and}$$

$$III = \int_{K'} |u_0(X^{-t}(x)) a_{\epsilon_k}(x,t) - u_0(X^{-t}(x)) a(x,t)|\ dx.$$

Since the $\{a_\epsilon\}$ are uniformly bounded by Lemma 4.6, I goes to zero as $k \to \infty$ by Lemma 4.7. III goes to zero as $\epsilon_k \to 0$ by Lemma 4.6, since the integration is over a compact set K'. Since the $\{a_\epsilon\}$ are uniformly bounded, it follows that $II \to 0$ as $k \to \infty$ once we show that

$$u_0(X_\epsilon^{-t}) \to u_0(X^{-t}) \quad \text{in} \quad L^1(\mathbb{R}^n).$$

To do this, we follow the procedure from [19], page 316. For arbitrary $\delta > 0$, we can find $u_\delta \in C_c(\mathbb{R}^n)$ such that $\|u_0 - u_\delta\|_{L^1(\mathbb{R}^n)} < \delta$. Note that

$$\begin{aligned}
&\|u_0(X_{\epsilon_k}^{-t}) - u_0(X^{-t})\|_{L^1(\mathbb{R}^n)} \\
&\le \|u_0(X_{\epsilon_k}^{-t}) - u_\delta(X_{\epsilon_k}^{-t})\|_{L^1(\mathbb{R}^n)} + \|u_\delta(X_{\epsilon_k}^{-t}) - u_\delta(X^{-t})\|_{L^1(\mathbb{R}^n)} + \|u_\delta(X^{-t}) - u_0(X^{-t})\|_{L^1(\mathbb{R}^n)} \\
&= (T1) + (T2) + (T3).
\end{aligned}$$

We can bound (T1) and (T3) using Lemma 4.7. For fixed δ, (T2) goes to zero as $\epsilon_k \to 0$ since $u_\delta \in C_c(\mathbb{R}^n)$ and $X_{\epsilon_k}^{-t}(x) \to X^{-t}(x)$ uniformly on compact subsets of $\mathbb{R}^n \times [0,T]$. This also shows that the convergence of the u_{ϵ_k} to u $L^1(\mathbb{R}^n)$ is uniform:

$$\lim_{k\to\infty} \sup_{0\le s\le T} \|u_{\epsilon_k}(\cdot,s) - u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

□

We now prove two corollaries of Proposition 3.1 and Theorem 3.1. Both corollaries are not surprising, since the construction of solutions in Theorem 3.1 is based on the method of characteristics. The first one states that the weak solutions constructed above belong to $C([0,T];L^1(\mathbb{R}^n))$.

Corollary 4.1. *(L^1 continuity in time) Let u be a weak solution of (1.1) with initial data (2.1), as constructed in Theorem 3.1. Then $u \in C([0,T];L^1(\mathbb{R}^n))$.*

Proof. First, note that the $u_\epsilon \in C([0,T];L^1(\mathbb{R}^n))$ by the existence theorem in [3]. The result then follows directly from (3.3).□

Our second corollary states that the $L^\infty(\mathbb{R}^n)$ norm of a weak solution of (1.1) is a continuous function of time.

Corollary 4.2. *(Continuity in time of L^∞ norm) Let u be a weak solution of (1.1) with initial data (2.1), as constructed in Theorem 3.1. The map $t \to \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is continuous.*

Proof. Let $g(s) = \|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}$ and fix $t \in [0,T]$; we first show that $\liminf_{s\to t} g(s) \geq g(t)$. Let $\delta > 0$ and set $\Omega_\delta = \{x : |u(x,t)| > g(t) - \delta\}$. We will show that for s sufficiently close to t, $\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} > \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} - 2\epsilon$. In order to do this, first note that $a(x,t)$ is uniformly continuous on $\mathbb{R}^n \times [0,T]$ since it is the pointwise limit of the $\{a_{\epsilon_j}\}$. Now, let $x \in \Omega_\delta$ and let $y = X^s(X^{-t}(x))$. Note that Lemma 4.7 implies that $m(X^s(X^{-t}(\Omega_\delta))) \geq \frac{1}{C^2} m(\Omega_\delta) > 0$. We now show

$$\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} > \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} - 2\delta \quad \text{for } s \text{ sufficiently close to } t. \tag{4.6}$$

Using Lemma 4.3, we can bound $|y-x|$:

$$|y-x| = |X^s(X^{-t}(x)) - X^s(X^{-s}(x))| \leq c|X^{-t}(x) - X^{-s}(x)| \leq c'|s-t|.$$

By (3.2), $|u(x,t) - u(y,s)| = |u_0(X^{-t}(x))(a(x,t) - a(y,s)| \to 0$ as $s \to t$ since a is uniformly continuous on $\mathbb{R}^n \times [0,T]$. This implies (4.6). Hence, $\liminf_{s\to t} g(s) \geq g(t)$.

