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Γ-CONVERGENCE OF GRAPH GINZBURG–LANDAU FUNCTIONALS

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Abstract. We study Γ-convergence of graph-based Ginzburg–Landau functionals, both the limit for zero diffusive interface parameter $\varepsilon \to 0$ and the limit for infinite nodes in the graph $m \to \infty$. For general graphs we prove that in the limit $\varepsilon \to 0$ the graph cut objective function is recovered. We show that the continuum limit of this objective function on 4-regular graphs is related to the total variation seminorm and compare it with the limit of the discretized Ginzburg–Landau functional. For both functionals we also study the simultaneous limit $\varepsilon \to 0$ and $m \to \infty$, by expressing ε as a power of m and taking $m \to \infty$. Finally we investigate the continuum limit for a nonlocal means-type functional on a completely connected graph.

1. Introduction

1.1. **The continuum Ginzburg–Landau functional.** In this paper we study an adaptation of the classical real Ginzburg–Landau (also called Allen–Cahn) functional to graphs. The Ginzburg–Landau functional is the object to be minimized[1] in a well-known phase-field model for phase separation in materials science, *e.g.* [47, 48], and is given by

$$F_\varepsilon^{GL}(u) := \varepsilon \int_\Omega |\nabla u(x)|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u(x)) \, dx, \qquad \varepsilon > 0, \tag{1.1}$$

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[1]Note that to avoid trivial minimizers an additional constraint needs to be added. In materials science it is common to add a mass constraint of the form $\int_\Omega u = M$ for a fixed $M > 0$. In image-analysis applications one often adds a fidelity term of the form $\lambda \|u - f\|^2_{L^2(\Omega)}$ to the functional F_ε^{GL}, where $\lambda > 0$ is a parameter and $f \in L^2(\Omega)$ is given data, often a noisy image which needs to be cleaned up; see [54].

where $u \in W^{1,2}(\Omega)$ is the phase field describing the different phases the material can be in and W is a double-well potential with two minima, *e.g.* $W(s) = s^2(s-1)^2$. Ω is a bounded domain in $\mathbb{R}^N$.

Recently [14] this functional has been adapted to weighted graphs in an application to machine learning and data clustering: An image is interpreted as a weighted graph, with the vertices corresponding to the pixels and the weights based on the similarities between the pixels' neighborhoods. The phase-separating nature of the Ginzburg–Landau functional then drives separation of the different features in the image.

The continuum functional F_ε^{GL} has been extensively used and studied, but a theoretical understanding of its equivalent on graphs is lacking. In this paper we use Γ-convergence [28, 15] to study the asymptotic behavior of minimizers of the graph-based Ginzburg–Landau functional when either $\varepsilon \to 0$ or the number of nodes in the graph $m \to \infty$. In Section 2.4 we discuss Γ-convergence in more detail. Its most important feature is that if a sequence of functions $\{f_n\}_{n=1}^\infty$ Γ-converges to a limit function f_∞ and in addition satisfies a specific compactness condition, then minimizers of f_n converge to minimizers of f_∞.

It has been proven [49, 47, 48][2] that F_ε^{GL} Γ-converges as $\varepsilon \to 0$ to the total variation functional

$$F_0^{GL}(u) := \sigma(W) \int_\Omega |\nabla u|, \tag{1.2}$$

where now u is restricted to functions of bounded variation taking on two values (corresponding to the minima of the potential W) almost everywhere and the surface tension coefficient $\sigma(W)$ is determined by the potential W (see Section 5.1 for more details). Because the total variation of a binary function is proportional to the length of the boundary between the regions where the function takes on different values, from this limit functional the phase-separating behavior can be seen clearly: u takes on one of two values, corresponding to the two different phases of the material, and by minimizing the BV seminorm of u the interface between the two phases gets minimized.

One of the results in this paper is a similar Γ-convergence statement for the graph Ginzburg–Landau functional:

$$f_\varepsilon(u) := \chi \sum_{i,j=1}^m \omega_{ij}(u_i - u_j)^2 + \frac{1}{\varepsilon} \sum_{i=1}^m W(u_i), \tag{1.3}$$

[2]As poster child for Γ-convergence the proof has been reproduced, clarified, and extended upon in various ways; see *e.g.* [10, 56, 33, 42, 11, 2, 3, 27, 4]

where u_i is the value of u on node i, ω_{ij} the weight of the edge connecting nodes i and j, m is the number of nodes in the graph, $\varepsilon > 0$, and $\chi \in (0, \infty)$ is a constant independent of ε and m, usually chosen to be $\chi = \frac{1}{2}$ so the first summation is the analogue of $\int |\nabla u|^2$ (see Section 2.2). The different terms in this functional and its scaling will be explained below.

The Euler–Lagrange equations for this functional are a nonlinear extension of the graph heat equation using the graph Laplacian [26]. Nonlinear elliptic equations on graphs were investigated in [50], and recently in [46] their well-posedness was studied.

We study not only the limit $\varepsilon \to 0$ in analogy with the classical continuum result, but also investigate the limit $m \to \infty$. For a graph embedded in $\mathbb{R}^n$ this can be interpreted as the limit for finer discretization or sampling scale. In order to make sense of this limiting process we need to assume some additional structure on the graph that tells us how nodes are added along a sequence of increasing m. In this paper we consider 4-regular graphs (*i.e.,* each node is connected to exactly 4 edges) with uniformly weighted edges in Sections 4 and 5, and a completely connected graph for the nonlocal means functional is studied in Section 6, but it is an interesting question if and how this can be extended to different types of graphs. Adaptation of our results to a 2-regular graph is fairly direct, but it is not clear at this moment how to extend our method to other graphs, even regular ones.

1.2. **Different scalings on a 4-regular graph.** The formulation of f_ε in (1.3) does not require the graph to be embedded in a surrounding space, although an embedding may exist as in the case of the 4-regular graph considered as an $N \times N$ square grid on the flat torus $\mathbb{T}^2$.

We study two natural scalings for the functional on this 4-regular graph. The first is a direct reformulation of the graph functional f_ε from (1.3) with $\chi = \frac{1}{2}$ and weights equal to N^{-1} on all existing edges and zero between two vertices that are not connected by an edge:

$$h_{N,\varepsilon}(u) := N^{-1} \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \varepsilon^{-1} \sum_{i,j=1}^{N} W(u_{i,j}). \quad (1.4)$$

The second we get from discretizing the functional F_ε^{GL} on the square grid using a forward finite difference scheme for the gradient and the trapezoidal rule for the integrals:

$$k_{N,\varepsilon}(u) := \varepsilon \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \varepsilon^{-1} N^{-2} \sum_{i,j=1}^{N} W(u_{i,j}). \quad (1.5)$$

The subscripts in $u_{i,j}$ denote the horizontal and vertical coordinates along the square grid.

We will consider Γ-limits of these functionals when $\varepsilon \to 0$ and $N \to \infty$ sequentially. We also prove results in the case where we set $\varepsilon = N^{-\alpha}$ for $\alpha > 0$ in a specified range and take $N \to \infty$. Based on the Γ-convergence result in the continuum case we expect $h_{N,\varepsilon}$ and $k_{N,\varepsilon}$ to converge to total variation functionals. This intuition turns out to be correct, but with a twist: $k_{N,\varepsilon}$ converges to the total variation functional $\int_{\mathbb{T}^2} |\nabla u|$, but $h_{N,\varepsilon}$ converges to the *anisotropic* total variation $\int_{\mathbb{T}^2} |u_x|+|u_y|$. It picks up the directionality of the grid. Precise results are stated and proved in Sections 4 and 5. These results fit in very well with the research on Γ-convergence of discrete functionals to continuum functionals, as in *e.g.* [15, 17, 5, 18, 6, 7, 8, 23]. In fact, many of the techniques used in Section 5 are inspired by [5] specifically.

We would like to point out that there is also a substantial literature on the convergence of graph Laplacians and their eigenvalues and eigenvectors to continuum limits. See *e.g.* [40, 36, 12, 41, 13, 44, 45] and references therein. The techniques used and the kind of results obtained in those papers are quite different from ours, but in a certain sense our results can be seen as nonlinear extensions of the graph Laplacian case.

In all cases we have to impose extra constraints on minimizers of the Ginzburg–Landau functional to avoid trivial minimizers. We prove results showing that in most cases the addition of a mass constraint or the addition of an L^p fidelity term to the functional is compatible with the Γ-convergence results.

1.3. **Asymptotic behavior of nonlocal means.** The functionals of nonlocal means type—or (anisotropic) nonlocal total variation type—we consider are built on the square grid in which the graphs are fully connected; see [21, 34, 35, 20]. Fix $\Phi \in C^\infty(\mathbb{T}^2)$. We study

$$g_N(u) := N^{-4} \sum_{i,j,k,l=1}^{N} (\omega_{L,N})_{i,j,k,l} |u_{i,j} - u_{k,l}|, \tag{1.6}$$

where $\omega_{L,N} := e^{-d^2_{L,N}/\sigma^2}$ with $\sigma, L > 0$ constants (possibly depending on N) and

$$(d^2_{L,N})_{i,j,k,l} := \sum_{r,s=-L}^{L} \left(\Phi\Big(\frac{i-r}{N}, \frac{j-s}{N}\Big) - \Phi\Big(\frac{k-r}{N}, \frac{l-s}{N}\Big)\right)^2. \tag{1.7}$$

If Φ is thought of as an image on $\mathbb{T}^2$, as in *e.g.* [14], then L gives the size of the pixel neighborhoods whose pairwise comparisons form the graph weights. As we will see in Section 3, g_N arises as the Γ-limit of f_ε in (1.3) as $\varepsilon \to 0$ on this particular fully connected graph. This is a natural class of problems for which to study Γ-convergence as $N \to \infty$.

1.4. **Structure of the paper.** This paper is structured as follows. Section 2 sets up notation and gives more background information about how to set up a PDE-to-graph "dictionary" used to find the graph analogue of the Ginzburg–Landau functional. It also gives more details about Γ-convergence. In Section 3 the Γ-convergence result for f_ε is proved. This result holds for general finite undirected weighted graphs. Next we turn our attention to the square grid on the torus. In Sections 4 and 5 the Γ-convergence results for $h_{N,\varepsilon}$ and $k_{N,\varepsilon}$ respectively are stated and proved. Γ-convergence for the nonlocal means-type functional g_N is discussed in Section 6. We close with a discussion of our results and open questions for future research in Section 7.

2. Setup

We will start with introducing some general graph theoretical notation.

2.1. **Graph notation.** Let $G = (V, E)$ be an undirected graph with vertex (or node) set V, $|V| = m \in \mathbb{N}$, and edge set $E \subset V^2$. Consider the space $\mathcal{V}$ of all functions $V \to \mathbb{R}$. A function $u \in \mathcal{V}$ can be seen as a labeling of the vertices of G. It is useful to number the vertices in V from 1 to m (in arbitrary but fixed order). We will write I_m for the set of integers i satisfying $1 \leq i \leq m$. If $u \in \mathcal{V}$ and $n_i \in V$ is the i^{th} vertex we will use the shorthand notation $u_i := u(n_i)$. Let $\mathcal{E}$ be the space of all functions $E \to \mathbb{R}$, which are skew-symmetric with respect to edge direction; *i.e.,* if $\varphi \in \mathcal{E}$ and $e_{ij} := (n_i, n_j) \in E$ is the edge between the i^{th} and j^{th} vertex in V, we write $\varphi_{ij} := \varphi(e_{ij})$ and demand $\varphi_{ij} = -\varphi_{ji}$[3]. Since the graph is undirected we have $e_{ij} \in E \Leftrightarrow e_{ji} \in E$. When no confusion arises we will abuse notation slightly and consider $e_{ij} = e_{ji}$[4]. In this paper we consider weighted graphs, which means we assume there is given a function $\omega : E \to (0, \infty)$, called the weight function, which assigns a positive weight to each edge. Because the graph is undirected the weight function is symmetric: $\omega_{ij} := \omega(e_{ij}) = \omega_{ji}$. It is often

[3]We impose skew-symmetry so that $\mathcal{E}$ can be viewed as the space of *flows* as defined in *e.g.* [22, Section 2.2]. An interesting topological structure arises in this setting [22, Section 3], but for our current purposes the demand of skew-symmetry neither hinders nor helps us.

[4]See note 3.

useful to extend ω to a function on V^2 instead of on $E \subset V^2$ by identifying the edge e_{ij} with the pair $(n_i, n_j) \in V^2$ of its end vertices and setting $\omega_{ij} = 0$ if and only if $e_{ij} \notin E$. In particular, if the graph has no self-loops, $\omega_{ii} = 0$ for all $i \in I_m$. In the same way we can extend $\varphi \in \mathcal{E}$ to a function $\varphi : V^2 \to \mathbb{R}$ by setting it to zero on node pairs that are not connected by an edge. We can incorporate unweighted graphs in this framework by viewing them as weighted graphs with the range of ω restricted to be $\{0, 1\}$. We define the degree of vertex n_i as $d_i := \sum_{j \in I_m} \omega_{ij}$. If G has no isolated vertices, then for every $i \in I_m$, $d_i > 0$.

2.2. **Graph Laplacians, Dirichlet energy, and total variation.** Our first goal is to define operators that serve as the graph-gradient and graph-divergence operators. Using these operators we can then define a graph Laplacian, a Dirichlet energy, and isotropic and anisotropic total variations on the graph. There are many possible choices to do this. Ours follow [41, Section 2] and [35] and are presented here. In Appendix A we give details and background on the justification of these choices. $\mathcal{V} \cong \mathbb{R}^m$ and[5] $\mathcal{E} \cong \mathbb{R}^{m(m-1)/2}$ are Hilbert spaces defined via the following inner products:

$$\langle u, v \rangle_{\mathcal{V}} := \sum_{i \in I_m} u_i v_i d_i^r, \quad \langle \varphi, \phi \rangle_{\mathcal{E}} := \frac{1}{2} \sum_{i,j \in I_m} \varphi_{ij} \phi_{ij} \omega_{ij}^{2q-1},$$

for some $r \in [0, 1]$ and $q \in [1/2, 1]$. Different choices of r and q are useful in different contexts, as will become clear later in this section. We also define the dot product as the operator from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{V}$ for $\varphi, \phi \in \mathcal{E}$ as follows:

$$(\varphi \cdot \phi)_i := \frac{1}{2} \sum_{j \in I_m} \varphi_{ij} \phi_{ij} \omega_{ij}^{2q-1}.$$

With the Hilbert space structure in place, if we define a difference operator, then all the other operators and functionals will follow naturally. We define the difference operator or gradient $\nabla : \mathcal{V} \to \mathcal{E}$ as

$$(\nabla u)_{ij} := \omega_{ij}^{1-q} (u_j - u_i).$$

Notice that the choice $q = 1$ makes the gradient operator nonlocal on the graph, because its dependence on ω_{ij} disappears. The locality reappears in the $\mathcal{E}$-"inner product" (or strictly speaking sesquilinear form), which is thus turned semidefinite. The opposite is the case for $q = \frac{1}{2}$.

[5]The factor $\frac{1}{2}$ in $m(m-1)/2$ comes in because the graph is undirected. Strictly speaking $\mathcal{E}$ is not isomorphic to $\mathbb{R}^{m(m-1)/2}$ if we impose skew-symmetry, but this distinction is not relevant for our purposes. The Hilbert space structure can be defined in any case.

The other graph objects of interest for this paper now follow:

- Norms:
  - $\|u\|_{\mathcal{V}} := \sqrt{\langle u,u\rangle_{\mathcal{V}}} = \sqrt{\sum_{i\in I_m} u_i^2 d_i^r}$,
  - $\|\varphi\|_{\mathcal{E}} := \sqrt{\langle \varphi,\varphi\rangle_{\mathcal{E}}} = \sqrt{\frac{1}{2}\sum_{i,j\in I_m} \varphi_{ij}^2 \omega_{ij}^{2q-1}}$,
  - $\|\varphi\|_i := \sqrt{(\varphi\cdot\varphi)_i} = \sqrt{\frac{1}{2}\sum_{j\in I_m} \varphi_{ij}^2 \omega_{ij}^{2q-1}}$ (note that $\|\cdot\|_{\mathcal{E},\mathrm{dot}} \in \mathcal{V}$),
  - $\|u\|_{\mathcal{V},\infty} := \max\{|u_i| : i\in I_m\}$ and $\|\varphi\|_{\mathcal{E},\infty} := \max\{|\varphi_{ij}| : i,j\in I_m\}$.
- The Dirichlet energy does not depend on r or q:

$$\frac{1}{2}\|\nabla u\|_{\mathcal{E}}^2 = \frac{1}{4}\sum_{i,j\in I_m} \omega_{ij}(u_i-u_j)^2.$$

- The divergence $\operatorname{div} : \mathcal{E}\to\mathcal{V}$ defined as the adjoint of the gradient[6]:

$$(\operatorname{div}\varphi)_i := \frac{1}{2d_i^r}\sum_{j\in I_m} \omega_{ij}^q(\varphi_{ji}-\varphi_{ij}).$$

- A family of graph Laplacians $\Delta_r := \operatorname{div}\circ\nabla : \mathcal{V}\to\mathcal{V}$ (not to be confused with the p-Laplacians from the literature): Writing out this definition gives

$$(\Delta_r u)_i := d_i^{1-r}u_i - \sum_{j\in I_m}\frac{\omega_{ij}}{d_i^r}u_j = \sum_{j\in I_m}\frac{\omega_{ij}}{d_i^r}(u_i-u_j).$$

If we view u as a vector in $\mathbb{R}^m$ we can also write

$$\Delta_r u = (D^{1-r} - D^{-r}W)u,$$

where D^r is the diagonal matrix with diagonal elements $D_{ii} = d_i^r$ and W is the weight matrix with elements $W_{ij} = \omega_{ij}$. We can recover two of the most frequently used graph Laplacians from the literature (*cf.* [26, 43, 41]) by choosing either $r=0$ or $r=1$. For $r=0$ we get the unnormalized graph Laplacian, and for $r=1$ we have the random-walk Laplacian, which also goes by the name of (asymmetric) normalized Laplacian. For the latter case, the connection with random walks comes from the fact that $D^{-1}W$ is a stochastic matrix; *i.e.,* the sum of the elements in each of its rows equals 1. Note that Δ_r is only symmetric if $r=0$. A third graph Laplacian which is often encountered in the literature is the symmetric normalized Laplacian $\mathbb{I} - D^{-\frac{1}{2}}WD^{-\frac{1}{2}}$, where $\mathbb{I}$ is the m-by-m identity matrix. However, this one does not fit well into the current framework and we will not consider it here.

[6]If the graph has an isolated node i for which $\omega_{ij} =$ for all j and $d_i = 0$, we interpret this definition as $(\operatorname{div}\varphi)_i = 0$.

- Total variations[7]:
  - The isotropic total variation $TV : \mathcal{V} \to \mathbb{R}$ defined by

$$\begin{aligned}\mathrm{TV}(u) &:= \max\{\langle \operatorname{div}\varphi, u\rangle_{\mathcal{V}} : \varphi \in \mathcal{E},\ \max_{i\in I_m} \|\varphi\|_i \leq 1\}\\ &= \frac{\sqrt{2}}{2}\sum_{i\in I_m}\sqrt{\sum_{j\in I_m}\omega_{ij}(u_i-u_j)^2}.\end{aligned}$$

  - A family of anisotropic total variations $\mathrm{TV}_{\mathrm{a}q} : \mathcal{V} \to \mathbb{R}$ defined by

$$\mathrm{TV}_{\mathrm{a}q}(u) := \max\{\langle \operatorname{div}\varphi, u\rangle_{\mathcal{V}} : \varphi \in \mathcal{E},\ \|\varphi\|_{\mathcal{E},\infty} \leq 1\} = \frac{1}{2}\sum_{i,j\in I_m}\omega_{ij}^q |u_i - u_j|.$$

  TV and $\mathrm{TV}_{\mathrm{a}\frac{1}{2}}$ appear in [35] as isotropic and anisotropic total variation respectively. In this paper we show that $\mathrm{TV}_{\mathrm{a}1}$ is the Γ-limit of a sequence of Ginzburg–Landau-type functionals (Theorem 3.1).

Because we can identify $u \in \mathcal{V}$ with a vector $\hat{u} \in \mathbb{R}^m$ and all norms on $\mathbb{R}^m$ are equivalent, we can express convergence in any of these norms. For definiteness we choose a simple norm, not dependent on the degree function d: For a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{V}$ and $u_\infty \in \mathcal{V}$ and corresponding vectors $\hat{u}_n, \hat{u}_\infty \in \mathbb{R}^m$ we define

$$u_n \to u_\infty \text{ as } n \to \infty \quad \text{if and only if} \quad |\hat{u}_n - \hat{u}_\infty|_2 \to 0 \text{ as } n \to \infty,$$

where $|\cdot|_p$, with $p \in \mathbb{N}$, is defined for $\hat{u} \in \mathbb{R}^m$ as

$$|\hat{u}|_p := \Big(\sum_{i\in I_m}\hat{u}_i^p\Big)^{\frac{1}{p}},$$

with the subscript i labeling the elements of the vector.

Where this does not lead to confusion, we will use the same notation u for both the function $u \in \mathcal{V}$ and the corresponding vector $\hat{u} \in \mathbb{R}^m$.

2.3. **The functionals.** A standard choice of double-well potential is $W(s) = s^2(s-1)^2$. This is a representative example in the class of potentials for which our results hold. We always assume that $W \in C^2(\mathbb{R})$, $W \geq 0$, and $W(s) = 0$ if and only if $s \in \{0, 1\}$. Different lemmas and theorems in this paper require different additional assumptions:

[7] An interesting question which falls outside the scope of this paper is in which respects, if any, the curvatures derived as "derivatives" from these total variations resemble the continuum case curvature.

(W_1) There exists two disjoint open intervals $\hat{I}_0$ and $\hat{I}_1$ containing 0 and 1 respectively, constants c_0, $c_1 > 0$, and a $\beta > 0$, such that

$$0 \leq \max\{W(s) : s \in \hat{I}\} < \min\{W(s) : s \in \hat{I}^c\}, \text{ where } \hat{I} := \hat{I}_0 \cup \hat{I}_1, \text{ and}$$

$$\forall s \in \hat{I}_0,\, W(s) \geq c_0|s|^\beta, \text{ and } \forall s \in \hat{I}_1,\, W(s) \geq c_1|s-1|^\beta. \tag{2.1}$$

(W_2) There exists a $c > 0$ such that for large $|s|$, $W(s) \geq c(s^2 - 1)$.
(W_3) There exist $c_1, c_2 > 0$ and $p > 0$ such that for large $|s|$, $c_1|s|^p \leq W(s) \leq c_2|s|^p$.
(W_4) There exist $c_3, c_4 > 0$ and $q > 0$ such that for large $|s|$, $c_3|s|^q \leq W'(s) \leq c_4|s|^q$.

(W_1) describes the behavior near the wells. It says that W is strictly bounded away from zero outside of neighborhoods of its wells and inside these neighborhoods W has a polynomial lower bound. We need it when we study the simultaneous scaling Γ-limit for h_N^α, and it gives us explicit estimates of how quickly sequences of functions with bounded Ginzburg–Landau "energy" approach the wells of the potential. Assumption (W_2) is a coercivity condition that will help establish compactness in some situations (it could be replaced by any assumption that allows the conclusion that $\int_{\mathbb{T}^2} W(u)$ is bounded from below by a function which is coercive in $\|u\|_{L^2(\mathbb{T}^2)}$). ($W_3$) with $p \geq 2$ is a condition needed to prove compactness in the classical Modica–Mortola Γ-convergence result for F_ε^{GL} (see *e.g.* [56, Proposition 3]). In addition, we will use its lower bound to prove equi-coerciveness (Definition 2.2) of the functional k_N^α, which is defined below in (2.4). Finally, we will need (W_4) to control the behavior of W in between grid points, when studying the simultaneous scaling Γ-limit for k_N^α. As is easily checked, the standard example $W(s) = s^2(s-1)^2$ satisfies all the above assumptions for correctly chosen constants.

We frequently encounter binary functions in $\mathcal{V}$ and write

$$\mathcal{V}^b := \{u \in \mathcal{V} : \forall i \in I_m,\, u_i \in \{0,1\}\}\,.$$

The graph Ginzburg–Landau functional $f_\varepsilon : \mathcal{V} \to \mathbb{R}$ from (1.3) can be defined in terms of the Dirichlet energy:

$$f_\varepsilon(u) = 2\chi\|\nabla u\|_{\mathcal{E}}^2 + \frac{1}{\varepsilon}\sum_{i=1}^m W(u_i), \qquad \text{with } \chi \in (0,\infty).$$

Let $\mathbb{T}^2$ be the two-dimensional flat unit torus. We construct a square grid with $m = N^2$ and nodes $G_N := N^{-1}\mathbb{Z}^2 \cap \mathbb{T}^2$. Interpreting G_N as a graph we can use the notation from Section 2.1 with subscript N, *e.g.* V_N are the vertices of G_N, $\mathcal{V}_N$ are the real-valued functions on V_N, $(\mathcal{V}^b)_N$ the binary

($\{0,1\}$-valued) functions on V_N, etc. We understand the vertices V_N to be embedded in $\mathbb{T}^2$. To distinguish the horizontal and vertical directions in our graph when working on G_N, instead of u_i we will write $u_{i,j} := u(n_{i,j})$, where $n_{i,j} := (i/N, j/N) \in V_N \subset \mathbb{T}^2$. We refer to a single square in the grid by

$$S_N^{i,j} := [i/N, (i+1)/N) \times [j/N, (j+1)/N). \tag{2.2}$$

We remind the reader that we introduced three different functionals on the square grid: the graph-theoretical Ginzburg–Landau functional $h_{N,\varepsilon} : \mathcal{V}_N \to \mathbb{R}$ in (1.4), the discretized Ginzburg–Landau functional $k_{N,\varepsilon} : \mathcal{V}_N \to \mathbb{R}$ in (1.5), and the "sharp interface" (*i.e.,* $\varepsilon \to 0$) nonlocal means functional $g_N : \mathcal{V}_N^b \to \mathbb{R}$ in (1.6).