Now, we show that $\limsup_{s\to t} g(s) \leq g(t)$; the argument is nearly the same as the above. Given $\delta > 0$, find s_δ close to t such that $g(s_\delta) > \limsup_{s\to t} g(s) - \delta$ and let $\Sigma_\delta = \{x : |u(x,s_\delta)| > g(s_\delta) - 2\delta\}$. Given $y \in \Sigma_\delta$, set $x = X^t(X^{-s_\delta}(y))$; we then have that $|u(x,t) - u(y,s_\delta)| = |u_0(X^{-s_\delta}(y))(a(x,t) - a(y,s_\delta))|$. Since $|y-x| \leq c'|t-s_\delta|$ by Lemma 4.3, we have, by the uniform continuity of a, that $g(t) > \limsup_{s\to t} g(s) - 3\delta$. Hence, $g(t) \geq \limsup_{s\to t} g(s)$. □

5. Uniqueness of weak solutions

In this section, we prove uniqueness of weak solutions of (1.1) with initial data (2.1), under a mild decay condition.

Theorem 5.1. *(Uniqueness of weak solutions) Let u_1, u_2 be two solutions of (1.1), defined on $\mathbb{R}^n \times [0,T]$, with initial data u_0 of the form (2.1) . Assume that the u_i obey the decay condition that for a.e. $t \in [0,T]$ we have the bound*

$$u_i(x,t) \leq C(1+|x|)^{-n-\alpha} \quad \text{for some } 0 < \alpha \text{ and all } x \in \mathbb{R}^n. \tag{5.1}$$

Then $u_1 = u_2$.

Before we prove Theorem 5.1, we outline the steps and discuss some necessary results. Set $\tilde{u} = u_1 - u_2$, let ϕ be the unique bounded solution of $\phi = -\Delta^{-1}\tilde{u}$, and let $v = \nabla\phi$. The proof of Theorem 5.1 consists of using the weak form of (1.1) to show that for some constant C and a.e. $t \in [0,T]$

$$\|v(t)\|^2_{L^2(\mathbb{R}^n)} \leq C \int_0^t \|v(s)\|^2_{L^2(\mathbb{R}^n)}\, ds. \tag{5.2}$$

Once (5.2) is established, an application of Grönwall's inequality implies that $v=0$ and hence $u_1=u_2$. This approach is analogous to the uniqueness argument for solutions to the 2-D vorticity equation (1.4) with the same class of initial data. In the case of (1.4), an inequality analogous to (5.2) is established for $\|\nabla^{\perp}\phi\|_{L^2(\mathbb{R}^n)}$. See [19], chapter 8, for details when the initial data is compactly supported.

In order to obtain (5.2), we rely on the identity (2.3), using ϕ as our test function. For this reason, we need to check that $\phi\in L^1([0,T];V)$. Our first step is to check that ϕ is given by $\phi=-N_n*(u_1-u_2)$, where N_n is the Newtonian potential. Clearly this is true in three or more space dimensions, due to the decay of N_n for large $|x|$ when $n\geq 3$. In order to show this for $n=2$, it suffices to show that $N_2*(u_1-u_2)$ is bounded. This follows from Lemmas 2.1 and 5.1.

LEMMA 5.1. *Let $f:\mathbb{R}^2\to\mathbb{R}$ satisfy $|f(x)|\leq C(1+|x|)^{-2-\alpha}$ for some $\alpha>0$ and assume $\int_{\mathbb{R}^n} f(y)\,dy\,=0$. Then $g=N_2*f\in L^{\infty}(\mathbb{R}^2)$.*

Proof. We have

$$2\pi g(x)=\int_{\mathbb{R}^n}\log|x-y|f(y)\,dy=\int_{|y|<\frac{1}{2}|x|}\log|x-y|f(y)\,dy+\int_{|y|>\frac{1}{2}|x|}\log|x-y|f(y)\,dy=I+II.$$

Since g is differentiable, it suffices to examine its behavior for large $|x|$. First note that $II\to 0$ as $|x|\to\infty$ because of (5.1). To bound I, we expand $\log(|x-y|)$ around x and find that

$$I=\log|x|\int_{|y|<\frac{1}{2}|x|}f(y)\,dy+\int_{|y|<\frac{1}{2}|x|}y\cdot\frac{1}{|x-\sigma y|}\frac{x-\sigma y}{|x-\sigma y|}f(y)\,dy=A+B\quad\text{for some } 0\leq\sigma\leq 1.$$

Now, note that $A\to 0$ as $|x|\to\infty$ by (5.1) and the zero mean condition, while $B\leq\|f(y)\|_{L^1(\mathbb{R}^n)}$. □

From Lemma 5.1, we know that $\phi=-\Delta^{-1}(u_1-u_2)=-N_n*(u_1-u_2)$. We are now ready to show that $\phi\in L^2([0,T];V)$.

LEMMA 5.2. *Let $\phi=-N_n*(u_1-u_2)$; then $\phi\in L^{\infty}([0,T];V)$.*

Proof. In order to establish the result, it suffices by Bochner's theorem (see page 133 of [24]) to show that ϕ is strongly $V-$measurable and $\operatorname{ess\,sup}_{0\leq s\leq T}\|\phi(\cdot,s)\|_V<\infty$. We first show that $\operatorname{ess\,sup}_{0\leq s\leq T}\|\phi(\cdot,s)\|_{L^{\infty}(\mathbb{R}^n)}<\infty$. Following the reasoning in Lemma 5.1, we can find R and L so that for $s\in[0,T]$, $|x|>R\Rightarrow\|\phi(x,s)\|_{L^{\infty}(\mathbb{R}^n)}<L$. We establish a bound on $|\phi(x,s)|$ for $|x|<R$ by breaking up the integral $N_2*(u_1-u_2)$ into two pieces. Because of (5.1), we can find C, L' and ϵ such that for a.e. $s\in[0,T]$

$$|x|<R\Rightarrow\left|\int_{|y|>C}\log|x-y|[u_1-u_2](y,s)\,dy\right|<\epsilon\text{ and }\left|\int_{|y|<C}\log|x-y|[u_1-u_2](y,s)\,dy\right|<L'.$$

Next, we show that $\sup_{0\leq s\leq T}\|\nabla\phi(\cdot,s)\|_{L^2(\mathbb{R}^n)}<\infty$ by establishing $\sup_{0\leq s\leq T}\|\widehat{\nabla\phi}(\cdot,s)\|_{L^2(\mathbb{R}^n)}<\infty$ and appealing to Parseval's theorem. Here, $\widehat{\ }$ denotes the Fourier transform in the space variables only. We begin by noting that $\hat{v}(\xi,s)_i=c\frac{\xi_i}{|\xi|^2}(\widehat{u_1-u_2})(\xi,s)$; this follows from $h(x)=x^{\beta}|x|^{-n}\Rightarrow\hat{h}(\xi)=c_n(i\partial_{\xi})^{\beta}\log(|\xi|)$ for any multi-index β [15]. Since we have uniform bounds on $\|(\widehat{u_1-u_2})(\cdot,s)\|_{L^2(\mathbb{R}^2)}$, it suffices to show that the $\hat{v}(\cdot,s)$ are uniformly locally L^2 integrable integrable at the origin. This follows from the uniform Hölder continuity of the $(\widehat{u_i})(\cdot,s)$ and $(\widehat{u_1-u_2})(0,s)=0$ for a.e. $s\in[0,T]$. Uniform Hölder continuity of the $\widehat{(u_i)(\cdot,s)}$ of exponent β, for all $\beta<\min(\alpha,1)$, follows from

$$|\hat{u}_i(\xi_1)-\hat{u}_i(\xi_2)|\leq\int_{\mathbb{R}^n}|u(x)(1-e^{ix\cdot(\xi_1-\xi_2)})|\;dx\leq 2^{1-\beta}|\xi_1-\xi_2|^\beta\int_{\mathbb{R}^n}|u(x)||x|^\beta\;dx. \tag{5.3}$$

while $\int_{\mathbb{R}^n}u_1(\cdot,s)-u_2(\cdot,s)\;dx=0\Rightarrow\widehat{(u_1-u_2)}(0,s)=0$ for a.e. $s\in[0,T]$.