We call $h_{N,\varepsilon}$ the graph-theoretical Ginzburg–Landau functional because it is equal to f_ε from (1.3) if we choose the weight ω in f_ε as

$$\omega(n_{i,j}, n_{k,l}) := \begin{cases} N^{-1} & \text{if } (|i-k| = 1 \wedge j = l) \vee (i = k \wedge |j-l| = 1), \\ 0 & \text{otherwise,} \end{cases}$$

and $\chi = \frac{1}{2}$. The $k_{N,\varepsilon}$ we get by using the trapezoidal rule and a standard finite-difference scheme to discretize the Ginzburg–Landau functional F_ε^{GL}. We have used the periodicity to relate the terms of the form $(u_{i,j} - u_{i-1,j})^2$ and $(u_{i,j} - u_{i,j-1})^2$ to $(u_{i+1,j} - u_{i,j})^2$ and $(u_{i,j+1} - u_{i,j})^2$ in the sum, respectively.

To study Γ-convergence for the "simultaneous" limits $\varepsilon \to 0$ and $N \to \infty$ of $h_{N,\varepsilon}$ and $k_{N,\varepsilon}$ we set $\varepsilon = N^{-\alpha}$, for $\alpha > 0$, and let $N \to \infty$ in the functionals

$$h_N^\alpha(u) := N^{-1} \sum_{i,j=1}^N (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + N^\alpha \sum_{i,j=1}^N W(u_{i,j}), \tag{2.3}$$

$$k_N^\alpha(u) := N^{-\alpha} \sum_{i,j=1}^N (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + N^{\alpha-2} \sum_{i,j=1}^N W(u_{i,j}). \tag{2.4}$$

We prove Γ-convergence results for h_N^α in Section 4.2 and for k_N^α in Section 5.2.

Note that $h_N^\alpha = N^{\gamma-1} k_N^\gamma$ if $\gamma = \frac{\alpha+3}{2}$. This shows that we do not expect h_N^α and k_N^γ to have the same limit, unless possibly if $\alpha = \gamma = 1$. This value falls outside the regimes for α we consider, and hence we do find different limits.

2.4. **Γ-convergence.** Γ-convergence was introduced by De Giorgi and Franzoni in [29]. It is a type of convergence for function(al)s that is tailored to the needs of minimization problems, as we will see below. A good introduction to the subject is [15], the standard reference work is [28].

Definition 2.1. *Let X be a metric space and let $\{F_n\}_{n=1}^\infty$ be a sequence of functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$. We say that F_n Γ-converges to the functional $F : X \to \mathbb{R} \cup \{\pm\infty\}$, denoted by $F_j \xrightarrow{\Gamma} F$, if for all $u \in X$ we have that*

- (LB) *for every sequence $\{u_n\}_{n=1}^\infty$ such that $u_n \to u$ it holds that $F(u) \leq \liminf\limits_{n\to\infty} F_n(u_n)$ and*
- (UB) *there exists a sequence $\{u_n\}_{n=1}^\infty$ such that $F(u) \geq \limsup\limits_{n\to\infty} F_n(u_n)$.*

The lower bound condition (LB) tells us that the values along the sequence $F_n(u_n)$ are bounded from below by $F(u)$; the upper bound (UB) shows that the value $F(u)$ is actually achieved. Combined with a compactness or equicoerciveness condition (Definition 2.2 below), this allows for conclusions on the minimizers of F_n and F.

It is useful to note that to prove (LB) for a given sequence $\{u_n\}_{n=1}^\infty$ we only need to prove it for a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ such that $\lim\limits_{n'\to\infty} F_{n'}(u_{n'}) = \liminf\limits_{n\to\infty} F_n(u_n)$. If (LB) is satisfied for such a sequence (which always exists), then $F(u) \leq \liminf\limits_{n'\to\infty} F_{n'}(u_{n'}) = \lim\limits_{n'\to\infty} F_{n'}(u_n') = \liminf\limits_{n\to\infty} F(u_n)$. Hence, when proving (LB) we will assume without loss of generality that $\{u_n\}_{n=1}^\infty$ is such a sequence. The uniqueness of the limit then implies that it suffices to prove (LB) for any subsequence. Clearly we can also assume that $\liminf F_n(u_n) < \infty$, and hence it suffices to prove (LB) for a specific subsequence $\{u_{n''}\}_{n''=1}^\infty$ for which there is a $C > 0$ such that $F_{n''}(u_{n''}) \leq C$. In practice this means that to prove (LB) we can assume a uniform bound on $F_n(u_n)$ and freely pass to subsequences when needed.

If we are working with functionals that depend on a continuous parameter, *e.g.* $\varepsilon \to 0$ or $N \to \infty$, we have to prove (LB) and (UB) for an arbitrary sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0$ as $n \to \infty$ (or $\{N_n\}_{n=1}^\infty$ with $N_n \to \infty$ as $n \to \infty$).

Definition 2.2. *Let X be a metric space and let $\{F_n\}_{n=1}^\infty$ be a sequence of functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$. We say the sequence is* equi-coercive *if for every $t \in \mathbb{R}$ there exists a compact set $K_t \subset X$ such that for every $n \in \mathbb{N}$ $\{u \in X : F_n(u) \leq t\} \subset K_t$.*

In practice equi-coerciveness is proved by showing that any sequence $\{u_n\}_{n=1}^\infty$ for which $F_n(u_n)$ is uniformly bounded has a convergent subsequence.

Γ-convergence combined with equi-coerciveness allows us to conclude the following result.

Theorem 2.3 (Chapter 7 in [28] and Theorem 1.21 in [15])**.** *Let X be a metric space, $\{F_n\}_{n=1}^\infty$ be a sequence of equi-coercive functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$, and let F be the Γ-limit of F_n for $n \to \infty$. Then there exists a minimizer of F in X and* $\min\{F(u) : u \in X\} = \lim_{n\to\infty} \inf\{F_n(u) : u \in X\}$. *Furthermore, if $\{u_n\}_{n=1}^\infty \subset X$ is a precompact sequence such that*

$$\lim_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} \inf\{F_n(u) : u \in X\},$$

then every cluster point of this sequence is a minimizer of F.

Following [5] for our purposes it turns out it is often more useful to reformulate Γ-convergence in terms of the Γ-lower limit

$$F'(u) := \inf\{\liminf_{n\to\infty} F_n(u_n) : u_n \to u\}$$

and the Γ-upper limit

$$F''(u) := \inf\{\limsup_{n\to\infty} F_n(u_n) : u_n \to u\},$$

[28, Definition 4.1]. It can be shown, [28, Remark 4.2, Proposition 8.1], [16], that our definition of Γ-convergence above is equivalent to the following two conditions. For each $u \in X$ we have that

(LB$'$) $F(u) \leq F'(u)$ and
(UB$'$) $F(u) \geq F''(u)$.

The benefit of this reformulation is that the functions F' and F'' are lower semicontinuous [28, Proposition 6.8], which comes in handy in Section 5. In fact, since conditions (LB) and (LB$'$) are equivalent, we will sometimes use the combination (LB)+(UB$'$) to prove Γ-convergence in this paper. Note that (UB) implies (UB$'$).

2.5. **Constraints.** It is common in semisupervised learning applications to have a mass constraint or an additional term in the functional corresponding to a fit to the known data. Moreover, such constraints are typically necessary to obtain nontrivial minimizers. We need to check that these constraints are compatible with the convergence.

First consider the case of adding a fidelity term of the form $\lambda|u - f|_p^p$ to the functional, where $f \in \mathcal{V}$ is a given function (usually representing some known data to which the minimizer should be similar) defined on some or all of the vertices in V and $\lambda > 0$ is a parameter. If p agrees with the topology of the Γ-convergence we can use the property that Γ-limits are stable under addition of a continuous term or a sequence of continuously convergent terms [28, Definition 4.7, Propositions 6.20–21] to conclude that the Ginzburg–Landau functionals plus fidelity term again Γ-converge. We will summarize the results that are relevant for us in the following definition and lemma, based on the cited definition and propositions in [28].

Definition 2.4. *Let X be a metric space, and let $\{F_n\}_{n=1}^\infty$ be a sequence of functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$. We say the sequence is* continuously convergent *to a function $F : X \to \mathbb{R} \cup \{\pm\infty\}$ if for every $u \in X$ and for every $\eta > 0$ there is an $\bar{N} \in \mathbb{N}$ and a $\delta > 0$ such that for all $n \geq \bar{N}$ and $v \in X$ with $\|u - v\| < \delta$ we have $|F_n(v) - F(u)| < \eta$.*

Lemma 2.5. *Let X be a metric space, and let $\{F_n\}_{n=1}^\infty$ be a sequence of functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$ which Γ-converges to $F : X \to \mathbb{R} \cup \{\pm\infty\}$ and $\{H_n\}_{n=1}^\infty$ be a sequence of functionals $H_n : X \to \mathbb{R} \cup \{\pm\infty\}$ which is continously convergent to $H : X \to \mathbb{R} \cup \{\pm\infty\}$; then $F_n + H_n$ Γ-converges to $F + H$. If $\hat{G} : X \to \mathbb{R} \cup \{\pm\infty\}$ is a continuous functional, then $F_n + \hat{G}$ Γ-converges to $F + \hat{G}$.*

If instead a mass constraint is imposed on the minimizers we have to check that each convergent sequence preserves the constraint (to make it compatible with the lower bound and compactness conditions) and that the recovery sequence from the upper bound condition does satisfy the constraint (or can be adapted to satisfy it, without violating the upper bound condition). Since we are dealing with L^p convergence, the former is usually trivially satisfied, but the latter does demand some more attention. Details for each of the functionals are provided in the relevant sections.

For functions $u \in \mathcal{V}^b$ we carefully need to determine the form of our mass constraint. The constraint $\sum_{i=1}^m u_i = M$ leads to shrinking support when $m \to \infty$, which is unwanted. An alternative condition is an average mass constraint of the form $\frac{1}{m} \sum_{i=1}^m u_i = M$. Note that $\sum_{i=1}^m u_i$ can take on only integer values between 0 and m, and hence mM should be of that form as well in order for the average mass constraint not to lead to an empty set of admissible minimizers. If we impose this for all m this is only possible if $M = 0$ or $M = 1$; however, specific subsequences of m can satisfy this condition for different values of M (*e.g.* if $M = \frac{1}{2}$ and we consider even m). Hence the choice of M can constrain the subsequences of m which are admissible. In order to avoid the possible difficulties with the average mass *equality*, one can also impose an average mass *inequality*. Since the arguments for the mass equality easily generalize to an inequality, we will not discuss this situation further.

3. Γ-CONVERGENCE OF f_ε

3.1. Γ-convergence and compactness.

In this section we prove Γ-convergence and compactness for the functional $f_\varepsilon : \mathcal{V} \to \mathbb{R}$ from (1.3).

Theorem 3.1 (Γ-convergence)**.** *$f_\varepsilon \xrightarrow{\Gamma} f_0$ as $\varepsilon \to 0$, where*

$$f_0(u) := \begin{cases} \chi \sum_{i,j\in I_m} \omega_{ij}|u_i - u_j| & \text{if } u \in \mathcal{V}^b, \\ +\infty & \text{otherwise} \end{cases} = \begin{cases} 2\chi TV_{a1}(u) & \text{if } u \in \mathcal{V}^b, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.2 (Compactness)**.** *Let W satisfy the coercivity condition (W_2), let $\{\varepsilon_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$, and let $\{u_n\}_{n=1}^\infty \subset \mathcal{V}$ be a sequence such that there exists a $C > 0$ such that for all $n \in \mathbb{N}$ $f_{\varepsilon_n}(u_n) < C$. Then there exists a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and a $u_\infty \in \mathcal{V}^b$ such that $u_{n'} \to u_\infty$ as $n \to \infty$.*

Although in F_ε^{GL} the first term is scaled by ε, the first term of f_ε contains no ε. The reason for this is that the Dirichlet energy in F_ε^{GL} is unbounded for the binary functions u that form the domain of the limit functional F_0^{GL}. However, the difference terms in f_ε are finite even for the binary functions and thus need no rescaling. The proof of Γ-convergence uses this fact to view the difference terms as a continuous perturbation of the functional

$$w_\varepsilon(u) := \frac{1}{\varepsilon} \sum_{i\in I_m} W(u_i).$$

Lemma 3.3. *The sequence of functionals g_ε Γ-converges: $w_\varepsilon \underset{\varepsilon\to 0}{\xrightarrow{\Gamma}} w_0$, where $w_0 : \mathcal{V} \to \{0, \infty\}$ is defined via*

$$w_0(u) := \begin{cases} 0 & \text{if } u \in \mathcal{V}^b, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. To prove the required lower bound (LB) let $u \in \mathcal{V}$ and consider sequences $\{\varepsilon_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ such that $\varepsilon_n \to 0$ and $u_n \to u$ as $n \to \infty$. If $u \in \mathcal{V}^b$, from $g_\varepsilon \geq 0$ it follows that $w_0(u) = 0 \leq \liminf_{n\to\infty} w_{\varepsilon_n}(u_n)$. If on the other hand $u \in \mathcal{V} \setminus \mathcal{V}^b$, then for large enough n, $\exists \bar{v} \in V$ such that $u_n(\bar{v}) \notin \{0, 1\}$ and hence

$$\liminf_{n\to\infty} w_{\varepsilon_n}(u_n) \geq \liminf_{n\to\infty} \frac{1}{\varepsilon_n} W(\bar{v}) = \infty = w_0(u).$$

For the upper bound (UB) we can assume without loss of generality that $u \in \mathcal{V}^b$. Define for every $n \in \mathbb{N}$ $u_n := u$; then trivially $u_n \to u$ if $n \to \infty$, and furthermore $\limsup\limits_{n\to\infty} w_{\varepsilon_n}(u_n) = 0 = w_0(u)$. $\square$

Proof of Theorem 3.1. Define

$$\hat{w}(u) := \chi \sum_{i,j\in I_m} \omega_{ij}(u_i - u_j)^2.$$

$\hat{w}$ is a polynomial on $\mathbb{R}^m$ and hence continuous. Γ-convergence is stable under continuous perturbations (*e.g.* [28, Proposition 6.21]). Since f_ε is a continuous perturbation of w_ε we have by Lemma 3.3 $f_\varepsilon \xrightarrow{\Gamma} \hat{w} + w_0$ as $\varepsilon \to 0$. We complete the proof by noting that if $u \in \mathcal{V}^b$, then

$$\hat{w}(u) = \chi \sum_{i,j\in I_m} \omega_{ij}|u_i - u_j|. \qquad \square$$

Proof of Theorem 3.2. By the uniform bound on $f_{\varepsilon_n}(u_n)$ we have

$$\sum_{i\in I_m} W((u_n)_i) \le C\varepsilon_n.$$

Combined with the coercivity condition (W_2) on W we conclude that for n large enough the sequence $\{u_n\}_{n=1}^\infty$ is bounded, and hence by the Bolzano–Weierstrass theorem there exists a converging subsequence with limit u_∞. Since $W \in C^2(\mathbb{R})$ and $\varepsilon_n \to 0$ as $n \to \infty$, we conclude that $W(u_\infty(v_i)) = 0$ for all $i \in I_m$ and hence $u_\infty \in \mathcal{V}^b$. $\square$

Remark 3.4. Since f_0 is defined on binary functions u, if we write

$$S_k := \{i \in I_m : u_i = k\} \quad \text{for } k \in \{0, 1\},$$

we can rewrite f_0 as

$$f_0(u) = \begin{cases} \chi \sum_{i\in S_0, j\in S_1} \omega_{i,j} & \text{if } u \in \mathcal{V}^b, \\ +\infty & \text{otherwise.} \end{cases}$$

Minimizing f_0 thus corresponds to finding a minimal graph cut of G, *i.e.*, dividing the graph into clusters with minimal edge weight between them. Such a minimization requires an extra constraint to avoid trivial minimizers. A common choice is to prescribe the number of clusters (in this case two) one wants, or to introduce a normalization into the sum of the weights based on the cluster sizes (*e.g.* normalized cut, normalized association [55], and Cheeger cut [57])). One could also minimize f_0 under a fixed-mass constraint (see Section 3.2 below). One often relaxes the problem of normalized graph-cut minimization by losing the binarity constraint, *e.g.* in spectral clustering [51, 43].

If the weight function ω is such that there is a nontrivial partition $A \cup B = I_m$ such that $\omega_{ij} = 0$ for $i \in A$ and $j \in B$, then a nontrivial minimizer of f_0 (without the mass constraint) is clearly given via $S_0 = A$ and $S_1 = B$. In this case we can write

$$u_i = \frac{\sum_{j \in I_m} \omega_{ij} u_j}{d_i},$$

which is of a form related to the image-denoising method known as nonlocal means; *cf.* [21].

Remark 3.5. In this paper we assume that W has wells at 0 and 1. The proofs in this section make no use of this fact and can easily be extended to potentials W with wells at values s_1 and s_2. In this case the set $\mathcal{V}^b$ needs to be redefined as the set of functions taking values in $\{s_1, s_2\}$, and the limit functional f_0 from Theorem 3.1 is multiplied by a factor $|s_1 - s_2|$, since $(u_i - u_j)^2 = |s_1 - s_2||u_i - u_j|$ for $u \in \mathcal{V}^b$.

3.2. **Constraints.** Next we show that the addition of a fidelity term $\lambda|u - f|_p^p$ or a mass constraint is compatible with the Γ-convergence.

Theorem 3.6 (Constraints)**.** *$f_\varepsilon + \lambda|\cdot - f|_p^p \xrightarrow{\Gamma} f_0 + \lambda|\cdot - f|_p^p$ as $\varepsilon \to 0$, where $p \in \mathbb{R}_+$, $\lambda > 0$, and a given function $f \in \mathcal{V}$ (or possibly a given function $f : U \to \mathbb{R}$ where U is a strict subset of the vertex set V and the sum in $|\cdot - f|_p^p$ is restricted to vertices in U). Compactness for $f_\varepsilon + \lambda|\cdot - f|_p^p$ as in Theorem 3.2 holds. If instead, for fixed $M > 0$, the domain of definition of f_ε is restricted to*

$$\mathcal{V}^M := \{u \in \mathcal{V} : \sum_{i \in I_m} u_i = mM\},$$

where M is such that mM is an integer between 0 and m, then the results of Theorems 3.1 and 3.2 remain valid, with the domain of f_0 restricted to $\mathcal{V}^M$.

Proof. The fidelity term $\lambda|u - f|_p^p$ is a polynomial; hence, a continuous perturbation independent of ε to f_ε and thus Γ-convergence follows by Theorem 3.1 and Lemma 2.5. The addition of this term does not affect the compactness property at all.

The mass constraint is compatible with the limit functional being defined on binary functions. The constraint is preserved under convergence in $\mathcal{V} \cong \mathbb{R}^m$, so it is compatible with (LB) from Definition 2.1 and compactness.

If $u \in \mathcal{V}^b$ satisfies the mass constraint, then trivially so does the recovery sequence $\{u_n\}_{n=1}^\infty$ used to prove (UB) from Definition 2.1 in Lemma 3.3. $\square$

4. Γ-LIMITS FOR THE GRAPH-BASED FUNCTIONAL $h_{N,\varepsilon}$

In this section we will study the convergence properties of $h_{N,\varepsilon}$ from (1.4). We consider two different cases. In the first we first take the limit $\varepsilon \to 0$ and then let $N \to \infty$; in the second we take both limits at once by substituting $\varepsilon = N^{-\alpha}$ for well-chosen $\alpha > 0$ and then considering the limit $N \to \infty$ for h_N^α in (2.3). These results have a similar feel as numerical convergence results; however, the lack of regularity of the binary limit functions complicates the results and proofs.

Remark 4.1. We note that, while we give the proofs for $\mathbb{T}^2$ in Sections 4.1 and 4.2, they can easily be generalized to $\mathbb{T}^d$ for any $d \in \mathbb{N}$, if we let the scaling factor in the first term of $h_{N,\varepsilon}$ in (1.4) be N^{1-d} instead of N^{-1} and we change h_N^α in (2.3) accordingly:

$$h_{N,\varepsilon}(u) := N^{1-d} \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \varepsilon^{-1} \sum_{i,j=1}^{N} W(u_{i,j}),$$

$$h_N^\alpha(u) := N^{1-d} \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + N^\alpha \sum_{i,j=1}^{N} W(u_{i,j}).$$

For the extra fidelity term in Theorem 4.13 the scaling factor then needs to be N^{-d} instead of N^{-2}.

4.1. Compactness and sequential Γ-limits: first $\varepsilon \to 0$, then $N \to \infty$.

By Theorem 3.1 we immediately have $h_{N,\varepsilon} \xrightarrow{\Gamma} h_{N,0}$ as $\varepsilon \to 0$, where $h_{N,0}$ is defined for $u \in \mathcal{V}$ as

$$h_{N,0}(u) := \begin{cases} N^{-1} \sum_{i,j=1}^N \left(|u_{i+1,j} - u_{i,j}| + |u_{i,j+1} - u_{i,j}|\right) & \text{if } u \in \mathcal{V}_N^b, \\ +\infty & \text{otherwise.} \end{cases}$$

In this section we prove that $h_{N,0} \xrightarrow{\Gamma} h_{\infty,0}$ as $N \to \infty$, where $h_{\infty,0}$ is defined for $u \in L^1(\mathbb{T}^2)$ as

$$h_{\infty,0}(u) := \begin{cases} \displaystyle\int_{\mathbb{T}^2} |u_x| + |u_y| & \text{if } u \in BV(\mathbb{T}^2; \{0,1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

The anisotropic total variation in $h_{\infty,0}$ is defined as

$$\int_{\mathbb{T}^2} |u_x| + |u_y| := \sup\left\{ \int_{\mathbb{T}^2} u \operatorname{div} v : v \in C_c^1(\mathbb{T}^2; \mathbb{R}^2),\ \forall x\ |v(x)|_\infty \leq 1 \right\},$$

where for a vector $v(x) = (v_1(x), v_2(x)) \in \mathbb{R}^2$, the norm $|\cdot|_\infty$ is defined by

$$|v(x)|_\infty := \max\{|v_1(x)|, |v_2(x)|\}.$$

For functions $u \in BV(\mathbb{T}^2; \{0,1\})$ this anisotropic total variation gives the length of the (reduced[8]) boundary of the set $\{u = 1\}$ projected onto the horizontal and vertical axes (counting multiplicities).

We prove a compactness and Γ-convergence result.

Theorem 4.2 (Compactness). *Let $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfy $N_n \to \infty$ as $n \to \infty$, and let $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ be a sequence for which there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $h_{N_n,0}(u_n) \leq C$. Then there exists a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and a $u \in BV(\mathbb{T}^2; \{0,1\})$ such that $u_{n'} \to u$ in $L^1(\mathbb{T}^2)$ as $n' \to \infty$.*

Theorem 4.3 (Γ-convergence). *$h_{N,0} \xrightarrow{\Gamma} h_{\infty,0}$ as $N \to \infty$ in the $L^1(\mathbb{T}^2)$ topology.*

The convergence of $h_{N,0}$ to an anisotropic total variation is reminiscent of the related, but different, results in [23].

Because the limit function $h_{\infty,0}$ is defined on $L^1(\mathbb{T}^2)$ functions, it will be useful to identify the binary functions on the square graph, *i.e.*, the functions in $\mathcal{V}_N^b$ with a subset of $L^1(\mathbb{T}^2)$, namely binary functions on $\mathbb{T}^2$ that are piecewise constant on the squares of the grid. Using the notation $S_N^{i,j}$ from (2.2) we define

$$\begin{aligned} \mathcal{A}_N &:= \left\{ u \in L^1(\mathbb{T}^2) : \forall (i,j) \in I_N^2,\ u \text{ is constant a.e. on } S_N^{i,j} \right\}, \qquad (4.1)\\ \mathcal{A}_N^b &:= \left\{ u \in \mathcal{A}_N : u \in L^1(\mathbb{T}^2; \{0,1\}) \right\}. \end{aligned}$$

We construct a bijection between the normed spaces $\mathcal{V}_N$ and $\mathcal{A}_N$ by identifying $u \in \mathcal{V}_N$ with the unique $\tilde{u} \in \mathcal{A}_N$ which satisfies $\tilde{u} \equiv u_{i,j}$ almost everywhere on $S_N^{i,j}$ for all $(i,j) \in I_N^2$. It is easy to check that convergence in $\mathcal{V}_N$ corresponds to L^p convergence in $\mathcal{A}_N$ $(1 \leq p < \infty)$ and that the bijection maps the subset $\mathcal{V}_N^b$ to $\mathcal{A}_N^b$ and vice versa (for its inverse). In what follows we will drop the tilde if this does not lead to confusion.

[8]For the definition of reduced boundary see [9, Definition 3.54].