We now establish that the map $t\to\phi(\cdot,t)$ is strongly $V-$measurable. By using ϕ as a test function in (2.2), we use this result in the proof of Theorem 5.1 to establish (5.4). It suffices to consider the case $n=2$ due to the decay of N_n for large $|x|$ when $n\geq 3$. From part 1) of Definition 2.1, there exists a sequence of finitely valued functions in $L^1(\mathbb{R}^n)$, $z_n:[0,T]\to L^1(\mathbb{R}^n)$, such that $z_n(t)\to u_1(\cdot,t)-u_2(\cdot,t)$ in $L^1(\mathbb{R}^n)$ for a.e. $t\in[0,T]$. The V-measurability of ϕ will follow from $\Delta^{-1}(z_n(\cdot,t))\to\phi(\cdot,t)=\Delta^{-1}(u_1(\cdot,t)-u_2(\cdot,t))$ in V for a.e. $t\in[0,T]$. We may assume that the z_n have zero mean and obey the same L^∞ and decay bounds as the u_i.

We first show that $\Delta^{-1}(z_n(\cdot,t))\to\Delta^{-1}(u_1(\cdot,t)-u_2(\cdot,t))$ in $L^\infty(\mathbb{R}^n)$ for a.e. $t\in[0,T]$. Given $\epsilon>0$ and $t\in[0,T]$ such that $z_n(\cdot,t)\to u_1(\cdot,t)-u_2(\cdot,t)$ in $L^1(\mathbb{R}^n)$, we use the reasoning from Lemma 5.1 to find R such that $|x|>R\Rightarrow|\Delta^{-1}[z_n(x,t)-(u_1(x,t)-u_2(x,t))]|<\epsilon$ for all n. We are left with finding n_0 so that $|x|\leq R\Rightarrow|\Delta^{-1}[z_n(t)-(u_1(x,t)-u_2(x,t))]|<\epsilon$ for $n\geq n_0$. Let $d_n(x,t)=z_n(x,t)-(u_1(x,t)-u_2(x,t))$; by (5.1), we can find C such that

$$\Big|\int_{|y|>C}\log|x-y|d_n(y,t)\;dy\Big|<\frac{\epsilon}{2}\quad\text{for}\quad|x|\leq R.$$

Now, since $d_n(\cdot,t)\to 0$ in L^1 and the d_n are uniformly bounded in L^∞, we can choose n_0 so that $n\geq n_0$ implies

$$\Big|\int_{|y|<C}\log|x-y|d_n(y,t)\;dy\Big|<\frac{\epsilon}{2}\quad\text{for}\quad|x|\leq R.$$

Next we show that $\nabla\Delta^{-1}(z_n(\cdot,t))\to\nabla\Delta^{-1}(u_1(\cdot,t)-u_2(\cdot,t))$ in L^2 as $n\to\infty$, which we write as $\nabla\Delta-1(d_n(\cdot,t))\to 0$. By Parseval's theorem, this is equivalent to showing $\nabla\widehat{\Delta^{-1}(d_n}(\cdot,t))\to 0$ in L^2. Since esssup ${}_{0\leq s\leq T}\|z_n(s)\|_{L^\infty(\mathbb{R}^n}<C$ independent of n, we have $z_n(t)\to 0$ in $L^2(\mathbb{R}^n)$. Recall from above that $\widehat{\nabla\Delta^{-1}d_n}(\xi,t)_i=c\frac{\xi_i}{|\xi|^2}\hat{d}_n(\xi,t)$, $\hat{d}_n(0,t)=0$. Since it follows from (5.3) that the $\widehat{d_n}(\cdot,t)$ are uniformly Hölder continuous for any exponent $0<\beta<\alpha$, the result follows. □

We now prove Theorem 5.1.

Proof. Clearly, ϕ satisfies $\int\nabla\psi\cdot\nabla\phi=\int\psi(u_1-u_2)$ for all $\psi\in H^1(\mathbb{R}^n)$. This identity extends to $\psi\in V$ by inserting as a test function $\chi_R\phi$ and letting $R\to\infty$. Here, χ_R is a smooth positive cutoff function identically equal to 1 on $|x|<R-1$ and supported in $|x|<R$. We then use the strong $V-$measurability of ϕ and follow the steps from pages 328-329 of [1] to find that

$$(\hat{u}(t),\hat{u}(t))^2_{H^{-1}}:=\int_{\mathbb{R}^n}\langle v,v\rangle\;dx=\frac{1}{2}\int_0^t\langle(u_1-u_2)_t,\phi\rangle\;ds. \tag{5.4}$$

for a.e. $t\in[0,T]$. Since u_1 and u_2 are weak solutions, the above can be re-written, by using ϕ as a test function in (2.3), as

$$(u_1(t)-u_2(t),u_1(t)-u_2(t))^2_{H^{-1}}=\int_0^t\int_{\mathbb{R}^n}[(\nabla K*u_2)u_2-(\nabla K*u_1)u_1\Big]v\;dx\;ds$$

$$=\int_0^t\int_{\mathbb{R}^n}\Big[(\nabla K*u_2)(u_2-u_1)+(\nabla K*(u_2-u_1))u_1\Big]v\;dx\;ds=\int_0^t I(s)\;ds+\int_0^t II(s)\;ds$$

for a.e. $t\in[0,T]$. In order to establish (5.2), we now estimate I and II. We first estimate II; integrating by parts, applying Young's inequality for convolutions and the Cauchy-Schwartz inequality, we arrive at

$$|II|\le\|v\|^2_{L^2(\mathbb{R}^n)}\|u_1\|_{L^\infty(\mathbb{R}^n}\|D^2K\|_{L^1(\mathbb{R}^n)}.$$

We may also integrate I by parts to arrive at

$$|I|\le C_n\|D_{ij}q\|_{L^\infty(\mathbb{R}^n)}\|\nabla\phi\|^2_{L^2(\mathbb{R}^2)}=\le C_n\|D_{ij}K*u\|_{L^\infty(\mathbb{R}^n)}\|\nabla\phi\|^2_{L^2(\mathbb{R}^n)}. \tag{5.5}$$

We first establish (5.5) in the two dimensional case and then outline the procedure in n dimensions. In two dimensions, we can re-write I as

$$I=\int_{\mathbb{R}^n}(q_x\phi_x+q_y\phi_y)(\phi_{xx}+\phi_{yy})\;dx$$

where $q=K*u_1$. Integrating by parts, we have

$$I=\int_{\mathbb{R}^n}-\frac{1}{2}q_{xx}(\phi_x)^2-\frac{1}{2}q_{yy}(\phi_y)^2-2q_{xy}\phi_x\phi_y-q_y\phi_{xy}\phi_x-q_x\phi_{xy}\phi_y\;dx$$

$$=\int_{\mathbb{R}^n}-\frac{1}{2}q_{xx}(\phi_x)^2-\frac{1}{2}q_{yy}(\phi_y)^2-2q_{xy}\phi_x\phi_y+\frac{1}{2}q_{yy}(\phi_x)^2+\frac{1}{2}q_{xx}(\phi_y)^2\;dx.$$

This implies (5.5) in the two dimensional case. This generalizes to n dimensions, in which case we have

$$I=\int_{\mathbb{R}^n}\langle\nabla q,\nabla\phi\rangle\Delta\phi\;dx=\int_{\mathbb{R}^n}-\frac{1}{2}\sum_{i=1}^n q_{ii}(\phi_i)^2-\sum_{i,j=1}^n q_{ij}\phi_i\phi_j-\sum_{i,j=1}^n q_i\phi_i\phi_{ij}\;dx$$

$$=\int_{\mathbb{R}^n}-\frac{1}{2}\sum_{i=1}^n q_{ii}(\phi_i)^2-\sum_{i,j=1}^n q_{ij}\phi_i\phi_j+\frac{1}{2}\sum_{i,j=1}^n q_{ij}(\phi_i)^2\;dx$$

which establishes (5.5) in the general case. ▯

6. Continuation of solutions and finite-time blow-up

In this section, we address the questions of continuation in time of the unique solution from the previous section and the occurrence of finite-time blow-up of solutions. We first show that weak solutions of (1.1) with initial data (2.1) either exist for all time or there exists a time T^*, beyond which the solution does not exist, such that $\lim_{t\to T^*}\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}=\infty$ for $p\in[2,\infty]$ if $n>2$ and $p\in(2,\infty]$ if $n=2$. This is the content of Theorem 6.1.