With this identification we write for $u \in \mathcal{A}_N$

$$h_{N,0}(u) = \begin{cases} \displaystyle\int_{\mathbb{T}^2} |u_x| + |u_y| & \text{if } u \in \mathcal{A}_N^b, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

In fact, for our Γ-convergence purposes, without loss of generality we extend the functional to all $u \in L^1(\mathbb{T}^2)$ such that $h_{N,0}(u) = +\infty$ if $u \in L^1(\mathbb{T}^2) \setminus \mathcal{A}_N^b$.

First we prove the compactness result.

Proof of Theorem 4.2. By the definition of the isotropic and anisotropic total variation and (4.2) we have for all $n \in \mathbb{N}$

$$\int_{\mathbb{T}^2} |\nabla u_n| \leq \int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y| \leq C.$$

In addition, for each $n \in \mathbb{N}$, $u_n \in \mathcal{A}_{N_n}^b$; hence, $\|u_n\|_{L^1(\mathbb{T}^2)} \leq 1$. We deduce that the sequence $\{u_n\}_{n=1}^\infty$ is uniformly bounded in the BV norm, and thus by compactness ([37, Theorem 1.19] or [32, 5.2.3 Theorem 4]) there exists a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and a $u \in BV(\mathbb{T}^2)$ such that $u_{n'} \to u$ in $L^1(\mathbb{T}^2)$ as $n' \to \infty$. Since all u_n take the values 0 and 1 almost everywhere, by pointwise almost-everywhere convergence (after possibly going to another subsequence) so does u. □

In the next lemma (LB) from Definition 2.1 is proved.

Lemma 4.4 (Lower bound)**.** *Let $u \in L^1(\mathbb{T}^2)$ and let $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ and $N_n \to \infty$ as $n \to \infty$. Then*

$$h_{\infty,0}(u) \leq \liminf_{n\to\infty} h_{N_n,0}(u_n).$$

Proof. First consider the case where $u \in BV(\mathbb{T}^2; \{0,1\})$; then without loss of generality we can assume that $u_n \in \mathcal{A}_{N_n}^b$. Analogous to the isotropic total variation also this anisotropic total variation is lower semicontinuous with respect to L^1 convergence [24, Lemma A.5]; hence, we find

$$h_{\infty,0}(u) = \int_{\mathbb{T}^2} |u_x| + |u_y| \leq \liminf_{n\to\infty} \int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y| = \liminf_{n\to\infty} h_{N_n,0}(u_n).$$

If $u \in L^1(\mathbb{T}^2) \setminus BV(\mathbb{T}^2; \{0,1\})$ and $u_n \to u$ in $L^1(\mathbb{T}^2)$, then

$$h_{\infty,0}(u) = \infty = \liminf_{n\to\infty} h_{N_n,0}(u_n),$$

which we prove via contradiction: Assume $\liminf\limits_{n\to\infty} h_{N_n,0}(u_n) < \infty$; then there is a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ for which $h_{N_{n'},0}(u_{n'})$ is uniformly bounded, and hence by Theorem 4.2 $u_{n'} \to \hat{u}$ in $L^1(\mathbb{T}^2)$ as $n' \to \infty$, where $\hat{u} \in$

$BV(\mathbb{T}^2;\{0,1\})$. By the uniqueness of limit $\hat{u}=u$, which is a contradiction. $\square$

To prove the lim sup inequality we use the following results from [24] (which we give here in a form adapted to our situation, which follows easily from the results in [24]).

Lemma 4.5 (Corollary A.4 and Theorem 4.1 from [24])**.** *For $u \in BV(\mathbb{T}^2)$ we have*

$$\begin{aligned}&\int_{\mathbb{T}^2} |u_x|+|u_y|\\&=\sup\left\{\int_{\mathbb{T}^2} u\,\mathrm{div}\, v : v\in L^\infty(\mathbb{T}^2;\mathbb{R}^2), \mathrm{div}\, v\in L^\infty(\mathbb{T}^2),\ |v(x)|_\infty \le 1 \text{ a.e.}\right\}.\end{aligned}$$

Furthermore, for each $u \in \mathcal{A}_N^b$ there exists a $v \in L^\infty(\mathbb{T}^2;\mathbb{R}^2)$ such that $|v(x)|_\infty \le 1$ almost everywhere,

$$-\int_{\mathbb{T}^2} u\,\mathrm{div}\, v = \int_{\mathbb{T}^2} |u_x|+|u_y|,$$

and $\|\mathrm{div}\, v\|_{L^\infty(\mathbb{T}^2)} = 4N$.

The first result in the above lemma says we can relax the condition on the admissible vector fields in the definition of the anisotropic total variation to L^∞ vector fields with essentially bounded divergence. The second result shows that the supremum is achieved by a specific vector field if $u \in \mathcal{A}_N^b$.

Lemma 4.6 (Upper bound)**.** *Let $u \in L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $N_n \to \infty$ as $n\to\infty$. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ as $n\to\infty$ and $h_{\infty,0}(u) \ge \limsup_{n\to\infty} h_{N_n,0}(u_n)$.*

Proof. Without loss of generality we can assume that $u \in BV(\mathbb{T}^2;\{0,1\})$. Construct u_n as follows. For $x \in S_{N_n}^{i,j}$ define

$$u_n(x) := \begin{cases} 1 & \text{if } (S_{N_n}^{i,j})^\circ \subset \mathrm{supp}\, u,\\ 0 & \text{otherwise.}\end{cases}$$

In words, u_n takes the value one on those squares of the grid whose interior lies completely in the support of u and zero on the other squares. By $\partial_* \mathrm{supp}\, u$ denote the reduced boundary of the set $\mathrm{supp}\, u$, *i.e.,* all points in $\partial\, \mathrm{supp}\, u$ for which there is a well-defined normal vector (see [9, Definition 3.54]). Since $\mathrm{supp}\, u_n$ is the maximal set which is both a union of squares on the grid and is contained in $\mathrm{supp}\, u$, the difference in area between $\mathrm{supp}\, u$

and $\operatorname{supp} u_n$ is bounded by the length of the reduced boundary of $\operatorname{supp} u$ times the area of a square; *i.e.*,

$$\int_{\mathbb{T}^2} |u_n - u| \leq \mathcal{H}^1(\partial^* \operatorname{supp} u) N_n^{-2}.$$

Because $u \in BV(\mathbb{T}^2; \{0,1\})$ the set $\operatorname{supp} u$ has finite perimeter, and hence $u_n \to u$ in $L^1(\mathbb{T}^2)$.

For each u_n let $v_n \in L^\infty(\mathbb{T}^2; \mathbb{R}^2)$ be the vector field whose existence is guaranteed by Lemma 4.5; then

$$\int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y| = -\int_{\mathbb{T}^2} u_n \operatorname{div} v_n = -\int_{\mathbb{T}^2} u \operatorname{div} v_n + \int_{\mathbb{T}^2} (u - u_n) \operatorname{div} v_n. \tag{4.3}$$

For the last term we have

$$\begin{aligned}\left| \int_{\mathbb{T}^2} (u - u_n) \operatorname{div} v_n \right| &\leq \|u - u_n\|_{L^1(\mathbb{T}^2)} \| \operatorname{div} v_n\|_{L^\infty(\mathbb{T}^2)} \\ &\leq 4\mathcal{H}^1(\partial^* \operatorname{supp} u) N_n^{-1} \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence, by the first statement of Lemma 4.5 we have

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y| &= \limsup_{n\to\infty} \Big(-\int_{\mathbb{T}^2} u \operatorname{div} v_n \Big) \\ &\leq \limsup_{n\to\infty} \int_{\mathbb{T}^2} |u_x| + |u_y| = \int_{\mathbb{T}^2} |u_x| + |u_y|,\end{aligned}$$

which proves the result. □

Proof of Theorem 4.3. Combining Lemmas 4.4 and 4.6 proves the Γ-convergence result in Theorem 4.3. □

Remark 4.7. Note that we could have used any L^p space instead of L^1 in the results above. Because $\mathbb{T}^2$ is bounded, convergence in L^p implies convergence in L^1, and so the result of Lemma 4.4 is easily recovered. For Theorem 4.2 and Lemma 4.6 we note that because u_n and u are binary functions taking values 0 and 1 almost everywhere, the bound on their L^p difference is the same as that on their L^1 difference, and the results follow again.

We end this section with an illustration of the preference for squares and rectangles of $h_{N,0}$.

Lemma 4.8 (Minimizers of $h_{N,0}$)**.** *Let $M \in [0,1]$ be such that $N^2 M = K^2$ for some $K \in \mathbb{N}$.*

If $M \in [0, \frac{1}{4})$, then u_0 is a minimizer of $h_{N,0}$ over all $u \in \mathcal{A}_N^b$ that satisfy $\int_{\mathbb{T}^2} u = M$ if and only if u_0 is the characteristic function of a square.

If $M \in (\frac{1}{4}, 1)$, then u_0 is a minimizer if and only if it is the characteristic function of a rectangle of the form $R = [a, b] \times [0, 1] \subset \mathbb{T}^2$ or $R = [0, 1] \times [a, b] \subset \mathbb{T}^2$ for a and b that satisfy the mass constraint.

If $M = \frac{1}{4}$, u_0 is a minimizer if and only if it is the characteristic function of a square or a rectangle R as above.

Proof. First consider the square grid G_N to be embedded in $\mathbb{R}^2$ instead of in $\mathbb{T}^2$ so that we do not have periodic boundary conditions on $[0, 1]^2$. Let u be the characteristic function of a set Ω. We can assume Ω is connected, because else $h_{N,0}(u)$ can be lowered by rearranging Ω to be connected without changing the mass. Let u_0 have the square with sides of length $L_0 = KN^{-1} = \sqrt{M}$ as support, and let Ω be contained in a rectangle with sides of lengths L and B. Then $\int_{[0,1]^2} u \leq LB$, and hence

$$h_{N,0}(u_0) = 4L_0 = 4\Big(\int_{\mathbb{T}^2} u\Big)^{1/2} \leq 4\sqrt{LB} \leq 2(L + B) \leq h_{N,0}(u),$$

with equality if and only if $L = B = L_0$. Hence characteristic functions of squares are the minimizers of $h_{N,0}$ if we ignore periodic boundary conditions.

However, on the periodic torus if Ω is a rectangle we can use periodicity to eliminate two sides of the rectangle. This can be done only if the other two sides have length 1 and hence $h_{N,0}(u) = 2$. This beats the square if $4L_0 = 4\sqrt{M} > 2$. $\square$

It is worth noting here that this asymptotic behavior of $h_{N,\varepsilon}$ is not accessible via numerical simulations of a gradient flow, since it is dependent on ε being small enough for the minimizers to be essentially binary and hence not differentiable.

4.2. **Simultaneous scaling Γ-limit for h_N^α.** In Section 4.1 we first took the limit $\varepsilon \to 0$ for $h_{N,\varepsilon}$ before letting $N \to \infty$. In this section we will consider the limit if we let both parameters go to their limit simultaneously. To this end we choose $\varepsilon = N^{-\alpha}$ for $\alpha > 0$ and consider the limit $N \to \infty$ of h_N^α in (2.3). We identify $\mathcal{V}_N$ with $\mathcal{A}_N$ in the sense of Section 5.1 and extend h_N^α to all of $L^1(\mathbb{T}^2)$ by setting $h_N^\alpha(u) := +\infty$ for $u \in L^1(\mathbb{T}^2) \setminus \mathcal{A}_N$.

We show that the Γ-limit is again given by $h_{\infty,0}$ if α is large enough, depending on the growth rate β of W around its wells given in assumption (W_1).

Theorem 4.9 (Γ-convergence)**.** *Assume that* W *satisfies condition* (W_1) *for some* $\beta > 0$ *in* (2.1)*, and let* $\alpha > \beta$*; then* $h_N^\alpha \xrightarrow{\Gamma} h_{\infty,0}$ *as* $N \to \infty$ *in the* $L^1(\mathbb{T}^2)$ *topology.*

Theorem 4.10 (Compactness)**.** *Assume that* W *satisfies condition* (W_1) *for some* $\beta > 0$ *in* (2.1)*, and let* $\alpha > \beta$*. Let* $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ *satisfy* $N_n \to \infty$ *as* $n \to \infty$*, and let* $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ *be a sequence for which there is a constant* $C > 0$ *such that for all* $n \in \mathbb{N}$*,* $h_{N_n}^\alpha(u_n) \le C$*. Then there exists a subsequence* $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ *and a* $u \in BV(\mathbb{T}^2; \{0,1\})$ *such that* $u_{n'} \to u$ *in* $L^1(\mathbb{T}^2)$ *as* $n' \to \infty$*.*

We first prove the lower bound for the Γ-limit.

Lemma 4.11 (Lower bound)**.** *Let* $u \in L^1(\mathbb{T}^2)$*, and let* $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ *and* $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ *be such that* $u_n \to u$ *in* $L^1(\mathbb{T}^2)$ *and* $N_n \to \infty$ *as* $n \to \infty$*. Assume that* W *satisfies condition* (W_1) *for some* $\beta > 0$ *in* (2.1)*, and let* $\alpha > \beta$*. Then*

$$h_{\infty,0}(u) \le \liminf_{n\to\infty} h_{N_n}^\alpha(u_n).$$

Proof. First consider the case where $u \in BV(\mathbb{T}^2; \{0,1\})$. Without loss of generality we may assume that $h_{N_n}^\alpha$ is uniformly bounded. Since W is nonnegative we deduce there is a $C > 0$ such that for all $(i,j) \in I_{N_n}^2$, $W((u_n)_{i,j}) \le C N_n^{-\alpha}$. Together with (2.1) in assumption (W_2) this implies that for n large enough and all $(i,j) \in I_{N_n}^2$ we have $(u_n)_{i,j} \in \hat{I}$. In addition, by the growth condition in (2.1) we get for n large enough and $s \in \{0,1\}$

$$c_s |(u_n)_{i,j} - s|^\beta \le W((u_n)_{i,j}) \le C N_n^{-\alpha}.$$

Hence, if we define $\delta_n := (C/\min(\{c_0, c_1\}))^{\frac{1}{\beta}} N_n^{-\frac{\alpha}{\beta}}$ we deduce that for n large enough $\delta_n \in (0, \frac{1}{2})$, and for all $(i,j) \in I_{N_n}^2$, $(u_n)_{i,j} \in (-\delta_n, \delta_n) \cup (1-\delta_n, 1+\delta_n)$. Define

$$\begin{aligned} X_n := \big\{(i,j) \in I_{N_n}^2 : \big((u_n)_{i,j} \in (-\delta_n, \delta_n) \wedge (u_n)_{i+1,j} \in (1-\delta_n, 1+\delta_n)\big) \\ \vee \big((u_n)_{i+1,j} \in (-\delta_n, \delta_n) \wedge (u_n)_{i,j} \in (1-\delta_n, 1+\delta_n)\big)\big\}; \end{aligned} \tag{4.4}$$

then for $(i,j) \in X_n$ we have $|(u_n)_{i+1,j} - (u_n)_{i,j}| \ge 1 - 2\delta_n$, hence

$$\begin{aligned} &N_n^{-1} \sum_{i,j=1}^{N_n} ((u_n)_{i+1,j} - (u_n)_{i,j})^2 \qquad (4.5) \\ \ge &N_n^{-1}(1-2\delta_n) \sum_{(i,j)\in X_n} |(u_n)_{i+1,j} - (u_n)_{i,j}| + N_n^{-1} \sum_{(i,j)\in X_n^c} ((u_n)_{i+1,j} - (u_n)_{i,j})^2 \end{aligned}$$

$$=N_n^{-1}\sum_{i,j=1}^{N_n}|(u_n)_{i+1,j}-(u_n)_{i,j}|-2\delta_n N_n^{-1}\sum_{(i,j)\in X_n}|(u_n)_{i+1,j}-(u_n)_{i,j}|$$
$$+N_n^{-1}\sum_{(i,j)\in X_n^c}((u_n)_{i+1,j}-(u_n)_{i,j})^2-N_n^{-1}\sum_{(i,j)\in X_n^c}|(u_n)_{i+1,j}-(u_n)_{i,j}|.$$

For the second summation in (4.5) we note that there are $c>0$ and $\tilde{C}>0$ such that

$$0\le 2\delta_n N_n^{-1}\sum_{(i,j)\in X_n}|(u_n)_{i+1,j}-(u_n)_{i,j}|\le c\delta_n N_n^{-1}N_n^2(1+2\delta_n)$$
$$\le \tilde{C}(N_n^{1-\alpha/\beta}+N_n^{1-2\alpha/\beta})\to 0 \quad \text{as } n\to\infty.$$

The third and fourth summation in (4.5) we combine into

$$\left|N_n^{-1}\sum_{(i,j)\in X_n^c}|(u_n)_{i+1,j}-(u_n)_{i,j}|\big(|(u_n)_{i+1,j}-(u_n)_{i,j}|-1\big)\right|$$
$$\le N_n^{-1}\sum_{(i,j)\in X_n^c}|(u_n)_{i+1,j}-(u_n)_{i,j}|\ \big||(u_n)_{i+1,j}-(u_n)_{i,j}|-1\big|$$
$$\le N_n^{-1}N_n^2 2\delta_n(2\delta_n-1)\le \tilde{C}(N_n^{1-2\alpha/\beta}-N_n^{1-\alpha/\beta})\to 0 \quad \text{as } n\to\infty,$$

for some $\tilde{C}>0$. As in Section 5.1 we now identify $\mathcal{V}_{N_n}$ with $\mathcal{A}_{N_n}$ to write for the first summation in (4.5)

$$N_n^{-1}\sum_{i,j=1}^{N_n}|(u_n)_{i+1,j}-(u_n)_{i,j}|=\int_{\mathbb{T}^2}|(u_n)_x|;$$

hence, from (4.5) and the computations that followed we deduce

$$N_n^{-1}\sum_{i,j=1}^{N_n}((u_n)_{i+1,j}-(u_n)_{i,j})^2\ge\int_{\mathbb{T}^2}|(u_n)_x|+\mathcal{O}(N_n^{1-\alpha/\beta}). \tag{4.6}$$

Analogously we have

$$N_n^{-1}\sum_{i,j=1}^{N_n}((u_n)_{i,j+1}-(u_n)_{i,j})^2\ge\int_{\mathbb{T}^2}|(u_n)_y|+\mathcal{O}(N_n^{1-\alpha/\beta}).$$

By the lower semicontinuity of the anisotropic total variation with respect to L^1 convergence [24, Lemma A.5] we have

$$\int_{\mathbb{T}^2}|u_x|+|u_y|\le\liminf_{n\to\infty}\int_{\mathbb{T}^2}|(u_n)_x|+|(u_n)_y|;$$

hence,

$$\begin{aligned}\int_{\mathbb{T}^2} |u_x| + |u_y| &\leq \liminf_{n\to\infty} N_n^{-1} \sum_{i,j=1}^{N_n} ((u_n)_{i+1,j} - (u_n)_{i,j})^2 + ((u_n)_{i,j+1} - (u_n)_{i,j})^2 \\ &\leq \liminf_{n\to\infty} h^{\alpha}_{N_n}(u_n),\end{aligned}$$

which proves the lower bound for the case where $u \in BV(\mathbb{T}^2; \{0,1\})$.

Now consider $u \in L^1(\mathbb{T}^2) \setminus BV(\mathbb{T}^2; \{0,1\})$. Let $u_n \to u$ in $L^1(\mathbb{T}^2)$; then an argument by contradiction (similar to that at the end of the proof of Lemma 4.4) and the compactness proved below in Theorem 4.10 prove that $h_{\infty,0}(u) = \infty = \liminf_{n\to\infty} h^{\alpha}_{N_n}(u_n)$. □

Next we prove the upper bound.

Lemma 4.12 (Upper bound)**.** *Let $u \in L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^{\infty} \subset \mathbb{N}$ be such that $N_n \to \infty$ as $n \to \infty$. Then there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset L^1(\mathbb{T}^2)$ such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ as $n \to \infty$ and*

$$h_{\infty,0}(u) \geq \limsup_{n\to\infty} h^{\alpha}_{N_n}(u_n).$$

Proof. Without loss of generality we can assume that $u \in BV(\mathbb{T}^2; \{0,1\})$. Construct $u_n \in \mathcal{A}_{N_n}$ as follows. For $x \in S^{i,j}_{N_n}$ define, as in the proof of Lemma 4.6,

$$u_n(x) := \begin{cases} 1 & \text{if } (S^{i,j}_{N_n})^{\circ} \subset \operatorname{supp} u, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 = |(u_n)_{i+1,j} - (u_n)_{i,j}|$, and hence, identifying $\mathcal{V}_{N_n}$ with $\mathcal{A}_{N_n}$,

$$N_n^{-1} \sum_{i,j=1}^{N_n} \big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 = N_n^{-1} \sum_{i,j=1}^{N_n} |(u_n)_{i+1,j} - (u_n)_{i,j}| = \int_{\mathbb{T}^2} |(u_n)_x|.$$

Similarly

$$N_n^{-1} \sum_{i,j=1}^{N_n} \big((u_n)_{i,j+1} - (u_n)_{i,j}\big)^2 = \int_{\mathbb{T}^2} |(u_n)_y|.$$

Since every u_n takes values in $\{0,1\}$ we can repeat the argument from the proof of Lemma 4.6 in and following (4.3) to prove that

$$\limsup_{n\to\infty} \int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y| \leq \int_{\mathbb{T}^2} |u_x| + |u_y|.$$

Since for every $n \in \mathbb{N}$ and $(i,j) \in I^2_{N_n}$ we have $W((u_n)_{i,j}) = 0$, we get the desired result. $\square$

Proof of Theorem 4.9. Combining Lemmas 4.11 and 4.12 we get the Γ-convergence result in Theorem 4.9. $\square$

Next we prove compactness.

Proof of Theorem 4.10. By the first part of the proof of Lemma 4.11 we have, after possibly going to a subsequence, that for all $n \in \mathbb{N}$ and all $(i,j) \in I^2_{N_n}$, $(u_n)_{i,j} \in \hat{I}$; hence, $\|u_n\|_{L^1(\mathbb{T}^2)}$ is uniformly bounded. By the same proof, in particular (4.6) and the uniform bound on $h^\alpha_{N_n}(u_n)$, we have for all $n \in \mathbb{N}$ that $\int_{\mathbb{T}^2} |(u_n)_x| + |(u_n)_y|$ is uniformly bounded. We deduce as in the proof of Theorem 4.2 a uniform bound on the BV norms of u_n, from which it follows by the compactness theorem in BV ([37, Theorem 1.19] or [32, 5.2.3 Theorem 4]) that there exists a subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and a $u \in BV(\mathbb{T}^2)$ such that $u_{n'} \to u$ in $L^1(\mathbb{T}^2)$ as $n' \to \infty$. By the arguments in the proof of Lemma 4.11, each $u_{n'}$ takes values in $(-\delta_{n'}, \delta_{n'}) \cup (1-\delta_{n'}, 1+\delta_{n'})$, where $\delta_{n'} \to 0$ as $n' \to \infty$. After possibly going to another subsequence, $u_{n'}$ converges pointwise almost everywhere to u; hence, u takes values in $\{0,1\}$ almost everywhere. $\square$

4.3. **Constraints.** In this section we show that addition of a fidelity term to the functional or imposing a mass constraint are compatible with the three Γ-limits we discussed; *i.e.*, $\varepsilon \to 0$ for $h_{N,\varepsilon}$, $N \to \infty$ for $h_{N,0}$, and $N \to \infty$ for h^α_N.

Theorem 4.13 (Constraints). (1) $h_{N,\varepsilon} + \lambda N^{-2} |\cdot - f|_p^p \xrightarrow{\Gamma} h_{N,0} + \lambda N^{-2} |\cdot - f|_p^p$ *for* $\varepsilon \to 0$, *where* $p \in \mathbb{N}$, $\lambda > 0$, *and a given function* $f \in \mathcal{V}_N$ (*or possibly a given function* $f : U \to \mathbb{R}$, *where* U *is a strict subset of the vertex set* V *and the sum in* $|\cdot - f|_p^p$ *is restricted to vertices in* U). *A compactness result for* $h_{N,\varepsilon} + \lambda N^{-2} |\cdot - f|_p^p$ *as in Theorem* 3.2 *holds.*

If instead, for fixed $M > 0$, *the domain of definition of* $h_{N,\varepsilon}$ *is restricted to* $\mathcal{V}_N^M$ (*i.e.*, $\mathcal{V}^M$ *from Theorem* 3.6 *on the grid* G_N), *where* M *is such that* $N^2 M$ *is an integer between 0 and* N^2, *then the* Γ-*convergence and compactness results for* $\varepsilon \to 0$ *remain valid, with the domain of* $h_{N,0}$ *restricted to* $\mathcal{V}^M$ *as well.*

(2) *Let* $p \in \mathbb{N}$, $\lambda > 0$, $f \in C^1(\mathbb{T}^2)$, *and* $f_N \in \mathcal{A}_N$ *the sampling of* f *on the grid* G_N (f *and* f_N *and their norms can also be defined on subsets of* $\mathbb{T}^2$ *and* G_N *as in part* 4.13); *then* $h_{N,0} + \lambda N^{-2} |\cdot - f_N|_p^p \xrightarrow{\Gamma} h_{\infty,0} + \lambda \|\cdot - f\|^p_{L^p(\mathbb{T}^2)}$

as $N \to \infty$ in the $L^p(\mathbb{T}^2)$ topology. A compactness result as in Theorem 4.2 and Remark 4.7 holds for $h_{N,0} + \lambda N^{-2}|\cdot - f_N|_p^p$.