Theorem 6.1. *(Maximal time interval of existence) Let u be a weak solution of (1.1) with initial data (2.1) which conserves mass. Then either u exists for all time or there exists a time T^*, after which the solution no longer exists, such that*

$$\lim_{t\to T^*}\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}=\infty\quad \textit{for } p\in[2,\infty] \textit{ if } n>2 \textit{ and } p\in(2,\infty] \textit{ if } n=2.$$

Proof. We first establish the result for $p=\infty$. If $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ remains bounded on an interval $[0,T)$, it follows from Theorem 3.1 that we can continue our solution for time $\triangle t$ which depends only on $\sup_{0\le s<T}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}$. Hence, if a solution cannot be continued past a time T^* we must have $\sup_{0\le s<T^*}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}=\infty$. It then follows from Theorem 3.1 and Corollary 4.2 that $\liminf_{s\to T^*}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}=\infty$.

We now consider the case $p\ge 2$ if $n>2$, $p>2$ if $n=2$. The result follows from the following bound on $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$:

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\le\|u_0\|_{L^\infty(\mathbb{R}^n)}e^{t\|\Delta K\|_{L^{q'}(\mathbb{R}^n)}\sup_{0\le s\le t}\|u(\cdot,s)\|_{L^q(\mathbb{R}^n)}} \tag{6.1}$$

where q' is conjugate to q. In order to establish (6.1), note that if u solves (1.1) on $[0,T]$, then by uniqueness of solutions and Proposition 3.1, there exists a finite sequence of times $0=T_1<T_2<\ldots<T_n=T$ such that for $t\in[T_i,T_{i+1}]$, $u(t)=\lim_{k\to\infty}u^{T_i}_{\epsilon_k}(t-T_i)$ in $L^1(\mathbb{R}^n)$ for some sequence $\{\epsilon_k\}$ converging to zero. Here, $u^{T_i}_\epsilon$ is the solution to (1.1) with initial data $u(\cdot,T_i)*J_\epsilon$. By (3.1) and Proposition 3.1 we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\le\|u(\cdot,T_i)\|_{L^\infty(\mathbb{R}^n)}e^{(t-T_i)\|\Delta K\|_{L^{q'}(\mathbb{R}^n)}\sup_{T_i\le s\le T_{i+1}}\|u(\cdot,s)\|_{L^q(\mathbb{R}^n)}}\ \text{for } T_i\le t\le T_{i+1}$$

where q' is conjugate to q. Combining these estimates results in (6.1). ∎

For the remainder of this section, we study finite time blow-up for kernels of the form $K(x)=e^{-|x|}+g(|x|^2)$, where $g\in C^2(\mathbb{R}^n)$. Our discussion uses results from [3]. We prove the following theorem.

THEOREM 6.2. *(Blow-up) Let K be a kernel with the property that $\nabla K(x)=\frac{x}{|x|}\Big(-1+g(|x|)\Big)$, where $g\in L^\infty(\mathbb{R}^n)$ is continuous at 0 and $g(0)=0$. Suppose that u is a weak solution of (1.1) with initial data (2.1) which is positive, radially symmetric, and compactly supported inside a small enough ball. Then the solution u ceases to exist after some time T^*.*

Define the energy (see [3] for more details)

$$E(u)=\int_{\mathbb{R}^n}u(x)K*u(x)\,dx. \tag{6.2}$$

Theorem 6.2 follows directly from Proposition 6.1, which gives an a priori bound upper bound on $E(u)$, and Proposition 6.3, which gives a lower bound on the rate of growth of $E(u)$.

PROPOSITION 6.1. *For all $u\in L^1(\mathbb{R}^n)$, we have*

$$E(u)\le\|K\|_{L^\infty(\mathbb{R}^n)}\|u\|^2_{L^1(\mathbb{R}^n)}.$$

Proof. We follow the calculation in [3].

$$E(u)=\int_{\mathbb{R}^n}u(x)K*u(x)\,dx\le\|K*u\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^1(\mathbb{R}^n)}\le\|K\|_{L^\infty(\mathbb{R}^n)}\|u\|^2_{L^1(\mathbb{R}^n)}.$$

∎

We now set up the machinery needed to prove Proposition 6.3, which provides a lower bound on the rate of growth of $E(u)$. Roughly speaking, we will prove Proposition 6.3 by approximating $E(u)$ by $E(u_\epsilon)$, where u_ϵ is a solution of (1.1) with smoothed out initial data, and obtain uniform estimates on the rate of change of the $E(u_\epsilon)$. In order to do this, we first note that if u is a classical solution of (1.1), then we have from [3] that

$$\frac{dE}{dt}(u) = \int_{\mathbb{R}^n} u|\nabla K * u|^2 \, dx. \tag{6.3}$$

We use the following proposition, from [3], to construct a lower bound for $\frac{dE}{dt}$.

Proposition 6.2. *[3] There exists a constant $C>0$ such that for all δ sufficiently small, we have, for any radially symmetric nonnegative $L^1(\mathbb{R}^n)$ function u of unit mass supported inside a ball of radius δ,*

$$\int_{\mathbb{R}^n} u|\nabla K * u|^2 \, dx \geq C.$$

We are now ready to prove a lower bound for the growth rate of $E(u)$ if u_0 is radially symmetric and supported inside a sufficiently small ball.

Proposition 6.3. *Let u be a weak solution of (1.1), defined on $\mathbb{R}^n \times [0,T]$ with initial data u_0 which is radially symmetric and supported inside a sufficiently small ball. Then there exists a constant $C>0$ such that*

$$E(u(\cdot,t)) - E(u_0) \geq Ct.$$

Proof. Since u solves (1.1) on the interval $[0,T]$, by Theorem 6.1 and Proposition 3.1 there exists a finite sequence of times $0 = T_0 < T_1 < ... < T_n = T$ such that for $t \in [T_i, T_{i+1}]$, $u(t) = \lim_{k\to\infty} u_{\epsilon_k}^{T_i}(t - T_i)$ in $L^1(\mathbb{R}^n)$ for some sequence $\{\epsilon_k\}$ converging to zero. Here, $u_\epsilon^{T_i}$ is the solution to (1.1) with initial data $u(\cdot,T_i) * J_\epsilon$. It follows from Proposition 3.1 and Young's inequality for convolutions that

$$E(u(\cdot,t)) = \lim_{k\to\infty} E(u_{\epsilon_k}^{T_i}(\cdot, t-T_i)) \quad \text{for } t \in [T_i, T_{i+1}]. \tag{6.4}$$

Next, note that u remains radially symmetric with support which does not grow in time. This follows from uniqueness of weak solutions and the fact that the $u_\epsilon^{T_i}$ remain radially symmetric with supports that do not grow, since characteristics of (1.1) point inward under our hypotheses on the kernel K. It then follows from Proposition 6.2 and (6.3) that

$$E(u_\epsilon^{T_i}(\cdot,h)) - E(u_\epsilon^{T_i}(\cdot,0)) \geq C(h) \quad \text{for} \quad \text{for } h \in [0, T_{i+1} - T_i]. \tag{6.5}$$

Combining (6.4) and (6.5), we have

$$E(u(\cdot,t)) - E(u_0) \geq Ct \quad \text{for } t \in [0,T].$$

□

7. Conclusion and discussion

We have proven existence and uniqueness of solutions of (1.1) with initial data belonging to (2.1) in space dimensions two and higher under a mild decay hypothesis. In the case of a potential with Lipschitz point at the origin (e.g. $K(x) = e^{-|x|}$), we have proven finite time blow-up of solutions which are initially radially symmetric and supported in a small ball around the origin.

Very little is understood about the structure of blow-up for solutions to (1.1) in more than one space dimension. Two important issues are the possibility of mass concentration and continuation of solutions beyond blow-up. Concentration of mass was shown to occur in one dimension in [4], using the asymptotic theory of shock waves. Although the mechanism for blow-up is not well understood in higher dimensions, the analysis in [3] does not rule out mass concentration. Variants of equation (1.1) with diffusion added have also been studied and in these cases blow-up is not observed. The model in [23] included degenerate density dependent diffusion and blow-up was not observed numerically, while [18] rigorously studied (1.1) with the addition of fractional linear diffusion. Another open area is blow-up for initial data which is not radially symmetric. Blow-up for a large class of initial data was proven in one spatial dimension by comparison with a Burgers-like dynamics in [4]; it was also studied, both analytically and computationally, in [17]. However, nothing is known for higher dimensions. Finally, we would like to have a better understanding of how blow-up is related to the smoothness and decay of the kernel $K(x)$. Blow-up of a modified version of (1.1), with diffusion added, was studied in the case of the highly singular kernel $K(x)=\frac{1}{|x|^{n-2}}$, the Newtonian potential, in [5].