If instead the domain of $h_{N,0}$ is restricted to $\mathcal{V}_N^M$ for a fixed M such that $N^2 M$ is an integer between 0 and N^2, then the compactness and lower bound results from Lemma 4.4, Theorem 4.2, and Remark 4.7 remain valid, with the domain of $h_{\infty,0}$ restricted to

$$BV_M(\mathbb{T}^2; \{0,1\}) := \Big\{ u \in BV(\mathbb{T}^2; \{0,1\}) : \int_{\mathbb{T}^2} u = M \Big\}.$$

With the same restriction, the upper-bound result from Lemma 4.6 is still valid if we restrict it to sequences $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ such that $N_n^2 M$ is an integer for each $n \in \mathbb{N}$.

(3) *If λ, f, and f_N are as in part 4.13 and $\alpha > \beta$ as in Theorem 4.9, then $h_N^\alpha + \lambda N^{-2}|\cdot - f_N|_1 \xrightarrow{\Gamma} h_N^\alpha + \lambda \|\cdot - f\|_{L^1(\mathbb{T}^2)}$ as $N \to \infty$ in the $L^1(\mathbb{T}^2)$ topology. A compactness result as in Theorem 4.10 holds for $h_N^\alpha + \lambda N^{-2}|\cdot - f_N|_1$.*

If instead the domain of h_N^α is restricted to $\mathcal{V}_N^M$ for a fixed $M \in [0,1]$, then the Γ-convergence and compactness results from Theorems 4.9 and 4.10 remain valid, with the domain of $h_{\infty,0}$ restricted to $BV_M(\mathbb{T}^2; \{0,1\})$.

We give a sketch of the proofs.

(1) This follows directly from Theorem 3.6.

(2) For the fidelity term first we note that for $v, f_N \in \mathcal{A}_N$,

$$N^{-2}|v - f_N|_p^p = \int_{\mathbb{T}^2} |v - f_N|^p.$$

Hence, for $v \in \mathcal{A}_N$, $u \in L^p(\mathbb{T}^2)$, $f \in C(\mathbb{T}^2)$, and $f_N \in \mathcal{A}_N$ the discretization of f on G_N, we have

$$\begin{aligned}\Big| \int_{\mathbb{T}^2} |v - f_N|^p - \int_{\mathbb{T}^2} |u - f|^p \Big| &\le \int_{\mathbb{T}^2} \big| |v - f_N| - |u - f| \big| \\ &\le \|v - f_N - u + f\|_{L^p(\mathbb{T}^2)}^p \le \|v - u\|_{L^p(\mathbb{T}^2)}^p + \|f - f_N\|_{L^p(\mathbb{T}^2)}^p.\end{aligned}$$

Since (by a Taylor-series argument) $f_N \to f$ in $L^p(\mathbb{T}^2)$ as $N \to \infty$, the sequence of fidelity terms is continuously convergent, and thus by Lemma 2.5 the Γ-limit follows. Clearly the compactness isn't harmed (even helped) by adding an extra term to the functional.

For the mass constraint, since $u \in \mathcal{A}_N^b$ the conditions on M are necessary as explained in Section 2.5. $L^p(\mathbb{T}^2)$ convergence preserves average mass, and hence the constraint is compatible with (LB) and compactness. For (UB) the recovery sequence $\{u_n\}_{n=1}^\infty$ used in the proof of Lemma 4.6 has support contained in the support of u, and hence if $\int_{\mathbb{T}^2} u = M$, then $\int_{\mathbb{T}^2} u_n \le M$. We can construct a similar recovery sequence $\{\hat{u}_n\}_{n=1}^\infty$, where $\hat{u}_n$ has support on all grid squares which intersect $\operatorname{supp} u$. This sequence satisfies all the required properties of a recovery sequence and has $\int_{\mathbb{T}^2} u_n \geq M$. Hence, under the assumption on $\{N_n\}_{n=1}^\infty$, which assures that the mass condition can be satisfied for each N_n, there exists a recovery sequence $\{\overline{u}_n\}_{n=1}^\infty$, where $\overline{u}_n$ takes the value 1 on $\operatorname{supp} u_n$ as well as on a select chosen number of squares which lie in $\operatorname{supp} \hat{u}_n \setminus \operatorname{supp} u_n$. For these combinations of $\{N_n\}_{n=1}^\infty$ and M, (UB) is compatible with the mass constraint as well.

(3) For the fidelity term we can use the same arguments as above.

For the mass constraint we note that now $u_n \in \mathcal{A}_{N_n}$ and the limit function $u \in BV(\mathbb{T}^2; \{0,1\})$. This means each choice $M \in [0,1]$ is allowed in the mass constraint. As above, because of the L^1 convergence this mass constraint is compatible with (LB) and compactness. For (UB) we note that the proof of Lemma 4.12 followed Lemma 4.6, so our argument here is very similar to that for $h_{N,0}$ above, with the added bonus that we do not need to restrict ourselves to specific combinations of $\{N_n\}_{n=1}^\infty$ and M. Let the recovery sequences $\{u_n\}_{n=1}^\infty$ and $\{\hat{u}_n\}_{n=1}^\infty$ be as above. Now construct another recovery sequence $\{\overline{u}_n\}_{n=1}^\infty$ by setting $\overline{u}_n = u_n$ on $\operatorname{supp} u_n$ and $\overline{u}_n = c_n \hat{u}_n$ on $(\operatorname{supp} u_n)^c$, where $c_n \in [0,1]$ is chosen such that for each n the average mass constraint is satisfied.

5. Γ-LIMITS FOR THE DISCRETIZED GINZBURG–LANDAU FUNCTIONAL $k_{N,\varepsilon}$

In this section we will study the convergence properties of $k_{N,\varepsilon}$ from (1.5). We first take the limit $N \to \infty$ and then $\varepsilon \to 0$. The resulting Γ-limits are given in Section 5.1. The simultaneous limit, obtained by substituting $\varepsilon = N^{-\alpha}$ for well-chosen $\alpha > 0$ and then considering the limit $N \to \infty$ for k_N^α in (2.3), is studied in Section 5.2.

There has been a series of (recent) papers dealing with convergence of discrete energies to integral energies among which are [17, 5, 18, 6, 7, 8], all expanding on the ideas in [15, Chapter 4]. For many of the proofs we will use ideas from [5], in which the authors prove a Γ-limit of integral form exists for a general class of grid-based functionals. Here we study a specific functional and hence can prove more explicit results.

Remark 5.1. In Sections 5.1 and 5.2 we prove the results for $\mathbb{T}^2$, but they can easily be generalized to $\mathbb{T}^d$ for any $d \in \mathbb{N}$, if we let the scaling factor in the first term of $k_{N,\varepsilon}$ in (1.5) be εN^{2-d} instead of εN^0 and the factor in the second term $\varepsilon^{-1} N^{-d}$ instead of $\varepsilon^{-1} N^{-2}$ and we change k_N^α in (5.2) accordingly:

$$k_{N,\varepsilon}(u) := \varepsilon N^{2-d} \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \varepsilon^{-1} N^{-d} \sum_{i,j=1}^{N} W(u_{i,j}), \tag{5.1}$$

$$k_N^\alpha(u) := N^{2-d-\alpha} \sum_{i,j=1}^{N} (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + N^{\alpha-d} \sum_{i,j=1}^{N} W(u_{i,j}). \tag{5.2}$$

The admissible range of α in Theorems 5.6 and 5.7 is dependent on the dimension d in a way which will be made precise in Remark 5.10. For the extra fidelity term in Theorem 5.12 the scaling factor then needs to be N^{-d} instead of N^{-2}.

5.1. **Sequential Γ-convergence and compactness: first $N \to \infty$, then $\varepsilon \to 0$.** We will prove $k_{N,\varepsilon} \xrightarrow{\Gamma} k_{\infty,\varepsilon}$ as $N \to \infty$, where $k_{\infty,\varepsilon}$ is defined for $u \in L^1(\mathbb{T}^2)$ as

$$k_{\infty,\varepsilon}(u) := \begin{cases} \varepsilon \int_{\mathbb{T}^2} |\nabla u|^2 + \varepsilon^{-1} \int_{\mathbb{T}^2} W(u) & \text{if } u \in W^{1,2}(\mathbb{T}^2), \\ +\infty & \text{otherwise.} \end{cases}$$

We see that $k_{\infty,\varepsilon}$ is the Ginzburg–Landau functional from (1.1). As explained in Section 1.1, it is known that this functional Γ-converges in either the $L^1(\mathbb{T}^2)$ or $L^2(\mathbb{T}^2)$ topology[9] as $\varepsilon \to 0$ to the total variation (1.2). To be precise, its Γ-limit is

$$k_{\infty,0}(u) := \begin{cases} \sigma(W) \displaystyle\int_{\mathbb{T}^2} |\nabla u| & \text{if } u \in BV(\mathbb{T}^2; \{0,1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\sigma(W) := 2\int_0^1 \sqrt{W(s)}\, ds > 0$ is a constant depending on the specific form of W, in particular on the transition between its wells; see [47]. The sequence of functionals is equi-coercive as well.

For fixed $u \in C^3(\mathbb{T}^2)$ we have pointwise convergence (in $C^3(\mathbb{T}^2)$) $k_{N,\varepsilon}(u) \to k_{\infty,\varepsilon}(u)$ as $N \to \infty$ (more details below), but the dependence of the discretization errors on derivatives of u prevents us from concluding uniform convergence. Γ-convergence offers a useful middle ground between pointwise

[9]Results are usually stated in the L^1 topology, but for example [53, 15, 25] note that the results can be stated in the L^2 topology as well.

and uniform convergence and can thus be seen as an extension of classical numerical analysis results. The pointwise convergence follows from combining the boundedness of $\mathbb{T}^2$ with the trapezoidal rule (for fixed i)

$$\int_{\mathbb{T}} f(x,y)\,dy = N^{-1}\sum_{j=1}^{N} f(x,j/N) + \frac{1}{12}N^{-2}\frac{\partial^2 f}{\partial y^2}(x,\xi), \quad \text{for } f\in C^2(\mathbb{T}^2),$$

for some f-dependent $\xi\in(0,1)$, and the finite difference approximation of the derivative

$$\begin{aligned}\frac{\partial u}{\partial x}(i/N,j/N) = N\big[u((i+1)/N,j/N) - u(i/N,j/N)\big] \\ - \frac{1}{2}N^{-1}\frac{\partial^2 u}{\partial x^2}((i+r_i)/N,j/N)\end{aligned} \tag{5.3}$$

for some $r_i\in[0,1]$. We see the dependence of the errors on derivatives of the functions f and u.

We prove the following Γ-convergence and compactness results.

Theorem 5.2 (Γ-convergence)**.** *$k_{N,\varepsilon} \xrightarrow{\Gamma} k_{\infty,\varepsilon}$ as $N\to\infty$ in the $L^1(\mathbb{T}^2)$ or $L^2(\mathbb{T}^2)$ topology.*

Theorem 5.3 (Compactness)**.** *Assume W satisfies (W_2). Let $\{N_n\}_{n=1}^\infty\subset\mathbb{N}$ satisfy $N_n\to\infty$ as $n\to\infty$, and let $\{u_n\}_{n=1}^\infty\subset L^1(\mathbb{T}^2)$ be a sequence for which there is a constant $C>0$ such that for all $n\in\mathbb{N}$, $k_{N_n,\varepsilon}(u_n)\le C$. Then there exists a subsequence $\{u_{n'}\}_{n'=1}^\infty\subset\{u_n\}_{n=1}^\infty$ and a $u\in W^{1,2}(\mathbb{T}^2)$ such that $u_{n'}\to u$ in $L^2(\mathbb{T}^2)$ as $n'\to\infty$.*

The proof of Γ-convergence adapts the ideas that are developed in an abstract general framework in [5] to our situation. In Section 4.1 we constructed a bijection between $\mathcal{V}_N$ and $\mathcal{A}_N$ from (4.1). In what follows we will identify $u\in\mathcal{V}_N$ with its counterpart $\tilde u\in\mathcal{A}_N$ and drop the tilde if no confusion arises.

For $u\in\mathcal{A}_N\subset L^1(\mathbb{T}^2)$ we define pointwise evaluation by identifying each u with its representative, which is piecewise constant on the squares $S_N^{i,j}$ and for which pointwise evaluation is well-defined. For $z\in\mathbb{T}^2$ we define the difference quotients as

$$D_N^k u(z) := N\big[u(z+e_k/N) - u(z)\big], \quad k\in\{1,2\}, \tag{5.4}$$

where e_k denotes the k^{th} standard basis vector of $\mathbb{R}^2$.

With the identification between $\mathcal{V}_N$ and $\mathcal{A}_N$ we extend the functional to all $u \in L^1(\mathbb{T}^2)$ as follows:

$$k_{N,\varepsilon}(u) = \begin{cases} \varepsilon \int_{\mathbb{T}^2} \left[(D_N^1 u)^2 + (D_N^2 u)^2\right] + \varepsilon^{-1} \int_{\mathbb{T}^2} W(u) & \text{if } u \in \mathcal{A}_N, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.5}$$

In the proof of the lim inf inequality we will use the slicing method (see [15, Chapter 15] and [16]), which uses the following notation. Remembering $\mathbb{T}^2 \cong [0,1)^2$ we define $\mathbb{T}_1^2 := \{(0,y) \in \mathbb{T}^2 : \exists t \in [0,1) : (t,y) \in \mathbb{T}^2\}$ and $\mathbb{T}_2^2 := \{(x,0) \in \mathbb{T}^2 : \exists t \in [0,1) : (x,t) \in \mathbb{T}^2\}$, and for $(0,y) \in \mathbb{T}_1^2$ and $(x,0) \in \mathbb{T}_2^2$ we define the sets $\mathbb{T}_{1,y}^2 := \{t \in [0,1) : (t,y) \in \mathbb{T}^2\}$ and $\mathbb{T}_{2,x}^2 := \{t \in [0,1) : (x,t) \in \mathbb{T}^2\}$ and the functions $u_{1,y}(t) := u(t,y)$ and $u_{2,x}(t) := u(x,t)$ on $\mathbb{T}_{1,y}^2$ and $\mathbb{T}_{2,x}^2$ respectively.

In what follows $\varepsilon > 0$ is fixed.

Lemma 5.4 (Lower bound)**.** *Let $u \in L^1(\mathbb{T}^2)$, and let $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ and $N_n \to \infty$ as $n \to \infty$. Then*

$$k_{\infty,\varepsilon}(u) \le \liminf_{n\to\infty} k_{N_n,\varepsilon}(u_n).$$

Proof. This proof is an application of arguments in [5, Proposition 3.4].

First consider the case where $u \in W^{1,2}(\mathbb{T}^2)$; then we can assume without loss of generality that $\{k_{N_n,\varepsilon}(u_n)\}_{n=1}^\infty$ is uniformly bounded, and thus $u_n \in \mathcal{A}_{N_n}$. For $(x,y) \in S_{N_n}^{i,j}$ we define

$$v_n^1(x,y) := u_n(i/N_n, j/N_n) + D_{N_n}^1 u_n(i/N_n, j/N_n) \cdot (x - i/N_n),$$
$$v_n^2(x,y) := u_n(i/N_n, j/N_n) + D_{N_n}^2 u_n(i/N_n, j/N_n) \cdot (y - j/N_n).$$

In what follows $k \in \{1,2\}$. Note $v_n^k \in BV(\mathbb{T}^2)$. We denote the densities of the absolutely continuous (with respect to Lebesgue measure) part of the measures $\nabla_x v_n^1$ and $\nabla_y v_n^2$ by $\frac{\partial v_n^1}{\partial x}$ and $\frac{\partial v_n^2}{\partial y}$ respectively. Then in the interior of $S_N^{i,j}$, for $(x,y) \in (S_N^{i,j})^\circ$, we have

$$\frac{\partial v_n^1}{\partial x}(x,y) = D_{N_n}^1 u_n(i/N_n, j/N_n) \quad \text{and} \quad \frac{\partial v_n^2}{\partial y}(x,y) = D_{N_n}^2 u_n(i/N_n, j/N_n).$$

We deduce $v_n^k \to u$ in $L^1(\mathbb{T}^2)$ as $n \to \infty$ from

$$\|v_n^k - u\|_{L^1(\mathbb{T}^2)} \le \|v_n^k - u_n\|_{L^1(\mathbb{T}^2)} + \|u_n - u\|_{L^1(\mathbb{T}^2)}.$$

The latter term converges to zero by assumption. The former we bound by $\|v_n^k - u_n\|_{L^2(\mathbb{T}^2)}$ using Hölder's inequality. We then note that from the uniform bound on $\{k_{N_n,\varepsilon}(u_n)\}_{n=1}^\infty$ we have

$$\sum_{i,j=1}^{N_n} \Big[D^k_{N_n} u_n(i/N_n, j/N_n)\Big]^2 \le CN_n^2$$

for some $C > 0$, and hence (if $k = 1$; similarly for $k = 2$)

$$\int_{\mathbb{T}^2} (v_n^1 - u_n)^2 = \sum_{i,j=1}^{N_n} \int_{S^{i,j}_{N_n}} \left[D^1_{N_n} u_n(i/N_n, j/N_n)(x - i/N_n)\right]^2 \, dy\, dx$$
$$= N_n^{-1} \sum_{i,j=1}^{N_n} \left[D^1_{N_n} u_n(i/N_n, j/N_n)\right]^2 \int_{i/N_n}^{(i+1)/N_n} (x - i/N_n)^2 \, dx \le \frac{C}{3} N_n^{-2}.$$

For $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$, the slice $(v_n^1)_{1,y} \in W^{1,2}(\mathbb{T}^2_{1,y})$. By Fubini's theorem and Fatou's lemma

$$\liminf_{n\to\infty} \int_{\mathbb{T}^2} \left(\frac{\partial v_n^1}{\partial x}\right)^2 = \liminf_{n\to\infty} \int_{\mathbb{T}^2_1} \int_{\mathbb{T}^2_{1,y}} |(v_n^1)'_{1,y}|^2 \, dt\, dy$$
$$\ge \int_{\mathbb{T}^2_1} \liminf_{n\to\infty} \int_{\mathbb{T}^2_{1,y}} |(v_n^1)'_{1,y}|^2 \, dt\, dy.$$

Because

$$\liminf_{n\to\infty} \int_{\mathbb{T}^2} \left(\frac{\partial v_n^1}{\partial x}\right)^2 \le \liminf_{n\to\infty} k_{N_n,\varepsilon}(u_{N_n}) < \infty, \tag{5.6}$$

we have that, after possibly going to a subsequence, for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ the sequence $\{\|(v_n^1)'_{1,y}\|_{L^2(\mathbb{T}^2_{1,y})}\}_{n=1}^\infty$ is bounded and hence weakly convergent in $L^2(\mathbb{T}^2_{1,y})$. Since for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ we have $(v_n^1)_{1,y} \to u_{1,y}$ in $L^1(\mathbb{T}^2_{1,y})$, we identify the limit as $(v_n^1)'_{1,y} \rightharpoonup u'_{1,y}$ in $L^2(\mathbb{T}^2_{1,y})$, for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ (see Lemma B.1 in Appendix B for details). Hence, by the weak lower semicontinuity of the L^2 norm, Fatou's lemma, and the completely analogous results for $u_{2,x}$ and $\frac{\partial v_n^2}{\partial y}$, we get

$$\liminf_{n\to\infty} \int_{\mathbb{T}^2} \Big[\Big(\frac{\partial v_n^1}{\partial x}\Big)^2 + \Big(\frac{\partial v_n^2}{\partial y}\Big)^2\Big] \ge \int_{\mathbb{T}^2_1} \int_{\mathbb{T}^2_{1,y}} |u'_{1,y}|^2 \, dt\, dy + \int_{\mathbb{T}^2_2} \int_{\mathbb{T}^2_{2,x}} |u'_{2,x}|^2 \, dt\, dy.$$

Putting the slices back together using [32, 4.9.2 Theorem 2]), we deduce that

$$\liminf_{n\to\infty} \int_{\mathbb{T}^2} \Big[\Big(\frac{\partial v_n^1}{\partial x}\Big)^2 + \Big(\frac{\partial v_n^2}{\partial y}\Big)^2\Big] \ge \int_{\mathbb{T}^2} |\nabla u|^2. \tag{5.7}$$

For the other term in the functional we use Fatou's lemma, the continuity of W, and the almost-everywhere pointwise convergence of u_n to u to find

$$\liminf_{n\to\infty} \int_{\mathbb{T}^2} W(u_n) \geq \int_{\mathbb{T}^2} \liminf_{n\to\infty} W(u_n) = \int_{\mathbb{T}^2} W(u).$$

This proves the result for $u \in W^{1,2}(\mathbb{T}^2)$.

Now consider the case where $u \in L^1(\mathbb{T}^2) \setminus W^{1,2}(\mathbb{T}^2)$. Assume that

$$\liminf_{n\to\infty} k_{N_n,\varepsilon}(u_n) < \infty;$$

then after possibly going to a subsequence for each $n \in \mathbb{N}$, $u_n \in \mathcal{A}_{N_n}$ and the sequences $\left\{\int_{\mathbb{T}^2} (D^k_{N_n} u_n)^2\right\}_{n=1}^\infty$ are bounded. We can follow the same slicing method as applied above up to equation (5.6). Again we find that for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ the sequence $\{\|(v^1_n)'_{1,y}\|_{L^2(\mathbb{T}^2_{1,y})}\}_{n=1}^\infty$ is bounded, and combined with $(v^1_n)_{1,y} \to u_{1,y}$ in $L^1(\mathbb{T}^2_{1,y})$ for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ we deduce that $u_{1,y} \in W^{1,2}(\mathbb{T}^2_{1,y})$ for $\mathcal{H}^1$-a.e. $y \in \mathbb{T}^2_1$ (see Lemma B.1 in Appendix B for details). We then continue as above to arrive at (5.7) and conclude that $u \in W^{1,2}(\mathbb{T}^2)$, which contradicts the assumption that $u \notin W^{1,2}(\mathbb{T}^2)$. Hence $\liminf_{n\to\infty} k_{N_n,\varepsilon}(u_n) = \infty$, which concludes the proof. □

Next we prove (UB′) (see Section 2.4).

Lemma 5.5 (Upper bound)**.** *Let $u \in L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $N_n \to \infty$ as $n \to \infty$, and let k''_ε be the Γ-upper limit of $k_{N_n,\varepsilon}$ as $n \to \infty$ with respect to the $L^2(\mathbb{T}^2)$ topology; then $k''_\varepsilon(u) \leq k_{\infty,\varepsilon}(u)$.*

Proof. This proof is an adaptation of the ideas in [5, Proposition 3.5].

The case where $u \in L^1(\mathbb{T}^2) \setminus W^{1,2}(\mathbb{T}^2)$ is trivial. For the case where $u \in W^{1,2}(\mathbb{T}^2)$ we first assume that $u \in C^\infty(\mathbb{T}^2)$. Define a sequence $\{u_n\}_{n=1}^\infty$ such that for each $n \in \mathbb{N}$, $u_n \in \mathcal{A}_{N_n}$ in the following way. If $(x, y) \in S^{i,j}_{N_n}$ then $u_n(x) := u(i/N_n, j/N_n)$. Then (by a Taylor series argument) $u_n \to u$ in $L^2(\mathbb{T}^2)$ as $n \to \infty$. Let $\nu^{i,j}_{N_n} := (i/N_n, j/N_n) \in G_{N_n}$; then

$$D^1_{N_n} u_n(\nu^{i,j}_{N_n}) = \int_0^1 \frac{\partial u}{\partial x}(\nu^{i,j}_{N_n} + se_1/N_n)\, ds.$$

Jensen's inequality then gives

$$(D^1_{N_n} u_n(\nu^{i,j}_{N_n}))^2 = \Big(\int_0^1 \frac{\partial u}{\partial x}(\nu^{i,j}_{N_n} + se_1/N_n)\, ds\Big)^2 \leq \int_0^1 \Big(\frac{\partial u}{\partial x}(\nu^{i,j}_{N_n} + se_1/N_n)\Big)^2 ds.$$

Because u is smooth the Taylor series with remainder gives for $z \in \mathbb{T}^2$

$$\frac{\partial u}{\partial x}(\nu^{i,j}_{N_n} + se_1/N_n) = \frac{\partial u}{\partial x}(z + se_1/N_n) + \nabla\frac{\partial u}{\partial x}((1-c)x + c\nu^{i,j}_{N_n}) \cdot (\nu^{i,j}_{N_n} - z),$$

for some $c \in [0,1]$. Using the fact that u and all its derivatives are bounded, we find for some constants $C_u > 0$ and $\tilde{C}_u > 0$ depending only on u that

$$N_n^{-2} \int_0^1 \left(\frac{\partial u}{\partial x}(\nu_{N_n}^{i,j} + se_1/N_n)\right)^2 ds = \int_{S_{N_n}^{i,j}} \int_0^1 \left(\frac{\partial u}{\partial x}(\nu_{N_n}^{i,j} + se_1/N_n)\right)^2 ds\, dx$$

$$\leq \int_{S_{N_n}^{i,j}} \int_0^1 \left[\left(\frac{\partial u}{\partial x}(z + se_1/N_n)\right)^2 + C_u|\nu_{N_n}^{i,j} - z| + \tilde{C}_u|\nu_{N_n}^{i,j} - z|^2\right] ds\, dz$$

$$\leq \int_0^1 \int_{S_{N_n}^{i,j}+se_1/N_n} \left(\frac{\partial u}{\partial x}\right)^2(z)\, dz\, ds + C_u\,(N_n)^{-3} + \tilde{C}_u\,(N_n)^{-4}.$$

Analogous estimates hold for $D_{N_n}^2 u_n(\nu_{N_n}^{i,j})$.