Numerical simulation may be an effective method for further study of blow-up. Computations performed in the one dimensional case [17] exhibit intersection of characteristics and concentration of mass at blow-up, both of which are consistent with the analysis in [4]. However, we do not know of any detailed computational results in space dimensions two or higher. Tools from optimal transport are also relevant to this problem. One can show that (1.1)is a gradient flow in a Wasserstein metric. This fact is used in [7] for example in one space dimension. It would be interesting to see if related approaches such as the geometric approach to gradient flows developed in [21] are useful for studying this problem.

8. Appendix

We now prove Lemma 4.5.

Proof. Since $\Delta K=\frac{1}{|x|}e^{-|x|}+e^{-|x|}+h(x)$, it suffices to prove that $m(x)=\frac{1}{|x|}e^{-|x|}*u$ is quasi-Lipschitz. Set $l(x)=\frac{1}{|x|}e^{-|x|}$ so that $m(x)=l*u$. Let $d\equiv|x_1-x_2|<1$ and let $B(x,r)$ be a ball of radius r centered at x. We have

$$|m(x_1)-m(x_2)|\leq\left[\int_{\mathbb{R}^2-B(x_1,2)}+\int_{B(x_1,2)-B(x_1,2d)}+\int_{B(x_1,2d)}\right]\times|l(x_1-y)-l(x_2-y)||u(y,t)|\;dy=$$

$$I_1+I_2+I_3$$

Since

$$\Big(\frac{1}{|x|}-\frac{1}{|y|}\Big)^2=\Big(\frac{|y|-|x|}{|x||y|}\Big)^2\leq\frac{|y-x|^2}{|x|^2|y|^2}$$

we have

$$|(l(x)-l(y))|\leq e^{-|x|}\left|\frac{1}{|x|}-\frac{1}{|y|}\right|+\left|\frac{1}{|y|}(e^{-|x|}-e^{-|y|})\right|$$

$$\leq e^{-|x|}\frac{|x-y|}{|x||y|}+\frac{1}{|y|}|-|x|+|y||\leq e^{-|x|}\frac{|x-y|}{|x||y|}+\frac{1}{|y|}|y-x|.$$

This estimate implies

$$I_1 \le c|x_1-x_2|\|u\|_{L^1(\mathbb{R}^2)}.$$

To bound I_2, apply the mean value theorem. Since $l(x)=\frac{1}{|x|}e^{-|x|}$, we find that

$$|l(x_1-y)-l(x_2-y)| \le |x_1-x_2|\sup{}_{0\le\theta\le1}|\nabla l(x_1-y+\theta(x_1-x_2)|$$

$$\le 2\frac{|x_1-x_2|}{|x_1-y|}+4\frac{|x_1-x_2|}{|x_1-y|^2} \qquad \text{for } y\in B(x_1,2)-B(x_1,2d)$$

¿From this, it follows that

$$I_2 \le c|x_1-x_2|\|u\|_{L^\infty(\mathbb{R}^2)}\int_{B(x_1,2)-B(x_1,2d)}\frac{1}{|x_1-y|}+\frac{1}{|x_1-y|^2}\ dy \le$$

$$\le c|x_1-x_2|\|u\|_{L^\infty(\mathbb{R}^2)}\Big(2+\int_{2d}^{2}\frac{dr}{r}\Big) \le c|x_1-x_2|\|u\|_{L^\infty(\mathbb{R}^2)}\Big(2-\ln^-|x_1-x_2|\Big).$$

We can bound I_3 just as in [19], page 317. Since $d=|x_1-x_2|$, we have

$$I_3 \le c\|u\|_{L^\infty(\mathbb{R}^2)}\Big(\int_{B(x_1,2d)}\frac{dy}{|x_1-y|}+\int_{B(x_1,2d)}\frac{dy}{|x_2-y|}\Big)$$

$$\le c\|u\|_{L^\infty(\mathbb{R}^2)}\Big(\int_0^{2d}dr+\int_0^{3d}dr\Big) \le c\|u\|_{L^\infty(\mathbb{R}^2)}|x_1-x_2|. \ \square$$

$\square$

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