Note that the sets $S_{N_n}^{i,j} + se_k/N_n$, $k \in \{1,2\}$, are just the squares $S_{N_n}^{i,j}$ shifted a distance s/N_n over the coordinate axes. Because we are working on the torus we get for some $\overline{C}_u > 0$ depending only on u

$$N_n^{-2} \sum_{i,j \in I_{N_n}} \left[\left(D_{N_n}^1 u_n\right)^2 (\nu_{N_n}^{i,j}) + \left(D_{N_n}^2 u_n\right)^2 (\nu_{N_n}^{i,j})\right]$$

$$\leq \int_0^1 \sum_{i,j \in I_{N_n}} \left[\int_{S_{N_n}^{i,j}+se_1/N_n} \left(\frac{\partial u}{\partial x}\right)^2(z)\, dz + \int_{S_{N_n}^{i,j}+se_2/N_n} \left(\frac{\partial u}{\partial y}\right)^2(z)\, dz\right] ds$$

$$+ \overline{C}_u\big((N_n)^{-1} + (N_n)^{-2}\big) = \int_{\mathbb{T}^2} |\nabla u|^2 + \overline{C}_u\big((N_n)^{-1} + (N_n)^{-2}\big).$$

We deduce that

$$\limsup_{n\to\infty} \int_{\mathbb{T}^2} \left(D_{N_n}^1 u_n\right)^2 + \left(D_{N_n}^2 u_n\right)^2 \leq \int_{\mathbb{T}^2} |\nabla u|^2.$$

Because $u \in C^\infty(\mathbb{T}^2)$, u is bounded on $\mathbb{T}^2$ and hence $\{|u_n|\}_{n=1}^\infty$ is uniformly bounded on $\mathbb{T}^2$. Therefore there exists $\hat{C} > 0$ such that for all $n \in \mathbb{N}$, $W(u_n) \leq \hat{C}$. Because the constant $\hat{C}$ is integrable on $\mathbb{T}^2$ we can use the dominated convergence theorem (or the reverse Fatou's lemma) and the continuity of W to deduce

$$\limsup_{n\to\infty} \int_{\mathbb{T}^2} W(u_n) \leq \int_{\mathbb{T}^2} \limsup_{n\to\infty} W(u_n) = \int_{\mathbb{T}^2} W(u)$$

(or use $\int_{\mathbb{T}^2} \big(W(u_n) - W(u)\big)\, dx = \int_{\mathbb{T}^2} \int_{u_n}^u W'(s)\, ds\, dx \leq C \int_{\mathbb{T}^2} |u_n - u|\, dx$).

Combining the two inequalities above leads to

$$\limsup_{n\to\infty} k_{N_n,\varepsilon}(u_n)$$

$$\leq \varepsilon \limsup_{n\to\infty} \int_{\mathbb{T}^2} \left(D^1_{N_n} u_n\right)^2 + \left(D^2_{N_n} u_n\right)^2 + \varepsilon^{-1} \limsup_{n\to\infty} \int_{\mathbb{T}^2} W(u_n)$$

$$\leq \varepsilon \int_{\mathbb{T}^2} |\nabla u|^2 + \varepsilon^{-1} \int_{\mathbb{T}^2} W(u) = k_{\infty,\varepsilon}(u). \tag{5.8}$$

In the terminology of Γ-upper limit of Section 2.4 we have proven that $k''_{\infty,\varepsilon}(u) \leq k_{\infty,\varepsilon}(u)$ for $u \in C^\infty(\mathbb{T}^2)$. Since $C^\infty(\mathbb{T}^2)$ is dense in $W^{1,2}(\mathbb{T}^2)$ (using $W^{1,2}(\mathbb{T}^2)$ convergence) we use the lower semicontinuity of the Γ-upper limit to conclude (UB′) for all $u \in W^{1,2}(\mathbb{T}^2)$ as follows. Let $\{u_n\}_{n=1}^\infty \subset C^\infty(\mathbb{T}^2)$ be a sequence such that $u_n \to u$ in $W^{1,2}(\mathbb{T}^2)$ as $n \to \infty$; then it also converges in $L^2(\mathbb{T}^2)$; hence,

$$k''_\varepsilon(u) \leq \liminf_{n\to\infty} k''_\varepsilon(u_n) \leq \liminf_{n\to\infty} k_{\infty,\varepsilon}(u_n).$$

Up to taking a subsequence, $u_n \to u$ pointwise almost everywhere; hence, $W(u_n) \to W(u)$ pointwise almost everywhere. Thus, by possibly redefining u on a set of measure zero for n large enough, we have that $W(u_n) \leq W(u)+\tilde{C}$ for some $\tilde{C} > 0$. We can assume $k_{\infty,\varepsilon}(u) < \infty$; hence, $W(u)$ is integrable on $\mathbb{T}^2$. Now we can again use the dominated convergence theorem or the reverse Fatou's lemma and the continuity of W to find

$$\limsup_{n\to\infty} \int_{T^2} W(u_n) \leq \int_{\mathbb{T}^2} \limsup_{n\to\infty} W(u_n) = \int_{\mathbb{T}^2} W(u).$$

Since $u_n \to u$ in $W^{1,2}(\mathbb{T}^2)$ we have

$$\limsup_{n\to\infty} \int_{\mathbb{T}^2} |\nabla u_n|^2 = \int_{\mathbb{T}^2} |\nabla u|^2;$$

hence,

$$k''_\varepsilon(u) \leq \liminf_{n\to\infty} k_{\infty,\varepsilon}(u_n) \leq \limsup_{n\to\infty} k_{\infty,\varepsilon}(u_n) \leq k_{\infty,\varepsilon}(u). \qquad \square$$

Proof of Theorem 5.2. Combining Lemmas 5.4 and 5.5 proves the Γ-convergence result. Note in particular that we have proven the lower bound for sequences converging in $L^1(\mathbb{T}^2)$, and the recovery sequence for the upper bound converges in $L^2(\mathbb{T}^2)$; hence, we can conclude Γ-convergence in both topologies. □

Using a technique from [31, 5.8.2 Theorem 3] we also get compactness for $k_{N,\varepsilon}$.

Proof of Theorem 5.3. In what follows, $k \in \{1, 2\}$. By (5.5) we have for all $n \in \mathbb{N}$, $u_n \in \mathcal{A}_{N_n}$ and

$$\varepsilon \int_{\mathbb{T}^2} \left[(D^1_{N_n} u_n)^2 + (D^2_{N_n} u_n)^2\right] + \varepsilon^{-1} \int_{\mathbb{T}^2} W(u_n) \leq C.$$

By assumption (W_2) on W we find that $\{\|u_n\|_{L^2(\mathbb{T}^2)}\}_{n=1}^\infty$ is uniformly bounded; hence, there is a subsequence of $\{u_n\}_{n=1}^\infty$ (again labelled by n) and a $u \in L^2(\mathbb{T}^2)$ such that $u_n \rightharpoonup u$ in $L^2(\mathbb{T}^2)$ as $n \to \infty$. Moreover, we see that $\{\|D_{N_n}^k u_n\|_{L^2(\mathbb{T}^2)}\}_{n=1}^\infty$ is uniformly bounded, and hence there is a further subsequence $\{u_{n'}\}_{n'=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and a $w \in L^2(\mathbb{T}^2;\mathbb{R}^2)$ such that $D_{N_{n'}}^k u_n \rightharpoonup w_k$ in $L^2(\mathbb{T}^2)$ as $n' \to \infty$. Let $\phi \in C_c^\infty(\mathbb{T}^2)$; then

$$\begin{aligned}\int_{\mathbb{T}^2} u_{n'} D_{N_{n'}}^k \phi &= N_{n'} \int_{\mathbb{T}^2} u_{n'}(x)[\phi(x+e_k/N_{n'}) - \phi(x)]\,dx \\ &= N_{n'} \int_{\mathbb{T}^2} [u_{n'}(x-e_k/N_{n'}) - u_{n'}(x)]\phi(x)dx = -\int_{\mathbb{T}^2} \phi D_{N_{n'}}^k u_{n'}.\end{aligned}$$

By (5.3) we have $\int_{\mathbb{T}^2} \left(D_{N_{n'}}^k \phi - \nabla\phi \cdot e_k\right)^2 = K N_{n'}^{-2}$, for some $K > 0$; hence, $D_{N_{n'}}^k \phi \to \nabla\phi \cdot e_k$ in $L^2(\mathbb{T}^2)$ as $n' \to \infty$. Combining this strong convergence for the difference quotient of ϕ with the weak convergence for $u_{n'}$ and its difference quotient, we deduce that

$$\int_{\mathbb{T}^2} u\nabla\phi \cdot e_k = \lim_{n'\to\infty} \int_{\mathbb{T}^2} u_{n'} D_{N_{n'}}^k \phi = -\lim_{n'\to\infty} \int_{\mathbb{T}^2} \phi\, D_{N_{n'}}^k u_{n'} = -\int_{\mathbb{T}^2} \phi\, w_k.$$

Hence $|\nabla u| = |w| \in L^2(\mathbb{T}^2)$. We conclude that $u \in W^{1,2}(\mathbb{T}^2)$.

Finally, the strong convergence $u_{n'} \to u$ in $L^2(\mathbb{T}^2)$ follows from the bound on $D_{N_{n'}}^k u_n$ by a discrete version of the Rellich–Kondrachov compactness theorem (see Lemma B.2 in Appendix B for details). $\square$

5.2. **Simultaneous scaling Γ-limit for k_N^α.** In Section 5.1 we studied the Γ-limits of $k_{N,\varepsilon}$ by first taking $N \to \infty$ and then $\varepsilon \to 0$. We will now show we can take both limits at once if we scale ε correctly in terms of N. This is particularly relevant for numerical applications. We set $\varepsilon = N^{-\alpha}$ for some $\alpha \in (0, \frac{2}{q+3})$, where q is the degree of polynomial growth of W' in condition (W_4), and take the limit $N \to \infty$ of k_N^α in (5.2).

Note that in contrast to the case for h_N^α the order of the limits $\varepsilon \to 0$ and $N \to \infty$ are reversed, and thus we have an upper bound on α instead of a lower bound.

We prove a compactness and Γ-convergence result.

Theorem 5.6 (Compactness). *Assume W satisfies (W_3) and (W_4) for given $p \geq 2$ and $q > 0$, and $\alpha \in (0, \frac{2}{q+3})$. Let $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfy $N_n \to \infty$ as $n \to \infty$, and let $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ be a sequence for which there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $k_{N_n}^\alpha(u_n) \leq C$. Then there exists a subsequence*

$\{u_{n'}\}_{n'=1}^{\infty} \subset \{u_n\}_{n=1}^{\infty}$ *and a* $u \in BV(\mathbb{T}^2, \{0,1\})$ *such that* $u_{n'} \to u$ *in* $L^2(\mathbb{T}^2)$ *as* $n' \to \infty$.

Theorem 5.7 (Γ-convergence)**.** *Let* W *satisfy* (W_3) *and* (W_4) *for given* $p, q > 0$, *and assume* $\alpha \in (0, \frac{2}{q+3})$. *Then* $k_N^\alpha \xrightarrow{\Gamma} k_{\infty,0}$ *as* $N \to \infty$ *in either the* $L^1(\mathbb{T}^2)$ *or* $L^2(\mathbb{T}^2)$ *topology.*

Using the difference quotient notation from (5.4) we can mimic (5.5) and write

$$k_{N_n}^\alpha(u) = \begin{cases} N^{-\alpha} \int_{\mathbb{T}^2} \left[(D_N^1 u)^2 + (D_N^2 u)^2\right] + N^\alpha \int_{\mathbb{T}^2} W(u) & \text{if } u \in \mathcal{A}_N, \\ +\infty & \text{otherwise.} \end{cases}$$

The proofs of this section make repeated use of the Modica–Mortola results [49, 47, 48, 56], which show (L^1 and L^2) compactness and convergence[10] for $F_{N^{-\alpha}}^{GL} \xrightarrow{\Gamma} F_0^{GL}$ as $N \to \infty$. Note that condition (W_3) with $p \geq 2$ on the double-well potential W is needed for the compactness result to hold (see *e.g.* [56, Proposition 3]).

Proof of Theorem 5.6. Let $\{u_n\}_{n=1}^{\infty} \subset L^1(\mathbb{T}^2)$ be a sequence such that $k_{N_n}^\alpha(u_n) \leq C$. Below we will prove the claim that there is a sequence $\{v_n\}_{n=1}^{\infty} \subset W^{1,2}(\mathbb{T}^2)$ such that $\|v_n - u_n\|_{L^2(\mathbb{T}^2)} \to 0$ as $n \to \infty$ and

$$k_{N_n}^\alpha(u_n) \geq F_{N_n^{-\alpha}}^{GL}(v_n) + R_n, \quad \lim_{n\to\infty} R_n = 0. \tag{5.9}$$

Given the veracity of this claim, it follows by the Modica–Mortola compactness result for $F_{N_n^{-\alpha}}^{GL}$ [49, 47, 48, 56] that there is a subsequence of $\{v_n\}_{n=1}^{\infty}$ (again labeled by n) such that $v_n \to u$ in $L^2(\mathbb{T}^2)$ for a $u \in BV(\mathbb{T}^2, \{0,1\})$ (here we need condition (W_3) on W with $p \geq 2$). Using the triangle inequality and the fact that $\|v_n - u_n\|_{L^2(\mathbb{T}^2)} \to 0$ as $n \to \infty$, we then conclude that there is a subsequence of $\{u_n\}_{n=1}^{\infty}$ converging in $L^2(\mathbb{T}^2)$ to u.

To prove the claim, first we show that $\|u_n\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(N_n^{\alpha/2})$. Assume not, and let $\gamma > \frac{\alpha}{2}$; then there is a subsequence (labeled again by n) such that

[10]As remarked in an earlier footnote, results are usually stated in the L^1 topology, but for example [53, 15, 25] note that the Γ-convergence can be stated in the L^2 topology as well. Compactness in L^2 follows from compactness in L^1 combined with the binary nature of the limit function. Finally, note that the original results on bounded domains are easily adapted for the torus. Compactness is not hindered by the periodicity because no regularity beyond $BV(\mathbb{T}^2; \{0,1\})$ is needed for the limit. The lower bound generalizes immediately by restriction to sequences of periodic functions. The important properties of the recovery sequence for the upper bound are local properties near the boundary of $\operatorname{supp} u$ and so are also satisfied on a periodic domain.

for each n there is a square $S_{N_n}^{i,j}$ on which $|(u_n)_{i,j}| \geq N_n^\gamma$. For definiteness assume $(u_n)_{i,j} = N_n^\gamma$. By the uniform bound on $k_{N_n}^\alpha(u_n)$, we have

$$N_n^{-\alpha}(N_n^\gamma - (u_n)_{i+1,j})^2 \leq C; \tag{5.10}$$

hence, $u_{i+1,j} = \Theta(N_n^\gamma)$. By induction over all the squares $S_{N_n}^{i,j}$ we find that $u_n = \Theta(N_n^\gamma)$. Therefore $\|u_n\|_{L^p(\mathbb{T}^2)}^p = \Theta(N_n^{p\gamma})$, but by the coercivity condition (W_3) on W (for any $p > 0$) the uniform bound on $k_{N_n}^\alpha(u_n)$ demands

$$c_1\|u_n\|_{L^p(\mathbb{T}^2)}^p \leq \int_{\mathbb{T}^2} W(u_n) \leq CN_n^{-\alpha},$$

which is a contradiction.

Now for any u_n let v_n be its bilinear interpolation: For $(x,y) \in S_{N_n}^{i,j}$ define

$$\begin{aligned}v_n(x,y) := N_n^2\Big[&(u_n)_{i,j}\Big(\frac{i+1}{N_n} - x\Big)\Big(\frac{j+1}{N_n} - y\Big)\\ &+ (u_n)_{i+1,j}\Big(x - \frac{i}{N_n}\Big)\Big(\frac{j+1}{N_n} - y\Big)\\ &+ (u_n)_{i,j+1}\Big(\frac{i+1}{N_n} - x\Big)\Big(y - \frac{j}{N_n}\Big) + (u_n)_{i+1,j+1}\Big(x - \frac{i}{N_n}\Big)\Big(y - \frac{j}{N_n}\Big)\Big],\end{aligned}$$

where for notational convenience we have identified $u_n \in \mathcal{A}_{N_n}$ with its counterpart in $\mathcal{V}_{N_n}$. Thus defined, v_n is continuous and $\|v_n\|_{L^\infty(\mathbb{T}^2)} = \|u_n\|_{L^\infty(\mathbb{T}^2)}$. A straightforward computation shows

$$\begin{aligned}\int_{\mathbb{T}^2}(v_n - u_n)^2 = \frac{1}{18}N_n^{-2}\sum_{i,j=1}^{N_n}\Big[&\frac{7}{2}\big((u_n)_{i,j} - (u_n)_{i+1,j}\big)^2\\ &+ \frac{7}{2}\big((u_n)_{i,j} - (u_n)_{i,j+1)}\big)^2 + 4\big((u_n)_{i,j} - (u_n)_{i+1,j+1}\big)^2\\ &- \frac{1}{2}\big((u_n)_{i+1,j} - (u_n)_{i,j+1}\big)^2 - \big((u_n)_{i+1,j} - (u_n)_{i+1,j+1}\big)^2\\ &- \big((u_n)_{i,j+1} - (u_n)_{i+1,j+1}\big)^2\Big].\end{aligned}$$

First we note that there is a $C > 0$ such that

$$\begin{aligned}&\big((u_n)_{i,j} - (u_n)_{i+1,j+1}\big)^2\\ &\leq C\Big[\big((u_n)_{i,j} - (u_n)_{i+1,j}\big)^2 + \big((u_n)_{i+1,j} - (u_n)_{i+1,j+1}\big)^2\Big],\\ &\big((u_n)_{i+1,j} - (u_n)_{i,j+1}\big)^2\\ &\leq C\Big[\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 + \big((u_n)_{i,j} - (u_n)_{i,j+1}\big)^2\Big].\end{aligned}$$

Next we use periodicity to deduce

$$\sum_{i,j=1}^{N_n} \big((u_n)_{i+1,j} - (u_n)_{i+1,j+1}\big)^2 = \sum_{i,j=1}^{N_n} \big((u_n)_{i,j} - (u_n)_{i,j+1}\big)^2,$$

and analogously for similar terms. Using the uniform bound on $k^\alpha_{N_n}(u_n)$ we find

$$\begin{aligned}\int_{\mathbb{T}^2} (v_n - u_n)^2 &\le C N_n^{-2} \sum_{i,j=1}^{N_n} \Big[\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 + \big((u_n)_{i,j+1} - (u_n)_{i,j}\big)^2\Big] \\ &\le C N_n^{-2+\alpha}.\end{aligned}$$

Thus, $\|v_n - u_n\|_{L^2(\mathbb{T}^2)} \to 0$ as $n \to \infty$. Another computation gives

$$\begin{aligned}\int_{\mathbb{T}^2} \Big(\frac{\partial v_n}{\partial x}\Big)^2 = \frac{1}{3} \sum_{i,j=1}^{N_n} \Big[&\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 + \big((u_n)_{i+1,j+1} - (u_n)_{i,j+1}\big)^2 \\ &+ \big((u_n)_{i+1,j} - (u_n)_{i,j}\big)\big((u_n)_{i+1,j+1} - (u_n)_{i,j+1}\big)\Big].\end{aligned}$$

Using periodicity as above in combination with the inequality

$$\begin{aligned}&\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)\big((u_n)_{i+1,j+1} - (u_n)_{i,j+1}\big) \\ &\le \frac{1}{2}\Big(\big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2 + \big((u_n)_{i+1,j+1} - (u_n)_{i,j+1}\big)^2\Big),\end{aligned}$$

we deduce

$$\int_{\mathbb{T}^2} \Big(\frac{\partial v_n}{\partial x}\Big)^2 \le \sum_{i,j=1}^{N_n} \big((u_n)_{i+1,j} - (u_n)_{i,j}\big)^2$$

and the analogous result for $\int_{\mathbb{T}^2} \big(\frac{\partial v_n}{\partial y}\big)^2$.

Finally we note that

$$N_n^\alpha \int_{\mathbb{T}^2} W(u_n) = N_n^\alpha \int_{\mathbb{T}^2} W(v_n) + R_n,$$

where

$$\begin{aligned}R_n = N_n^\alpha \int_{\mathbb{T}^2} \big(W(u_n) - W(v_n)\big) &\le N_n^\alpha M_n^{W'} \|u_n - v_n\|_{L^1(\mathbb{T}^2)} \\ &\le N_n^\alpha M_n^{W'} \|u_n - v_n\|_{L^2(\mathbb{T}^2)} \le M_n^{W'} N_n^{\frac{3}{2}\alpha - 1}.\end{aligned} \tag{5.11}$$

Here

$$M_n^{W'} := \max_{x\in\mathbb{T}^2} \max_{s\in[v_n(x),u_n(x)]} |W'(s)|.$$

By construction, since $\|u_n\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(N_n^{\alpha/2})$, we have

$$\|v_n\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(N_n^{\alpha/2}).$$

Hence, the maximum over s is achieved for some $s = \mathcal{O}(N_n^{\alpha/2})$. By the regularity of W' and its polynomial growth condition (W_4), we then have $M_n^{W'} = \mathcal{O}(N_n^{\alpha q/2})$; hence, $R_n \to 0$ as $n \to \infty$ by the choice of α.

The inequalities above prove the claim and hence finish the proof. $\square$

Remark 5.8. Clearly, the $L^2(\mathbb{T}^2)$ convergence in Theorem 5.6 can be replaced by $L^1(\mathbb{T}^2)$ convergence if desired.

Lemma 5.9 (Lower bound)**.** *Assume W satisfies (W_3) and (W_4) for given $p > 0$ and $q > 0$, and $\alpha \in (0, \frac{2}{q+3})$. Let $u \in L^1(\mathbb{T}^2)$, and let $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ and $N_n \to \infty$ as $n \to \infty$; then*

$$k_{\infty,0}(u) \leq \liminf_{n\to\infty} k_{N_n}^\alpha(u_n).$$

Proof. Without loss of generality we can assume that $k_{N_n}^\alpha(u_n)$ is uniformly bounded. In the proof of Theorem 5.6 we established that then estimate (5.9) follows, where the v_n are the bilinear interpolations of u_n that converge to u in $L^1(\mathbb{T}^2)$. Using the Γ-convergence result of Modica and Mortola [49, 47, 48, 56], specifically their lower bound, we find

$$\liminf_{n\to\infty} k_{N_n}^\alpha(u_n) \geq \liminf_{n\to\infty} \left(F_{N_n^{-\alpha}}^{GL}(v_n) + R_n\right) \geq k_{\infty,0}(u). \qquad \square$$

Remark 5.10. As noted earlier in Remark 5.1 our results (and proofs) generalize to $\mathbb{T}^d$ if the terms in k_N^α are rescaled properly depending on the dimension d. In this case we carefully need to reexamine the admissible range for α in Theorem 5.6 and Lemma 5.9 (and hence by extension Theorem 5.7). In particular, (5.10) becomes

$$N_n^{2-d-\alpha}(N_n^\gamma - (u_n)_{i+1,j})^2 \leq C,$$

and hence we need to choose $\gamma > \frac{\alpha+d-2}{2}$ and deduce that $\|u_n\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(N_n^{\frac{\alpha+d-2}{2}})$. Generalizing (5.11) and the discussion that follows then leads to the conclusion that the admissible range of α is $\alpha \in \left(0, \frac{2q-2(d-2)}{q(q+3)}\right)$. In particular, q in condition (W_4) should be chosen larger than $d-2$.

Lemma 5.11 (Upper bound)**.** *Let $\alpha \in (0,1)$, $u \in L^1(\mathbb{T}^2)$, and $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $N_n \to \infty$ as $n \to \infty$, and let k'' be the Γ-upper limit of $k_{N_n}^\alpha$ as $n \to \infty$ with respect to the $L^2(\mathbb{T}^2)$ topology; then $k''(u) \leq k_{\infty,0}(u)$.*

Proof. The case where $u \in L^1(\mathbb{T}^2) \setminus BV(\mathbb{T}^2; \{0,1\})$ is trivial, so assume $u \in BV(\mathbb{T}^2; \{0,1\})$. First assume that $\operatorname{supp} u$ has smooth boundary $\partial \operatorname{supp} u$.

By the classical Modica–Mortola results used before, $k_{\infty,\varepsilon}$ Γ-converges in both the $L^1(\mathbb{T}^2)$ and $L^2(\mathbb{T}^2)$ topology to $k_{\infty,0}$. Let $\{v_n\}_{n=1}^\infty$ be the recovery sequence for this convergence with $\varepsilon = N_n^{-\alpha}$ (see *e.g.* [47, Proposition 2], [16, §7.2.1]); then each $v_n \in W^{1,2}(\mathbb{T}^2)$ is a Lipschitz-continuous function, $v_n \to u$ in $L^2(\mathbb{T}^2)$ as $n \to \infty$, and $\limsup_{n\to\infty} k_{\infty,N_n^{-\alpha}}(v_n) \le k_{\infty,0}(u)$. We denote the Lipschitz constant of v_n by K_n and note that $K_n = \mathcal{O}(N_n^\alpha)$ as $n \to \infty$.

We now follow a similar line of reasoning as in the proof of Lemma 5.5, but need to be more careful to deal with the lesser regularity of v_n (only Lipschitz continuous instead of C^∞). For each $i, j \in I_N$ and each $n \in \mathbb{N}$, define

$$\nu_{N_n}^{i,j} := \operatorname*{argmin}_{z \in \overline{S_{N_n}^{i,j}}} \Big[(D_{N_n}^1 v_n(z))^2 + (D_{N_n}^2 v_n(z))^2 \Big],$$

where we used the difference-quotient notation from (5.4). Note that since v_{N_n} is continuous and $\overline{S_{N_n}^{i,j}}$ is compact, the minimum is attained. Note that it may be that $\nu_{N_n}^{i,j} \in \overline{S_{N_n}^{i,j}} \setminus S_{N_n}^{i,j}$. Now define a sequence $\{u_n\}_{n=1}^\infty$ such that for each $n \in \mathbb{N}$, $u_n \in \mathcal{A}_{N_n}$ in the following way. If $x \in S_{N_n}^{i,j}$, then $u_n(x) := v_n(\nu_{N_n}^{i,j})$. Note that by construction

$$(D_{N_n}^1 u_n(x))^2 + (D_{N_n}^2 u_n(x))^2 \le (D_{N_n}^1 v_n(x))^2 + (D_{N_n}^2 v_n(x))^2.$$

First we check that $u_n \to u$ in $L^2(\mathbb{T}^2)$. We estimate $\|u_n - u\|_{L^2(\mathbb{T}^2)} \le \|u_n - v_n\|_{L^2(\mathbb{T}^2)} + \|v_n - u\|_{L^2(\mathbb{T}^2)}$. We know that $v_n \to u$ in $L^2(\mathbb{T}^2)$; for the first term on the right we use the Lipschitz continuity of v_n to compute

$$\begin{aligned}
\|u_n - v_n\|_{L^2(\mathbb{T}^2)}^2 &= \int_{\mathbb{T}^2} |u_n(x) - v_n(x)|^2 dx = \sum_{i,j \in I_{N_n}} \int_{S_{N_n}^{i,j}} |v_n(\nu_{N_n}^{i,j}) - v_n(x)|^2 \, dx \\
&\le K_n^2 \sum_{i,j \in I_{N_n}} \int_{S_{N_n}^{i,j}} |\nu_{N_n}^{i,j} - x|^2 \, dx \le \sqrt{2} K_n^2 N_n^2 N_n^{-2} N_n^{-2}.
\end{aligned}$$

Here we have used that there are N_n^2 nodes in the grid, the area of each square $S_{N_n}^{i,j}$ is N_n^{-2}, and for $x \in S_{N_n}^{i,j}$ we have $|\nu_{N_n}^{i,j} - x| \le \sqrt{2} N_n^{-1}$. Since $K_n = \mathcal{O}(N_n^\alpha)$ as $n \to \infty$ and $\alpha < 1$, we have $\|u_n - v_n\|_{L^2(\mathbb{T}^2)}^2 \to 0$ as $n \to \infty$, and hence $u_n \to u$ in $L^2(\mathbb{T}^2)$.

Since $v_{N_n} : \mathbb{T}^2 \to \mathbb{R}$ is Lipschitz continuous, if we fix either $y \in \mathbb{T}$ or $x \in \mathbb{T}$ so are $v_{N_n}(\cdot, y) : \mathbb{T} \to \mathbb{R}$ and $v_{N_n}(x, \cdot) : \mathbb{T} \to \mathbb{R}$. Therefore by Rademacher's theorem the partial derivatives of v_{N_n} exist almost everywhere on horizontal and vertical lines; hence, we have, for almost every $x \in \mathbb{T}^2$,

$$D^1_{N_n} v_n(x) = \int_0^1 \frac{\partial v_n}{\partial x}(x+se_1/N_n)\, ds, \quad D^2_{N_n} v_n(x) = \int_0^1 \frac{\partial v_n}{\partial y}(x+se_2/N_n)\, ds.$$

By Jensen's inequality,

$$(D^1_{N_n} v_n(x))^2 = \left(\int_0^1 \frac{\partial v_n}{\partial x}(x+se_1/N_n)\, ds\right)^2 \leq \int_0^1 \left(\frac{\partial v_n}{\partial x}(x+se_1/N_n)\right)^2 ds,$$

and similarly for $D^2_{N_n} v_n(x)$. For the finite-difference terms in $k^\alpha_{N_n}(u_n)$ we now find, by construction of u_n,

$$\begin{aligned}
&\int_{\mathbb{T}^2} \left[(D^1_{N_n} u_n(x))^2 + (D^2_{N_n} u_n(x))^2\right] dx \\
&= \sum_{i,j\in I_{N_n}} \int_{S^{i,j}_{N_n}} \left[(D^1_{N_n} u_n(x))^2 + (D^2_{N_n} u_n(x))^2\right] dx \\
&\leq \sum_{i,j\in I_{N_n}} \int_{S^{i,j}_{N_n}} \left[(D^1_{N_n} v_n(x))^2 + (D^2_{N_n} v_n(x))^2\right] dx \\
&\leq \sum_{i,j\in I_{N_n}} \int_{S^{i,j}_{N_n}} \int_0^1 \left[\left(\frac{\partial v_n}{\partial x}(x+se_1/N_n)\right)^2 + \left(\frac{\partial v_n}{\partial y}(x+se_2/N_n)\right)^2\right] ds\, dx \\
&= \int_0^1 \sum_{i,j\in I_{N_n}} \left[\int_{S^{i,j}_{N_n}+se_1/N_n} \left(\frac{\partial v_n}{\partial x}(x)\right)^2 dx + \int_{S^{i,j}_{N_n}+se_2/N_n} \left(\frac{\partial v_n}{\partial y}(x)\right)^2 dx\right] ds \\
&= \int_{\mathbb{T}^2} |\nabla v_n|^2,
\end{aligned}$$

where we have used Fubini's theorem and the fact that we are working on a torus, so the union of all sets of the form $S^{i,j}_{N_n} + se_k/N_n$ (for either $k=1$ or $k=2$) is the same as the union of all $S^{i,j}_{N_n}$, *i.e.*, $\mathbb{T}^2$.

To deal with the double-well potential term in $k^\alpha_{N_n}$, we first note that for each $x \in \mathbb{T}^2$ and each $n \in \mathbb{N}$, $v_n(x) \in [0,1]$; hence, W' is bounded on intervals of the form $[v_n(x), v_n(\nu^{i,j}_{N_n})]$ by some $C > 0$. Therefore, for $x \in S^{i,j}_{N_n}$,

$$\begin{aligned}
W(u_n(x)) - W(v_n(x)) = W(v_n(\nu^{i,j}_{N_n})) - W(v_n(x)) &= \int_{v_n(x))}^{v_n(\nu^{i,j}_{N_n})} W'(s)\, ds \\
&\leq C|v_n(\nu^{i,j}_{N_n}) - v_n(x)| \leq CK_n|\nu^{i,j}_{N_n} - x|.
\end{aligned}$$

We thus find

$$\int_{\mathbb{T}^2} W(u_n)(x)\,dx = \int_{\mathbb{T}^2} W(v_n(x))\,dx + CK_n \sum_{i,j\in I_{N_n}} \int_{S^{i,j}_{N_n}} |\nu^{i,j}_{N_n} - x|\,dx$$
$$\leq \int_{\mathbb{T}^2} W(v_n(x))\,dx + CK_n N_n^2 N_n^{-2} N_n^{-1}.$$

Hence $\int_{\mathbb{T}^2} W(u_n)(x)\,dx \leq \int_{\mathbb{T}^2} W(v_n(x))\,dx + R_n$, where $R_n = \mathcal{O}(N_n^{\alpha-1})$. Since $\alpha < 1$, $R_n \to 0$ as $n \to \infty$.

Combining both terms in $k^\alpha_{N_n}$ we find $k^\alpha_{N_n}(u_n) \leq k_{\infty,N_n^{-\alpha}}(v_n) + R_n$. We already know that $\limsup_{n\to\infty} k_{\infty,N_n^{-\alpha}}(v_n) \leq k_{\infty,0}(u)$, so we have proved that for $u \in BV(\mathbb{T}^2;\{0,1\})$ with smooth $\partial \operatorname{supp} u$ we have $\limsup_{n\to\infty} k^\alpha_{N_n}(u_n) \leq k_{\infty,0}(u)$, or in terms of the Γ-upper limit, $k''(u) \leq k_{\infty,0}(u)$.

Now let $u \in BV(\mathbb{T}^2;\{0,1\})$, not necessarily with the smoothness condition on the boundary. Then by [9, Theorem 3.42] there is a sequence $\{\hat{u}_n\}_{n=1}^\infty \subset BV(\mathbb{T}^2;\{0,1\})$ such that each $\operatorname{supp} u_n$ has smooth boundary and $\lim_{n\to\infty} k_{\infty,0}(\hat{u}_n) = k_{\infty,0}(u)$. Hence by lower semicontinuity of the Γ-upper limit k'' we conclude

$$k''(u) \leq \liminf_{n\to\infty} k''(\hat{u}_n) \leq \liminf_{n\to\infty} k_{\infty,0}(\hat{u}_n) = k_{\infty,0}(u). \qquad \square$$

Proof of Theorem 5.7. Since we have used $L^1(\mathbb{T}^2)$ convergence in (LB) in Lemma 5.9 and $L^2(\mathbb{T}^2)$ convergence in Lemma 5.11 for (UB′), we can now conclude Γ-convergence in either of these two topologies. □

5.3. **Discussion of the range of α.** The range of admissible α in the results in the previous section is not only of theoretical interest, but is also important for computations. In simulations, choosing ε of the right order is a hard problem. If ε is too small in gradient flow simulation, this leads to the phenomenon of "pinning," where the initial condition gets pinned down into the wells of W without changing its geometry. On the other hand, an ε which is too large leads to immediate diffusion of the initial condition and loss of some relevant features. Our results do not directly address the gradient flow, but are in the same spirit.

In the proof of compactness and the lower bound above we have assumed $\alpha \in (0, \frac{2}{q+3})$, where q is the degree of polynomial growth of W'. There are some reasons to believe this restriction could possibly be relaxed to $\alpha \in (0, \frac{2}{q+2})$, but we have not found a proof for this statement.

First note that the dependence on q in the range of α comes from the discrete nature of the problem. Fundamentally it can be traced back to the lack of a chain rule for discrete differentiation. Trying to copy the classical Modica–Mortola result, we can define $w_n := \int_0^{u_n} \sqrt{W(s)}\, ds$ and estimate

$$\int_{\mathbb{T}^2} |\nabla w_n| = N_n^{-1} \sum_{i,j=1}^{N_n} |(w_n)_{i+1,j} - (w_n)_{i,j}| + |(w_n)_{i,j+1} - (w_n)_{i,j}|$$

$$\leq 2N_n^{-1} \sum_{i,j=1}^{N_n} \left(\left[((w_n)_{i+1,j} - (w_n)_{i,j})^2 + ((w_n)_{i,j+1} - (w_n)_{i,j})^2 \right] \right)^{\frac{1}{2}}$$

$$= 2N_n^{-1} \sum_{i,j=1}^{N_n} \left(\Big(\int_{(u_n)_{i,j}}^{(u_n)_{i+1,j}} \sqrt{W(s)}\, ds \Big)^2 + \Big(\int_{(u_n)_{i,j}}^{(u_n)_{i,j+1}} \sqrt{W(s)}\, ds \Big)^2 \right)^{\frac{1}{2}}$$

$$= 2N_n^{-1} \sum_{i,j=1}^{N_n} \Big(\Big[((u_n)_{i+1,j} - (u_n)_{i,j})^2 W((u_n^*)_{i+1,j})$$

$$+ ((u_n)_{i,j+1} - (u_n)_{i,j})^2 W((u_n^*)_{i,j+1}) \Big] \Big)^{\frac{1}{2}},$$

where $(u_n^*)_{i+1,j} \in [(u_n)_{i,j}, (u_n)_{i+1,j}]$ and $(u_n^*)_{i,j+1} \in [(u_n)_{i,j}, (u_n)_{i,j+1}]$ come from the mean-value theorem. If we had control over the behavior of W in between grid points, we could use Cauchy's inequality to bound the expression above by $k_{N_n}^{\alpha}(u_n)$. Without condition (W_4) this control is lacking. We could do without this condition if we would somehow have an *a priori* L^∞ bound for the sequence $\{u_n\}_{n=1}^\infty$. This reflects a similar situation in the continuum case [56, Remark 1.35], where condition (W_3) with $p \geq 2$ can be dropped from the assumptions needed for compactness if an *a priori* L^∞ bound is available. By construction, a uniform bound on $\|u_n\|_{L^\infty(\mathbb{T}^2)}$ gives a similar bound on $\|v_n\|_{L^\infty(\mathbb{T}^2)}$; hence, under such an *a priori* bound we could drop conditions (W_3) and (W_4) from our assumptions and eleminate q (*i.e.,* $q = 0$) from the restriction on α. The difference with the continuum case is that in our case (in the absence of an L^∞ bound) we need control over W and its derivative W'. The scale of the discretization, N^{-1}, should be fine enough to resolve the variations in W'.

Next note that in the proof of the upper bound we only use $\alpha \in (0, 1)$. This restriction has a natural interpretation: If we interpret N^{-1} as the discretization spacing, then $\varepsilon = N^{-\alpha}$ for $0 < \alpha < 1$ tells us that the discretization should be fine enough to "resolve the diffuse interface," which, in the continuum case, has width of order ε. The following example shows that for $\alpha > 1$ the lower bound fails. Let $u \in BV(\mathbb{T}^2, \{0, 1\})$ be equal to zero on half the torus (say on $[0,1/2]\times[0,1)$) and equal to one on the other half, and let $\{u_n\}_{n=1}^{\infty}$ be a sequence converging to u in $L^1(\mathbb{T}^2)$ obtained by simply discretizing u on the grid G_{N_n} for a sequence $\{N_n\}_{n=1}^{\infty}$, $N_n\to\infty$ as $n\to\infty$. Then $W(u_n)\equiv 0$ for all n. The finite difference term in $k_{N_n}^{\alpha}(u_n)$ only has nonzero contributions along the boundary between the parts of the torus where $u=0$ and $u=1$. Thus there are $2N$ jumps of order 1, and hence $k_{N_n}^{\alpha}(u_n)=2N_n^{1-\alpha}$. If $\alpha>1$ this converges to zero, but the limit functional $k_{\infty,0}(u)>0$, which contradicts (LB).

Finally, note that in (5.11) our estimate is not sharp since we use Hölder's inequality to go from $\|u_n-v_n\|_{L^1(\mathbb{T}^2)}$ to $\|u_n-v_n\|_{L^2(\mathbb{T}^2)}$. If instead a bound $\|u_n-v_n\|_{L^1(\mathbb{T}^2)}=\mathcal{O}(N_n^{-1})$ could be proved, possibly using the uniform bound on $k_{N_n}^{\alpha}(u_n)$, then in (5.11) the condition on α relaxes to $\alpha\in(0,\frac{2}{q+2})$, which in the absence of q would reduce to $\alpha\in(0,1)$. We therefore conjecture that $\alpha\in(0,\frac{2}{q+2})$ is in fact the natural restriction for α (on $\mathbb{T}^2$; see Remark 5.10 for a discussion about the range of α in general dimensions), or, if an *a priori* L^∞ bound is available, $\alpha\in(0,1)$. However, it might be the case that a bilinear interpolation is not the right interpolation to attain this bound.

5.4. **Constraints.** In this section we show that addition of a fidelity term or imposing a mass constraint are compatible with the three Γ-limits we established, *i.e.,* $N\to\infty$ for $k_{N,\varepsilon}$, $\varepsilon\to 0$ for $k_{\infty,\varepsilon}$, and $N\to\infty$ for k_N^{α}.

The Modica–Mortola limit $k_{\infty,\varepsilon}\xrightarrow{\Gamma}k_{\infty,0}$ as $N\to\infty$ in the $L^p(\mathbb{T}^2)$ topology, $p\in\{1,2\}$, is known to be compatible with a mass constraint, *e.g.* [47, Proposition 2], [56, Theorem 1], and [15, Proposition 6.6]. Furthermore, since an $L^p(\mathbb{T}^2)$ fidelity term, $p\in\{1,2\}$, is clearly continuous with respect to $L^p(\mathbb{T}^2)$ convergence, it is also compatible with the Γ-limit. The theorem below addresses the other two Γ-limits for $k_{N,\varepsilon}$ and k_N^{α}.

Theorem 5.12 (Constraints)**.** (1) $k_{N,\varepsilon}+\lambda N^{-2}|\cdot-f_N|_p^p\xrightarrow{\Gamma}k_{\infty,\varepsilon}+\lambda\int_{\mathbb{T}^2}|\cdot-f|_p^p$ *for $N\to\infty$ in the $L^p(\mathbb{T}^2)$ topology, where $p\in\{1,2\}$, $\lambda>0$, $f\in C^1(\mathbb{T}^2)$, and $f_N\in\mathcal{A}_N$ is the sampling of f on the grid G_N (f, f_N, and their norms can also be defined on subsets of $\mathbb{T}^2$ and G_N as in Theorem* 4.13, *part* 4.13). *A compactness result for $k_{N,\varepsilon}+\lambda N^{-2}|\cdot-f|_p^p$ as in Theorem* 5.3 *holds.*

If instead, for fixed $M\in[0,1]$, the domain of definition of $k_{N,\varepsilon}$ is restricted to $\mathcal{V}_N^M$ (i.e., $\mathcal{V}^M$ from Theorem 3.6 *on the grid G_N), then the Γ-convergence and compactness results for $N\to 0$ remain valid, with the domain of $k_{\infty,0}$ restricted to $\mathcal{V}^M$.*

(2) $k_N^{\alpha}+\lambda N^{-2}|\cdot-f_N|_p^p\xrightarrow{\Gamma}k_{\infty,0}+\lambda\int_{\mathbb{T}^2}|\cdot-f|_p^p$ *for $N\to\infty$ in the $L^p(\mathbb{T}^2)$ topology, where $p\in\{1,2\}$; λ, f, and f_N are as in part* 5.12; *and $\alpha\in(0,\frac{2}{q+3})$*

is as in Theorem 5.7. *A compactness result for* $k_N^\alpha + \lambda N^{-2}|\cdot - f|_p^p$ *as in Theorem* 5.6 *and Remark* 5.8 *holds.*

If instead, for fixed $M \in [0,1]$*, the domain of definition of* k_N^α *is restricted to* $\mathcal{V}_N^M$*, then the* Γ*-convergence and compactness results for* $N \to 0$ *remain valid, with the domain of* $k_{\infty,0}$ *restricted to* $\mathcal{V}^M$*.*

We give a sketch of the proofs.

(1) The compatibility of the fidelity term with the Γ-convergence and compactness follows as in the proof of Theorem 4.13, part 4.13. As in that theorem, the mass constraint is preserved under $L^p(\mathbb{T}^2)$ convergence and so is compatible with both (LB) and compactness[11]. To show that the mass constraint is compatible with (UB$'$) as well, we need to check two conditions. First, the recovery sequence which was constructed (in the proof of Lemma 5.5) for $u \in C^\infty(\mathbb{T}^2)$ should either satisfy or be able to be adapted to satisfy the mass constraint. Second, for each $u \in W^{1,2}(\mathbb{T}^2)$ there should be an approximating sequence $\{u_n\}_{n=1}^\infty \subset C^\infty(\mathbb{T}^2)$ which has constant mass. The latter follows directly by the use of normalized mollifiers to construct the approximating sequence. For the former condition we follow an argument reminiscent of the proof that a mass constraint is compatible with the Modica–Mortola Γ-convergence result for the continuum Ginzburg–Landau functional; see *e.g.* [47, 56, 15]. Assume that $\int_{\mathbb{T}^2} u = M$ for some $M > 0$. Because u is smooth it has bounded derivatives on $\mathbb{T}^2$; hence (using the notation $S_{N_n}^{i,j}$ from (2.2)),

$$\begin{aligned}\int_{\mathbb{T}^2} u_n - u &= \sum_{i,j=1}^{N_n} \int_{S_{N_n}^{i,j}} \left[u(i/N_n, j/N_n) - u(x)\right] dx\\ &\le C_u \sum_{i,j=1}^{N_n} \int_{S_{N_n}^{i,j}} |(i/N_n, j/N_n) - x|_2 \, dx \le \tilde{C}_u N_n^{-1},\end{aligned}$$

for some constants C_u and $\tilde{C}_u$, depending only on u. Hence for each $n \in \mathbb{N}$ there is a $\delta_n = \mathcal{O}(N_n^{-1})$ such that $\tilde{u}_n := u_n + \delta_n$ satisfies $\int_{\mathbb{T}^2} \tilde{u}_n = M$. For this new proposed recovery sequence we compute

$$k_{N_n,\varepsilon}(\tilde{u}_n) = k_{N_n,\varepsilon}(u_n) + \varepsilon^{-1} \int_{\mathbb{T}^2} \left[W(\tilde{u}_n) - W(u_n)\right].$$

[11]Both the fidelity term and the mass constraint are not only compatible with the compactness result, but even help with concluding uniform boundedness of either $L^1(\mathbb{T}^2)$ or $L^2(\mathbb{T}^2)$ norm and hence can replace assumption (W_2) in Theorem 5.3 when compactness with respect to the correct topology is considered.

By Taylor's theorem, for $x \in \mathbb{T}^2$, $W(\tilde{u}_n(x)) - W(u_n(x)) = W'(c_n(x))\delta_n$, where $c_n(x) \in [u_n, u_n + \delta_n]$. Since u is continuous and hence bounded on $\mathbb{T}^2$, the sequence $\{u_n\}_{n=1}^\infty$ is equibounded, and hence, because $W \in C^2(\mathbb{R})$, $W'(c_n)$ is equibounded. Therefore, the sequence $\{\tilde{u}_n\}_{n=1}^\infty$ is indeed a recovery sequence for (UB) with $u \in C^\infty(\mathbb{T}^2)$ and satisfies the mass constraint.

(2) As in part 5.12, the addition of fidelity terms is compatible with the Γ-limit and compactness and the mass constraint is compatible with both (LB) and compactness. For compatibility with (UB′) again we check two things: First that the recovery sequence which was constructed (in the proof of Lemma 5.11) for $u \in BV(\mathbb{T}^2; \{0,1\})$ with $\partial \operatorname{supp} u$ smooth either satisfies or can be adapted to satisfy the mass constraint, and second that for each $u \in BV(\mathbb{T}^2; \{0,1\})$ an approximating sequence $\{u_n\}_{n=1}^\infty \subset BV(\mathbb{T}^2; \{0,1\})$ can be chosen for which $\partial \operatorname{supp} u_n$ is smooth and which has constant mass. The latter condition is satisfied if we use the approximating sequence as in [9, Theorem 3.42] and then introduce a small dilation, diminishing along the sequence, of the support of each u_n so that the mass remains fixed (see *e.g.* [52, Proposition 7.1]). The former condition follows in a way similar to that of construction above in the case $N \to \infty$ for $k_{N,\varepsilon}$.

5.5. **Gradient flow for $k_{N,\varepsilon}$ with constraints.** To minimize $k_{N,\varepsilon}$ either under a mass constraint or with a fidelity term we can use a gradient flow. First consider the latter case: $k_{N,\varepsilon,\lambda} := k_{N,\varepsilon} + \lambda|\cdot - f|_2^2$. For $u, v \in \mathcal{V}_N$ we compute $\operatorname{grad}(k_{N,\varepsilon,\lambda})(u) \in \mathcal{V}_N$ via

$$\frac{d}{dt} k_{N,\varepsilon,\lambda}(u + tv)\Big|_{t=0} = \langle \operatorname{grad}(k_{N,\varepsilon\,\lambda})(u), v\rangle_{\mathcal{V}}$$

and then set for all $i, j \in I_N$

$$\frac{\partial u_{i,j}}{\partial t} = -\big(\operatorname{grad}(k_{N,\varepsilon,\lambda})(u)\big)_{i,j}.$$

This leads to the equation

$$\frac{\partial u_{i,j}}{\partial t} = -4^{-r}\Big[2\varepsilon \sum_{(k,l)\in\mathcal{N}(i,j)} (u_{i,j} - u_{k,l}) + \varepsilon^{-1} N^{-2} W'(u_{i,j}) + 2\lambda(u_{i,j} - f_{i,j})\Big], \quad (5.12)$$

where the set of indices of neighbors of (i,j) is given by $\mathcal{N}(i,j) = \{(i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$. The overall prefactor 4^{-r} comes from the factor d_{ij}^{-r}, which is needed to cancel the factor $d_{i,j}^r$ in the $\mathcal{V}_N$ inner product. Here we assume the weights in this case to be equal to 1 (on existing edges). Equation (5.12) is the discretized analogue of the continuum Allen–Cahn equation with data fidelity

$$\frac{\partial u}{\partial t} = 2\varepsilon \Delta u - \varepsilon^{-1} W'(u) - 2\lambda(u - f),$$

which is the L^2 gradient flow of $F_\varepsilon^{GL}(u) + \lambda \|u - f\|^2_{L^2(\mathbb{T}^2)}$.

If instead of the fidelity term a mass constraint is imposed; the term $2\lambda(u_{i,j} - f_{i,j})$ gets replaced by a Lagrange multiplier

$$\kappa = \varepsilon^{-1} N^{-4} \sum_{i,j=1}^{N} W'(u_{i,j}).$$

To illustrate we show simulation results using a fidelity term with f the characteristic function of a square. We use a one-step forward-in-time finite-difference scheme to discretize the time derivative, *i.e.,*

$$u_{i,j}^{n+1} = u_{i,j}^n - 4^{-r} dt \Big[2\varepsilon \sum_{(k,l) \in \mathcal{N}(i,j)} (u_{i,j}^n - u_{k,l}^n) + \varepsilon^{-1} N^{-2} W'(u_{i,j}^n) + 2\lambda(u_{i,j}^n - f_{i,j}) \Big].$$

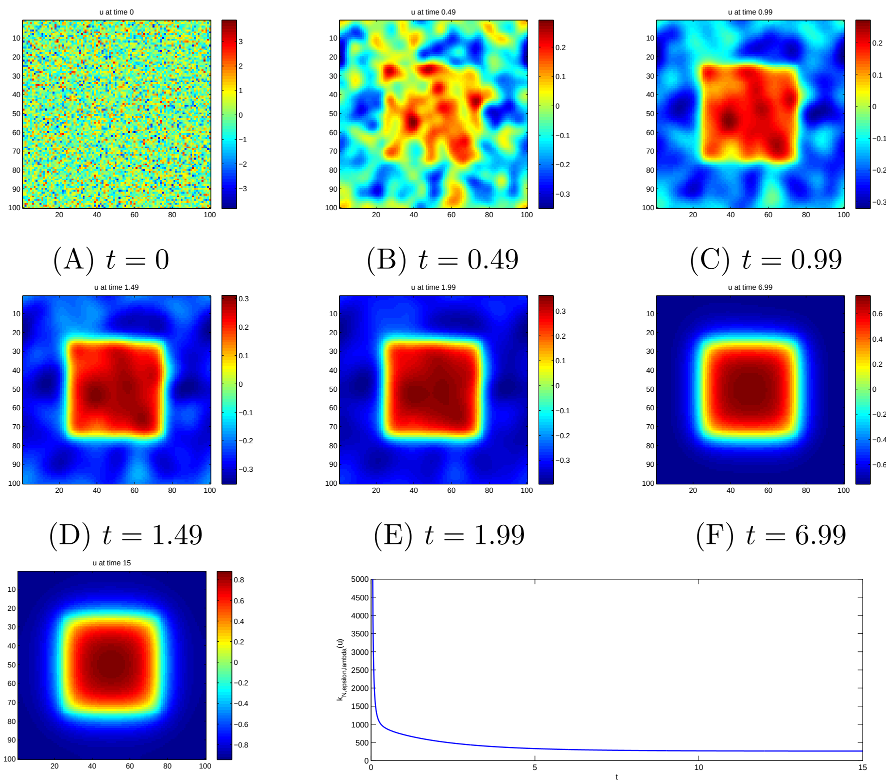
Here dt is the discrete time step and the superscript n labels the time step. We start with a random initial condition u^0. We use the inner product structure on $\mathcal{V}_N$ corresponding to the unnormalized Laplacian ($r = 0$). Using the structure corresponding to the random-walk Laplacian ($r = 1$) instead only gives an overall multiplicative factor $\frac{1}{4}$ in the right-hand side of the gradient flow in (5.12), and hence is effectively just a time rescaling leading to qualitatively the same behavior.

In Figure 1 we use the following parameter values: $N = 100$, $\varepsilon = 5$, $\lambda = 0.1$, and $dt = 0.01$. We use $W(s) = s^2(s-1)^2$ for the potential and f is data prescribed to be 1 in a square region and 0 outside that region.

Note that W' satisfies the growth condition (W_4) with $q = 3$; hence, according to Theorems 5.6 and 5.7, N and ε should satisfy the relation $N^{-\alpha} = \varepsilon$ for an $\alpha \in (0, 1/3)$. The combination $N = 100$ and $\varepsilon = 5$ used in Figure 1 falls outside this range ($\alpha \approx -0.35$), but the simulations still give a good result. Our theoretical results give a good guideline for choosing α (especially the upper bound), but in practice different values of α, and hence ε, can produce good gradient flow simulations. We have chosen this larger value of ε for our figures to show a stronger diffusion.

6. The continuum limit of nonlocal means

In this section we study the nonlocal means functional g_N from (1.6) for a given fixed $\Phi \in C^\infty(\mathbb{T}^2)$. Remember that the weights are $\omega_{L,N} := e^{-d_{L,N}^2/\sigma^2}$



(A) $t = 0$ (B) $t = 0.49$ (C) $t = 0.99$

(D) $t = 1.49$ (E) $t = 1.99$ (F) $t = 6.99$

(G) $t = 15$ (H)

FIGURE 1. (A)–(G) show snapshots of the gradient flow (5.12) using the parameters in the text. (H) shows the corresponding time evolution of $k_{N,\varepsilon,\lambda}(u)$.

with $\sigma, L > 0$ constants (possibly depending on N) and $d_{L,N}$ defined in (1.7). We define the limit weights $\omega_{L,\sigma}, \omega_{\ell,c} \in L^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$ as

$$\omega_{L,\sigma}(x,y) := e^{-\frac{4L^2}{\sigma^2}\left(\Phi(x)-\Phi(y)\right)^2}, \qquad \omega_{\ell,c}(x,y) := e^{-c^2 \int_{S_\ell}\left(\Phi(x+z)-\Phi(y+z)\right)^2 dz},$$

where $\ell, c > 0$ and $S_\ell := \{z \in \mathbb{R}^2 : |z_1| + |z_2| \le \ell\}$. The limit functionals $g_\infty^L : L^1(\mathbb{T}^2) \to \mathbb{R}$ and $g_\infty^\ell : L^1(\mathbb{T}^2) \to \mathbb{R}$ are

$$g_\infty^L(u) := \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_{L,\sigma}(x,y)|u(x)-u(y)|\,dx\,dy,$$

$$g_\infty^\ell(u) := \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_{\ell,c}(x,y)|u(x)-u(y)|\,dx\,dy.$$

We prove a Γ-convergence result.

Theorem 6.1 (Γ-convergence)**.** (1) *If $\sigma, L > 0$ are fixed, then $g_N \xrightarrow{\Gamma} g_\infty^L$ as $N \to \infty$, in the $L^1(\mathbb{T}^2)$ topology.*

(2) *If $\sigma = \frac{N}{c}$ for some $c > 0$ and L is such that $L/N \to \ell$ as $N \to \infty$ for some $\ell \in (0, 1/2)$, then $g_N \xrightarrow{\Gamma} g_\infty^\ell$ as $N \to \infty$, in the $L^1(\mathbb{T}^2)$ topology.*

As explained in Remark 6.5 we do *not* have compactness in this case.

Note that g_N is the functional f_0 from Theorem 3.1, where the graph G is the grid G_N and the choices $\chi = N^{-4}$ and $\omega = \omega_{L,N}$ have been made. Two main differences between this functional and the previous functionals on the grid G_N we considered are that the graph is now completely connected and the weights are not uniform over the edges. For the latter reason it is useful to introduce notation for the space of graph weights on G_N. Given a weight function ω and nodes $n_{i,j}, n_{k,l} \in V_N$ we write $\omega_{i,j,k,l} := \omega(n_{i,j}, n_{k,l})$. Define

$$\mathcal{W}_N := \{\omega : V_N \times V_N \to [0,\infty) : \text{ for all } i,j,k,l \in I_{N^2},\ \omega_{i,j,k,l} = \omega_{k,l,i,j}\}.$$

Completely analogous to the identification between $\mathcal{V}_N$ and $\mathcal{A}_N$ that was introduced in Section 4.1, we can identify $\mathcal{W}_N$ with

$$\Omega_N := \{\omega : \mathbb{T}^2 \times \mathbb{T}^2 \to [0,\infty) : \text{ for all } x,y \in \mathbb{T}^2,\ \omega(x,y) = \omega(y,x)\}.$$

Identifying $\omega_{L,N} \in \mathcal{W}_N$ with the corresponding $\omega_{L,N} \in \Omega_N$ and $u \in \mathcal{V}_N^b$ with the corresponding $u \in \mathcal{A}_N^b$, we can write

$$g_N(u) = \begin{cases} \displaystyle\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_{L,N}(x,y)|u(x)-u(y)|\,dx\,dy & \text{if } u \in \mathcal{A}_N^b, \\ +\infty & \text{if } u \in L^1(\mathbb{T}^2) \setminus \mathcal{A}_N^b. \end{cases}$$

We prove Theorem 6.1 in two steps. First we show that uniform convergence of the weights suffices for Γ-convergence of g_N, and then we show that uniform convergence holds.

Lemma 6.2. *Let $\{\omega_N\}_{N=1}^\infty$ be such that $\omega_N \in \Omega_N$ and $\omega_N \to \omega$ uniformly as $N \to \infty$ for some $\omega \in L^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$. For $N \in \mathbb{N}$ define the functional $g_\infty : L^1(\mathbb{T}^2) \to \mathbb{R}$ by*

$$g_\infty(u) := \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega(x,y)|u(x)-u(y)|\,dx\,dy;$$

then $g_N \xrightarrow{\Gamma} g_\infty$ as $N \to \infty$ in the $L^1(\mathbb{T}^2)$ topology.

Proof. Let $\{N_n\}_{n=1}^\infty \subset \mathbb{N}$ be such that $N_n \to \infty$ as $n \to \infty$, and let $u \in L^1(\mathbb{T}^2)$ and $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T}^2)$ be such that $u_n \to u$ in $L^1(\mathbb{T}^2)$ as $n \to \infty$. Assume that for each $n \in \mathbb{N}$, $u_n \in \mathcal{A}^b_{N_n}$. Then

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_{N_n}(x,y)|u_n(x)-u_n(y)|\,dx\,dy - \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega(x,y)|u(x)-u(y)|\,dx\,dy\Big| \\
&= \Big| \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} (\omega_{N_n}(x,y)-\omega(x,y))|u_n(x)-u_n(y)|\,dx\,dy \\
&\qquad + \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega(x,y)\left(|u_n(x)-u_n(y)|-|u(x)-u(y)|\right)\,dx\,dy\Big| \\
&\le I_1 + \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega(x,y)|u_n(x)-u_n(y)-u(x)+u(y)|\,dx\,dy \le I_1 + 2I_2,
\end{aligned}$$

where for simplicity we have used the notation

$$\begin{aligned}
I_1(u) &:= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |\omega_{N_n}(x,y)-\omega(x,y)||u_n(x)-u_n(y)|\,dx\,dy \\
I_2(u) &:= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega(x,y)|u_n(x)-u(x)|\,dx\,dy.
\end{aligned}$$

Since $\omega_{N_n} \to \omega$ uniformly as $n \to \infty$, there is a sequence of constants $C_n > 0$ such that for n large enough $|\omega_{N_n}(x,y) - \omega(x,y)| \le C_n$ and $C_n \to 0$ as $n \to \infty$. Hence,

$$I_1 \le C_n \int_{\mathbb{T}^2}\int_{T^2} |u_n(x)-u_n(y)|\,dx\,dy.$$

Because $u_n \in \mathcal{A}^b_{N_n}$, we have $|u_n(x)-u_n(y)| \le 2$ for almost all $(x,y) \in \mathbb{T}^2\times\mathbb{T}^2$, and hence $I_1 \to 0$ as $n \to \infty$. Furthermore, $I_2 \le \|\omega\|_{L^\infty(\mathbb{T}^2)}\|u_n - u\|_{L^1(\mathbb{T}^2)}$, and thus $I_2 \to 0$ as $n \to \infty$. We conclude that $\lim_{n\to\infty} g_{N_n}(u_n) = g_\infty(u)$ for any sequence $\{u_n\}_{n=1}^\infty$ such that $u_n \in \mathcal{A}^b_{N_n}$ and $u_n \to u$ in $L^1(\mathbb{T}^2)$ as $n \to \infty$. In particular, (LB) is proven. To prove (UB′), all that remains is to show that there exists such a sequence.

First assume $u \in C^\infty(\mathbb{T}^2)$ and define $u_n(x) := u(i/N_n, j/N_n)$, where $i,j \in I_N$ are such that $x \in S^{i,j}_{N_n}$ from (2.2). This sequence satisfies the required conditions; hence, (UB′) is proved for $u \in C^\infty(\mathbb{T}^2)$. We conclude the argument by using the lower semicontinuity of the upper Γ-limit and density of $C^\infty(\mathbb{T}^2)$ in $L^1(\mathbb{T}^2)$ as in the proof of Theorems 5.5 and 5.11 to deduce (UB′) for $u \in L^1(\mathbb{T}^2)$. $\square$

Remark 6.3. Note that g_N does not converge uniformly to g_∞, because if $u \in L^1(\mathbb{T}^2)$ is such that for all $N \in \mathbb{N}$ $u \notin \mathcal{A}^b_N$, then $|g_N(u) - g_\infty(u)| = \infty$.

If we would restrict the domains of g_N and g_∞ to continuous u and define g_N to be

$$g_N(u) := \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_N(x,y)|u_N(x)-u_N(y)|\,dx\,dy,$$

where $u_N(x) = u(i/N, j/N)$ where i and j are such that $x \in S^{i,j}_{N_n}$, then $g_N \to g_\infty$ uniformly by the estimates in the proof of Lemma 6.2. By [28, Proposition 5.2], g_N then Γ-converges to the lower-semicontinuous envelope of g_∞.

Lemma 6.4. (1) *If $\sigma, L > 0$ are fixed, then $\omega_{L,N} \to \omega_{L,\sigma}$ uniformly as $N \to \infty$.*

(2) *If $\sigma = \frac{N}{c}$ for some $c > 0$ and L is such that $L/N \to \ell$ as $N \to \infty$ for some $\ell \in (0, 1/2)$, then $\omega_{L,N} \to \omega_{\ell,c}$ uniformly as $N \to \infty$.*

We defer the relatively straightforward proof to Appendix B.

Proof of Theorem 6.1. Combining Lemmas 6.2 and 6.4, the result follows directly. $\square$

Remark 6.5. It is important to note that for g_N we do *not* have a compactness result in the $L^1(\mathbb{T}^2)$ topology. If we have sequences $\{N_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ such that $N_n \to \infty$ as $n \to \infty$ and $u_n \in \mathcal{A}^b_{N_n}$, the bound on $\|u_n\|_{L^\infty(\mathbb{T}^2)}$ allows us to conclude that $u_n \overset{*}{\rightharpoonup} u$ for some subsequence (labelled again by n) and some $u \in L^\infty(\mathbb{T}^2)$. In order to deduce $L^1(\mathbb{T}^2)$ convergence we would need some information on the derivatives (or finite differences), which we do not have when the weights ω are nonsingular. A uniform bound on $g_{N_n}(u_n)$ adds no useful information since $g_{N_n}(u_n) \le 1$ per definition, if $u_n \in \mathcal{A}^b_{N_n}$.

A simple counterexample is the case where Φ is constant, hence the graph weight function $\omega \equiv 1$. Let $u_N \in \mathcal{A}^b_N$ be a checkerboard pattern on G_N (i.e., as a function in $\mathcal{V}^b_N$, $(u_N)_{0,0} = 0$ and $(u_N)_{i,j} \neq (u_N)_{i+1,j} = (u_N)_{i,j+1}$ for all i, j); then

$$g_N(u_N) = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |u_N(x)-u_N(y)|\,dx\,dy = 2|\operatorname{supp} u_N|\ |(\operatorname{supp} u_N)^c| \le 2.$$

However, no subsequence of $\{u_N\}_N$ converges in $L^1(\mathbb{T}^2)$, as can be seen as follows. Let $M \gg N$; then the square $S^{i,j}_N$ contains $\mathcal{O}\Big(\big(\frac{M}{N}\big)^2\Big)$ squares of size M^{-1} by M^{-1}. On approximately half (at least $\mathcal{O}(1)$) of these $u_M \neq u_N$, so $\displaystyle\int_{\mathbb{T}^2} |u_N - u_M| = N^2 M^{-2} \mathcal{O}\Big(\big(\frac{M}{N}\big)^2\Big)\mathcal{O}(1) = \mathcal{O}(1).$

7. Discussion and open questions

In this paper we have shown various Γ-convergence results. The convergence of f_ε in Section 3 shows that we can extend the classical Modica–Mortola Γ-convergence result for the Ginzburg–Landau functional to graphs, if we are careful about the precise scaling. The discrete nature of a graph forces us to not include an ε in the finite-difference term, unlike the ε in the gradient term for the continuum Ginzburg–Landau functional. As has been shown on a specific regular square grid in Section 4, this has consequences for the limit functional, which now behaves like an anisotropic instead of isotropic total variation. We recovered the isotropic total variation for the regular grid case in Section 5 by taking an approach reminiscent of classical numerical analytic results, instead of graph-based results. Specifically, to do this we need to choose a scaling in line with standard finite-difference and quadrature methods and make the limit $N \to \infty$ dominant over the limit $\varepsilon \to 0$ such that, in a sense, we first get back to the continuum case, before passing to the total variation. The lesson in here is twofold. On the one hand it shows that one has to be careful when discretizing on a grid not to pick up grid direction, which has been known to numerical analysts for a long time. On the other hand, however, it entices us to look at graph-based functionals and nonlinear partial differential equations in their own right, because they can behave in surprising ways if the topology of the graph is allowed to interact with the functional or PDE. This conclusion is reminiscent of the behavior which is found in [41]. In that paper the authors study the limit of the graph Laplacian on a graph which is constructed by sampling points from a manifold. They find the limit is independent of the sampling distribution only for a specific scaling of the graph Laplacian.

In Section 6 we studied the limit of a functional of nonlinear means type on graphs, showing that while the limit exists, the nonlocal nature of the functional leads to loss of compactness. This is not expected to be a specific problem of the graph-based nature of the functional, but of the nonlocality, and as such is expected to be present for a continuum version of g_N as well.

One question raised in Section 5.3 is whether the range of α under which Γ-convergence and compactness of k_N^α can be proven, can be extended to $(0, 1)$. This is an important question in practice when running gradient flow simulations. A choice of ε which is too small or large can lead to either pinning or too-fast diffusion respectively.

The Γ-convergence results for $h_{N,\varepsilon}$ and $k_{N,\varepsilon}$ naturally lead to the question of the limit behavior for other graphs. In order to have a good interpretation for that question it is in the first place necessary to have a structured way in which to increase m, the number of nodes for the graph. A triangulation might be the natural next step. A random Erdős–Rényi graph [30] also carries a natural rule on how to connect new nodes to the graph and may be an interesting exploration into the question whether the Ginzburg–Landau functional on a graph without explicit spatial embedding can possibly have "continuum" limit. For arbitrary graphs it is less clear how to add new nodes in a structured way. One option could be to construct a sequence of graphs where in each next step each existing edge is bisected by a new node.

A question that is very relevant for the applications of the Ginzburg–Landau functional is that of stability of minimizers with respect to perturbations of the graph (*e.g.,* perturb the weights or add or delete nodes). For example, in data analysis, if the nodes represent data points and the edge weights measure similarity, it is quite likely that noise is present in the weights.

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Appendix

A. Choice of Hilbert space and difference structure

In this section we will give some background information and derivations concerning the choices made in Section 2.2 that defined our graph operators and functionals.

We start by associating $\mathcal{V}$ and $\mathcal{E}$ with the finite-dimensional vector spaces $\mathbb{R}^m$ and $\mathbb{R}^{m(m-1)/2}$ respectively. We will turn these vector spaces into Hilbert spaces by defining inner products on them. We follow the procedure described in [41, Section 2][12]. For $u, v \in \mathcal{V}$ and $\varphi, \phi \in \mathcal{E}$ we define

$$\langle u, v\rangle_{\mathcal{V}} := \sum_{i\in I_m} u_i v_i \alpha(d_i), \qquad \langle \varphi, \phi\rangle_{\mathcal{E}} := \frac{1}{2}\sum_{i,j\in I_m} \varphi_{ij}\phi_{ij}\beta(\omega_{ij}),$$

[12]We slightly deviate from [41]. Instead of sums $\sum_{i=1}^m$ they use averages $\frac{1}{m}\sum_{i=1}^m$, which is a choice not unanimously adopted in the literature, but which leads to cleaner convergence statements in [41]. We could adopt this convention in this paper, but it would not significantly alter our results, *mutatis mutandis.*

where $\alpha, \beta : [0,\infty) \to [0,\infty)$ are functions yet to be determined. Note that *a priori* we allow α and β to take the value zero, which means that the above "inner products" might be positive semidefinite and not positive definite. We will get back to this issue after we have decided on our choices of α and β[13].

As in [35] we also define the dot product for $\varphi, \phi \in \mathcal{E}$ as

$$(\varphi \cdot \phi)_i := \frac{1}{2} \sum_{j\in I_m} \varphi_{ij}\phi_{ij}\beta(\omega_{ij}).$$

Note that $\varphi \cdot \phi \in \mathcal{V}$. As explained in [41] we can now define the difference operator or gradient $\nabla : \mathcal{V} \to \mathcal{E}$ as

$$(\nabla u)_{ij} := \gamma(\omega_{ij})(u_j - u_i),$$

where $\gamma : [0,\infty] \to [0,\infty]$ is a third yet-to-be-determined function. With this choice for the gradient we find that its adjoint, the divergence $\operatorname{div} : \mathcal{E} \to \mathcal{V}$, is given by [41, Lemma 3]:

$$(\operatorname{div} \varphi)_i := \frac{1}{2\alpha(d_i)} \sum_{j\in I_m} \beta(\omega_{ij})\gamma(\omega_{ij})(\varphi_{ji} - \varphi_{ij}).$$

This expression follows from the defining property of the adjoint: $\langle \nabla u, \varphi\rangle_{\mathcal{E}} = \langle u, \operatorname{div}\varphi\rangle_{\mathcal{V}}$ for all $u \in \mathcal{V}$ and all $\varphi \in \mathcal{E}$.

Now that we have inner products, a gradient operator, and a divergence operator, we can define the following objects:

- Inner product norms $\|u\|_{\mathcal{V}} := \sqrt{\langle u, u\rangle_{\mathcal{V}}}$ and $\|\varphi\|_{\mathcal{E}} := \sqrt{\langle \varphi, \varphi\rangle_{\mathcal{E}}}$.
- Maximum norms[14] $\|u\|_{\mathcal{V},\infty} := \max\{|u_i| : i \in I_m\}$ and $\|\varphi\|_{\mathcal{E},\infty} := \max\{|\varphi_{ij}| : i, j \in I_m\}$.
- The norm corresponding to the dot product $\|\varphi\|_i := \sqrt{(\varphi\cdot\varphi)_i}$. Note that $\|\cdot\|_{\mathcal{E},\mathrm{dot}} \in \mathcal{V}$.
- The Dirichlet energy $\frac{1}{2}\|\nabla u\|_{\mathcal{E}}^2$.

[13]Note that if α and β are such that positive definiteness is satisfied, these inner products do indeed turn $\mathcal{V}$ and $\mathcal{E}$ into Hilbert spaces, since convergence with respect to the induced $\mathcal{E}$ norm preserves skew-symmetry.

[14]To justify these definitions and convince ourselves that there should be no β or γ included in the maximum norms we define $\|\varphi\|_{\mathcal{E},p}^p := \frac{1}{2}\sum_{i,j\in I_m} \varphi_{ij}^2\beta(\omega_{ij})$. Adapting the proofs in the continuum case in *e.g.* [1, Theorems 2.3 and 2.8] to the graph situation, we can prove a Hölder inequality $\|\varphi\phi\|_{\mathcal{E},1} \le \|\varphi\|_{\mathcal{E},p}\|\phi\|_{\mathcal{E},q}$ for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, an embedding theorem of the form $\|\varphi\|_{\mathcal{E},p} \le \left(\frac{1}{2}\sum_{i,j\in I_m} \beta(\omega_{ij})\right)^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_{\mathcal{E},q}$ for $1 \le p \le q \le \infty$, and the limit $\lim_{p\to\infty} \|\varphi\|_{\mathcal{E},p} = \|\varphi\|_{\mathcal{E},\infty}$. A similar result holds for the norms on $\mathcal{V}$.

• The graph Laplacian $\Delta := \operatorname{div} \circ \nabla : \mathcal{V} \to \mathcal{V}$. So

$$(\Delta u)_i := \frac{1}{\alpha(d_i)} \sum_{j \in I_m} \beta(\omega_{ij}) \gamma^2(\omega_{ij})(u_i - u_j).$$

• The isotropic and anisotropic total variation $TV : \mathcal{V} \to \mathbb{R}$ and $TV_a : \mathcal{V} \to \mathbb{R}$ respectively:

$$\mathrm{TV}(u) := \max\{\langle \operatorname{div} \varphi, u \rangle_{\mathcal{V}} : \varphi \in \mathcal{E},\ \max_{i \in I_m} \|\varphi\|_i \leq 1\}.$$
$$\mathrm{TV_a}(u) := \max\{\langle \operatorname{div} \varphi, u \rangle_{\mathcal{V}} : \varphi \in \mathcal{E},\ \|\varphi\|_{\mathcal{E},\infty} \leq 1\}.$$

We note that by the property of the adjoint we can also use $\langle \nabla u, \varphi \rangle_{\mathcal{E}}$ in the definitions above instead of $\langle \operatorname{div} \varphi, u \rangle_{\mathcal{V}}$. In this finite-dimensional setting these maxima over unit balls will be achieved; hence, we are justified in using max instead of sup.

Before we start making specific choices for α, β, and γ, it is interesting to make some general observations which do not depend on these choices.

• We can alternatively derive the Laplacian via the variational principle from the Dirichlet energy as follows. Consider $u, v \in \mathcal{V}$ and $t \in \mathbb{R}$; then

$$\begin{aligned}
\frac{d}{dt} \frac{1}{2} \|\nabla u + tv\|_{\mathcal{E}}^2 \bigg|_{t=0} &= \frac{1}{2} \sum_{i,j \in I_m} \beta(\omega_{ij}) \gamma^2(\omega_{ij})(u_i - u_j)(v_i - v_j) \\
&= \sum_{i,j \in I_m} \beta(\omega_{ij}) \gamma^2(\omega_{ij})(u_i - u_j) v_i \\
&= \sum_{i,j \in I_m} \frac{\beta(\omega_{ij}) \gamma^2(\omega_{ij})}{\alpha(d_i)} (u_i - u_j) v_i \alpha(d_i) = \langle \Delta u, v \rangle_{\mathcal{V}}.
\end{aligned}$$

If we choose $v = u$, this also shows that an analogue of "integration by parts" holds:

$$\langle \Delta u, u \rangle_{\mathcal{V}} = \frac{1}{2} \sum_{i,j \in I_m} \beta(\omega_{ij}) \gamma^2(\omega_{ij})(u_i - u_j)^2 = \|\nabla u\|_{\mathcal{E}}^2.$$

• We can also give a variational TV-type formulation of the Dirichlet energy itself via

$$\|\nabla u\|_{\mathcal{E}} = \max\{\langle \operatorname{div} \varphi, u \rangle_{\mathcal{V}} : \varphi \in \mathcal{E},\ \|\varphi\|_{\mathcal{E}} \leq 1\}.$$

To see why this holds we first remember that $\langle \nabla u, \varphi \rangle_{\mathcal{E}} = \langle \operatorname{div} \varphi, u \rangle_{\mathcal{V}}$. Then we see that by the Cauchy–Schwarz inequality,

$$\langle \nabla u, \varphi \rangle_{\mathcal{E}} \leq \|\nabla u\|_{\mathcal{E}} \|\varphi\|_{\mathcal{E}} \leq \|\nabla u\|_{\mathcal{E}}.$$

Equality is achieved when

$$\varphi = \varphi^{\mathcal{E}}(u) := \begin{cases} \frac{\nabla u}{\|\nabla u\|_{\mathcal{E}}} & \text{if } \|\nabla u\|_{\mathcal{E}} \neq 0, \\ 0 & \text{if } \|\nabla u\|_{\mathcal{E}} = 0, \end{cases}$$

which is permissible since $\|\varphi^{\mathcal{E}}(u)\|_{\mathcal{E}} \leq 1$.

We will now make particular choices for α, β, and γ. Our choices will be driven by the desire to satisfy the following properties.

(1) We will consider a family of graph Laplacians indexed by a parameter $r \leq 1$ (not to be confused with the p-Laplacian from the literature). As it turns out, only the choice of α is influenced by the choice of r. The Laplacians we consider are

$$(\Delta_r u)_i := d_i^{1-r} u_i - \sum_{j \in I_m} \frac{\omega_{ij}}{d_i^r} u_j = \sum_{j \in I_m} \frac{\omega_{ij}}{d_i^r}(u_i - u_j).$$

As explained in Section 2.2, by choosing either $r = 0$ or $r = 1$ we recover the unnormalized or random-walk Laplacian respectively. To construct the symmetric normalized Laplacian as it appears in the literature requires a gradient of the form

$$(\nabla u)_{ij} = \gamma(\omega_{ij})\Big(\frac{u_j}{\sqrt{d_j}} - \frac{u_i}{\sqrt{d_i}}\Big)$$

(*cf.* [41]). This falls outside our current framework, and hence we will not consider it here.[15] Some discussion of the pros and cons of different graph Laplacians can be found in *e.g.* [43, 58].

(2) The Dirichlet energy is given by $\frac{1}{2}\|\nabla u\|_{\mathcal{E}}^2 = \frac{1}{4}\sum_{i,j \in I_m} \omega_{ij}(u_i - u_j)^2$, independently of the choice of r in the Laplacian.

(3) The isotropic total variation is

$$\mathrm{TV}(u) = \sum_{i \in I_m} \|\nabla u\|_i = \frac{1}{2}\sqrt{2} \sum_{i \in I_m} \sqrt{\sum_{j \in I_m} \omega_{ij}(u_i - u_j)^2}$$

(*cf.* [35] where TV is called nonlocal TV because the graph is assumed to be embedded in a Euclidean space and so what is local on the graph (neighboring vertices) might not be local in the embedding space).

[15] As a word of caution we note that the use of the symmetric Laplacian in Allen-Cahn type equations in combination with a double-well potential W with wells that are not symmetrically placed around 0 (as in the case where the wells are at 0 and 1) causes an asymmetry between the two phases that is typically unwanted.

(4) We will consider a family of anisotropic total variations parametrized by the parameter $q \in [1/2, 1]$[16]:

$$\mathrm{TV}_{\mathrm{a}q}(u) = \langle \nabla u, \mathrm{sgn}(\nabla u)\rangle_{\mathcal{E}} = \frac{1}{2}\sum_{i,j\in I_m} \omega_{ij}^q |u_i - u_j|.$$

The parameter q comes in via the definitions of β and γ and the signum function is understood to act element-wise on the elements of ∇u.

Let us consider all the points above to find out what conditions we have to impose on α, β, and γ to satisfy this list of requirements.

(1) As can be seen in the definition of the Laplacian (*cf.* also [41, Definition 7]), in order to get the desired Laplacians we have to choose α, β, and γ such that for each ω_{ij} and each d_i,

$$\frac{\beta(\omega_{ij})\gamma^2(\omega_{ij})}{\alpha(d_i)} = \frac{\omega_{ij}}{d_i^r}.$$

Specifically, $\beta(\omega_{ij})\gamma^2(\omega_{ij}) = \omega_{ij}$ for any choice of r and $\alpha(d_i) = d_i^r$. We will see below that the choice of α is irrelevant for the points 2–4, and hence all choices of r are compatible with what follows.

(2) For the Dirichlet energy we compute

$$\frac{1}{2}\|\nabla u\|_{\mathcal{E}}^2 = \frac{1}{4}\sum_{i,j\in I_m}(u_i - u_j)^2\beta(\omega_{ij})\gamma^2(\omega_{ij}) = \frac{1}{4}\sum_{i,j\in I_m}\omega_{ij}(u_i - u_j)^2.$$

Since the graph Laplacian appears as the natural operator in the Euler–Lagrange equation associated with the Dirichlet energy, it is not surprising that we do not get any extra conditions on α, β, or γ from the Dirichlet energy that we didn't already get from the Laplacian. It is interesting to note, though, that the Dirichlet energy does not depend on the choice of α (and hence r in the Laplacian) at all.

(3) For the isotropic total variation TV we use the Cauchy–Schwarz inequality on the dot product norm to get $\langle \nabla u, \varphi\rangle_{\mathcal{E}} = \sum_{i\in I_m}(\nabla u \cdot \varphi)_i \leq \sum_{i\in I_m}\|\nabla u\|_i\|\varphi\|_i \leq \sum_{i\in I_m}\|\nabla u\|_i$. To achieve equality[17] let $\varphi_{ij} = \varphi_{ij}^{\mathrm{TV}}(u) := \begin{cases} \frac{(\nabla u)_{ij}}{\|\nabla u\|_i} & \text{if } \|\nabla u\|_i \neq 0, \\ 0 & \text{if } \|\nabla u\|_i = 0 \end{cases}$. Again, we do not require

[16]We can take $q \in \mathbb{R}$ if we interpret ω_{ij}^q as zero whenever $\omega_{ij} = 0$.

[17]Note that demanding φ^{TV} to achieve equality does not determine it uniquely on the set of vertices for which $\|\nabla u\|_i = 0$.

extra conditions on β and γ. They will be determined by the last requirement[18].

(4) To compute $\mathrm{TV_a}$ we use the bound on the maximum norm of φ to find

$$\begin{aligned}\langle \nabla u, \varphi\rangle_{\mathcal{E}} &= \frac{1}{2}\sum_{i,j\in I_m} \varphi_{ij}(u_j-u_i)\beta(\omega_{ij})\gamma(\omega_{ij})\\ &\le \frac{1}{2}\sum_{i,j\in I_m} |\varphi_{ij}||u_i-u_j|\beta(\omega_{ij})\gamma(\omega_{ij})\\ &\le \frac{1}{2}\sum_{i,j\in I_m} |u_i-u_j|\beta(\omega_{ij})\gamma(\omega_{ij}).\end{aligned}$$

To achieve equality we can choose $\varphi=\varphi^a := \mathrm{sgn}(\nabla u)$; *i.e.*, $\varphi^a_{ij} = \mathrm{sgn}(u_j-u_i)$ for i and j such that $\gamma(\omega_{ij})>0$ and $\varphi^a_{ij}=0$ otherwise[19]. Hence

$$\mathrm{TV_a}(u) = \frac{1}{2}\sum_{i,j\in I_m} |u_i-u_j|\beta(\omega_{ij})\gamma(\omega_{ij}).$$

If we now choose $\beta(\omega_{ij}) = \omega_{ij}^{2q-1}$ and $\gamma(\omega_{ij}) = \omega_{ij}^{1-q}$, then $\mathrm{TV_a} = \mathrm{TV_{a\mathit{q}}}$ while β and γ satisfy the necessary condition $\beta(\omega_{ij})\gamma^2(\omega_{ij}) = \omega_{ij}$ from the previous points.

These choices for α, β, and γ lead to the inner products (or semidefinite sesquilinear forms), operators, and functions presented in Section 2.2.

It is interesting to consider the conditions under which $\|\nabla u\|_{\mathcal{E}}=0$ and $\|\nabla u\|_i = 0$:

$$\|\nabla u\|_{\mathcal{E}} = 0 \Leftrightarrow \sum_{i,j\in I_m} \omega_{ij}(u_i-u_j)^2 \Leftrightarrow \forall (i,j)\in I_m^2\ [\omega_{ij}=0 \vee u_i=u_j].$$

This means that $\|\nabla u\|_{\mathcal{E}}=0$ if and only if u is constant on connected components of the graph. Similarly,

$$\|\nabla u\|_i = 0 \Leftrightarrow \sum_{j\in I_m} \omega_{ij}(u_i-u_j)^2 \Leftrightarrow \forall j\in I_m\ [\omega_{ij}=0 \vee u_i=u_j];$$

[18]If we had defined the dot product $(\varphi\cdot\phi)_i := \frac{1}{2}\sum_{j\in I_m}\varphi_{ij}\phi_{ij}\delta(\omega_{ij})$ for a function δ possibly different than β, the requirement on TV would have led to the condition $\frac{\beta^2(\omega_{ij})\gamma^2(\omega_{ij})}{\delta(\omega_{ij})} = \omega_{ij}$. Together with $\beta(\omega_{ij})\gamma^2(\omega_{ij}) = \omega_{ij}$ from point 1 this gives $\delta=\beta$ as we have assumed all along.

[19]Note that we can change φ^a on the set of vertices for which $\nabla u = 0$ without losing equality. See also footnote 17.

hence, $\|\nabla u\|_i = 0$ if and only if u is constant on the set $\{n_j \in V : j = i \vee e_{ij} \in E\}$ consisting of neighboring vertices of the i^{th} vertex plus the i^{th} vertex itself.

We see that these conditions do what we would hope and expect them to do, even if the choice of q has made the $\mathcal{E}$-sesquilinear form semidefinite; *i.e.*, $\|\nabla u\|_{\mathcal{E}} = 0$ gives global (per connected component) constants and $\|\nabla u\|_i = 0$ gives local constancy.

B. Deferred proofs

The next lemma was used in the proof of Lemma 5.4.

Lemma B.1. *Let* $\{u_n\}_{n=1}^\infty \subset L^1(\mathbb{T})$ *be a such that* $u_n \to u$ *in* $L^1(\mathbb{T})$ *for a* $u \in L^1(\mathbb{T}^2)$ *and* $u_n' \rightharpoonup v$ *in* $L^2(\mathbb{T})$ *for a* $v \in L^2(\mathbb{T})$. *Then* $v = u'$ *(and thus* $u \in W^{1,2}(\mathbb{T})$*).*

Note in the proof below that this result in fact does not depend on the dimension and holds on $\mathbb{T}^d$.

Proof of Lemma B.1. Let $\varphi \in C_c^\infty(\mathbb{T})$; then

$$0 = \lim_{n\to\infty} \int_{\mathbb{T}} \varphi(v - u_n') = \lim_{n\to\infty} \int_{\mathbb{T}} \left[\varphi v - \varphi' u - \varphi u_n' + \varphi' u\right].$$

There is a $C > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{T}} \left[-\varphi' u - \varphi u_n'\right] = \lim_{n\to\infty} \int_{\mathbb{T}} \varphi' \left[u_n - u\right] \le \lim_{n\to\infty} C\|u_n - u\|_{L^1(\mathbb{T})} = 0;$$

hence, we conclude $\int_{\mathbb{T}} \varphi v = -\int_{\mathbb{T}} \varphi' u$. □

The next lemma is a discrete Rellich–Kondrachov-type compactness result used in the proof of Theorem 5.3.

Lemma B.2 (Discrete Rellich-Kondrachov compactness result)**.** *Let* $\{u_n\}_{n=1}^\infty \subset L^2(\mathbb{T}^2)$ *and* $\{N_n\}_{n=1}^\infty \subset (0,\infty)$ *be sequences such that as* $n \to \infty$ *we have* $N_n \to \infty$, $u_n \rightharpoonup u$ *in* $L^2(\mathbb{T}^2)$ *for some* $u \in L^2(\mathbb{T}^2)$, *and the difference quotients* (*see* (5.4)) $\|D_{N_n}^k u_n\|_{L^2(\mathbb{T}^2)}$ $(k \in \{1,2\})$ *are uniformly bounded. Then* $u_n \to u$ *in* $L^2(\mathbb{T}^2)$.

Proof. For $\varepsilon > 0$, let $u_n^\varepsilon := J_\varepsilon u_n \in C^\infty(\mathbb{T}^2)$ be a mollified function on the torus, defined to be the solution to the heat equation after time ε with initial condition u_n:

$$J_\varepsilon u_n(x) = \sum_{k\in\mathbb{Z}^2} \widehat{u_n}(k) e^{-\varepsilon^2|k|^2 + 2\pi i k\cdot x},$$

where $\widehat{u_n}(k) = \int_{\mathbb{T}^2} u_n(x)e^{-2\pi i k\cdot x}\,dx$. We proceed in two steps. First we need to prove some properties of the mollifier, then we will prove the statement of the lemma.

Step 1: From [38, Appendix B] and [39, Lemma1], we get that there is a $C>0$ such that $\|u_n^\varepsilon\|_{L^2(\mathbb{T}^2)} \le C\|u_n\|_{L^2(\mathbb{T}^2)}$. Additionally, J_ε is a linear operator. [38, 39] also give $\|J_\varepsilon f - f\|_{L^2(\mathbb{T}^2)} \le \varepsilon\|f\|_{H^1(\mathbb{T}^2)}$ for $f\in H^1(\mathbb{T}^2)$. u_n is not regular enough to use this estimate, so we need a discrete version of this.

As in the references above, using that $1-e^{-\theta^2}\le|\theta|^2$ for $\theta\in\mathbb{C}$, we find that

$$\Big|\frac{(1-e^{-\varepsilon^2|k|^2})^2}{1+|k|^2}\Big| < \begin{cases} \varepsilon^2\delta^2 & \text{if } |k|<\delta,\\ \delta^{-2} & \text{if } |k|\ge\delta.\end{cases}$$

Hence,

$$\begin{aligned}\|u_n^\varepsilon - u_n\|^2_{L^2(\mathbb{T}^2)} &= \sum_{k\in\mathbb{Z}^2}\big(1-e^{-\varepsilon^2|k|^2}\big)^2|\widehat{u_n}(k)|^2\\ &\le \Big(\sup_{k\in\mathbb{Z}^2}\Big|\frac{(1-e^{-\varepsilon^2|k|^2})^2}{1+|k|^2}\Big|\Big)\sum_{k\in\mathbb{Z}^2}\big(1+|k|^2\big)\,|\widehat{u_n}(k)|^2\\ &\le \varepsilon^2\sum_{k\in\mathbb{Z}^2}\big(1+|k|^2\big)\,|\widehat{u_n}(k)|^2.\end{aligned}$$

By Plancherel's/Parseval's identity we get immediately $\sum_{k\in\mathbb{Z}^2}|\widehat{u_n}(k)|^2 = \|u_n\|_{L^2(\mathbb{T}^2)}$. Furthermore,

$$\begin{aligned}\|D_{N_n}u_n\|^2_{L^2(\mathbb{T}^2)} &= \int_{\mathbb{T}^2}\Big[\big(D^1_{N_n}u_n(x)\big)^2 + \big(D^2_{N_n}u_n(x)\big)^2\Big]\,dx\\ &= N_n^2\sum_{k\in\mathbb{Z}^2}\Big(\Big|\widehat{(u_n^{+1}-u_n)}(k)\Big|^2 + \Big|\widehat{(u_n^{+2}-u_n)}(k)\Big|^2\Big),\end{aligned}$$

where $u_n^{+j}(x) := u_n(x+N_ne_j)$ for standard basis vectors e_j, $j\in\{1,2\}$. It's easily computed that $\widehat{u_n^{+j}}(k) = \widehat{u_n}(k)e^{2\pi i k_j/N_n}$; hence,

$$\|D_{N_n}u_n\|^2_{L^2(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2}|\widehat{u_n}(k)|^2\Big[\Big|N_n(e^{2\pi i k_1/N_n}-1)\Big|^2 + \Big|N_n(e^{2\pi i k_2/N_n}-1)\Big|^2\Big].$$

Recognizing the difference quotient

$$N_n\big(e^{2\pi i k_1/N_n}-1\big) = 2\pi i k_1\frac{e^{2\pi i k_1/N_n}-e^0}{2\pi i k_1/N_n} = 2\pi i k_1 + \mathcal{O}(N_n^{-1}),$$

we deduce

$$\|D_{N_n}u_n\|^2_{L^2(\mathbb{T}^2)} = 4\pi^2 \sum_{k\in\mathbb{Z}^2} |k|^2|\widehat{u_n}(k)|^2 + C_n \sum_{k\in\mathbb{Z}^2} |\widehat{u_n}(k)|^2,$$

where $C_n = \mathcal{O}(N_n^{-1})$. On the last term we can use again Parseval's formula. By the uniform bounds on $\|u_n\|_{L^2(\mathbb{T}^2)}$ and $\|D_{N_n}u_n\|_{L^2(\mathbb{T}^2)}$ we then find that

$$\|u_n^\varepsilon - u_n\|^2_{L^2(\mathbb{T}^2)} \le C\varepsilon^2, \quad \text{uniformly in } n \text{ for } n \text{ large enough.} \tag{B1}$$

Next we compute an estimate on the derivatives of u_n^ε:

$$\frac{\partial}{\partial x_1} u_n^\varepsilon(x) = 2\pi i \sum_{k\in\mathbb{Z}^2} k_1 \hat{u}_n(k) e^{-\varepsilon^2|k|^2+2\pi i k\cdot x},$$

and hence, since $|\hat{u}_n(k)| \le \|u_n\|_{L^1(\mathbb{T}^2)}$,

$$\left|\frac{\partial}{\partial x_1} J_\varepsilon u_n(x)\right| \le 2\pi\|v\|_{L^1(\mathbb{T}^2)} \sum_{k\in\mathbb{Z}^2} |k_1| e^{-\varepsilon^2|k|^2}.$$

We compute

$$\sum_{k_2\in\mathbb{Z}} e^{-\varepsilon^2 k_2^2} \le 2\sum_{k_2=0}^{\infty} e^{-\varepsilon^2 k_2^2} \le 2\int_0^\infty e^{-\varepsilon^2 k_2^2}\, dk_2 + 1 = \sqrt{\pi}\varepsilon^{-1} + 1$$

and

$$\sum_{k_1\in\mathbb{Z}} |k_1| e^{-\varepsilon^2 k_1^2} = 2\sum_{k_1=0}^{\infty} k_1 e^{-\varepsilon^2 k_2^2} \le 2\int_0^\infty k_1 e^{-\varepsilon^2 k_1^2} dk_1 = \varepsilon^{-2}\int_0^\infty x^2 e^{-x^2} dx = \varepsilon^{-2}.$$

Because $\|u_n\|_{L^1(\mathbb{T}^2)} \le \|u_n\|_{L^2(\mathbb{T}^2)}$ is uniformly bounded, we conclude (for ε small enough) there is a $C > 0$ such that $\|\nabla J_\varepsilon u_n(x)\cdot e_k\|_{L^\infty(\mathbb{T}^2)} \le C\varepsilon^{-3}$, $k \in \{1, 2\}$.

Step 2: Let $\eta > 0$, and let n be large enough for the bounds proved in Step 1 to hold. Fix $\varepsilon > 0$ small enough such that, by (B1), for each n we have $\|u_n - u_n^\varepsilon\|_{L^2(\mathbb{T}^2)} \le \eta/3$.

By the bounds from Step 1, both $\|u_n^\varepsilon\|_{L^2(\mathbb{T}^2)}$ and $\|\nabla J_\varepsilon u_n(x)\cdot e_k\|_{L^2(\mathbb{T}^2)}$ are uniformly (in n, for fixed ε) bounded, and hence by the Rellich–Kondrachov compactness theorem [31, Section 5.7 Theorem 1], [1, Theorem 6.2] the sequence $\{u_n^\varepsilon\}_{n=1}^\infty$ converges strongly in $L^2(\mathbb{T}^2)$ as $n \to \infty$. In particular, it is a Cauchy sequence in $L^2(\mathbb{T}^2)$, so choose $M_\varepsilon > 0$ such that for all $n, m > M_\varepsilon$ we have $\|u_n^\varepsilon - u_m^\varepsilon\|_{L^2(\mathbb{T}^2)} \le \eta/3$. Then for such n and m,

$$\|u_n - u_m\|_{L^2(\mathbb{T}^2)} \le \|u_n - u_n^\varepsilon\|_{L^2(\mathbb{T}^2)} + \|u_m - u_m^\varepsilon\|_{L^2(\mathbb{T}^2)} + \|u_n^\varepsilon - u_m^\varepsilon\|_{L^2(\mathbb{T}^2)} \le \eta.$$

Thus, $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{T}^2)$ and therefore converges strongly in $L^2(\mathbb{T}^2)$. By the uniqueness of the limit it converges to u. □

Proof of Lemma 6.4. In both cases we assume without loss of generality that N is large enough such that $\frac{L}{N} < \frac{1}{2}$. For fixed $N > 0$ and $i, j, k, l \in I_{N^2}$, let $x = (i/N, j/N)$ and $y = (k/N, j/N)$. For $z \in \mathbb{T}^2$, write $f^{x,y}(z) = \big(f(x-z) - f(y-z)\big)^2$; then from (1.7),

$$(d^2_{L,N})_{i,j,k,l} = \sum_{r,s=-L}^{L} f^{x,y}(r/N, s/N).$$

Also define $S_{L,N} := \{z \in \mathbb{T}^2 : |z_1| + |z_2| \leq L/N\}$. By repeated use of the trapezoidal rule for approximating integrals, we then find

$$\begin{aligned} N^{-2}(d^2_{L,N})_{i,j,k,l} &= \int_{S_{L,N}} f^{x,y}(z)\,dz - \frac{L}{6N^3}\int_{-L/N}^{L/N} \frac{\partial^2 f^{x,y}}{\partial z_2^2}(z_1, \zeta_2)\,dz_1 \\ &\quad - \frac{L}{6N^4}\sum_{s=-L}^{L} \frac{\partial^2 f^{x,y}}{\partial z_1^2}(\zeta_1, s/N) \\ &=: \int_{S_{L,N}} f^{x,y}(z)\,dz + R_{L,N}, \end{aligned} \tag{B2}$$

where $(\zeta_1, \zeta_2) \in S_{L,N}$. By smoothness of f and compactness of $\mathbb{T}^2$, we have $|R_{L,N}| \leq C_f \frac{L^2}{N^4}$ for some constant $C_f > 0$ depending on f.

For the first statement in the lemma we now find

$$\begin{aligned} (d^2_{L,N})_{i,j,k,l} &= N^2 \int_{S_{L,N}} f^{x,y}(z)\,dz + N^2 R_{L,N} = \frac{4L^2}{|S_{L,N}|}\int_{S_{L,N}} f^{x,y}(z)\,dz + N^2 R_{L,N} \\ &\to 4L^2 f^{x,y}(0) = 4L^2\big(f(x) - f(y)\big)^2 \quad \text{uniformly as } N \to \infty. \end{aligned}$$

The uniformity of the convergence follows by the bound of the smooth f on the compact domain $\mathbb{T}^2$. This proves the claim (since the composition of a continuous function and a uniformly converging sequence of functions is uniformly converging to the composition of the continuous function and the limit of the sequence).

For the second statement the bound on f allows us to conclude that

$$\int_{S_{L,N}} f^{x,y}(z)\,dz \to \int_{S_\ell} f^{x,y}(z)\,dz \quad \text{uniformly as } N \to \infty.$$

The claim then follows by taking the limit $N \to \infty$ in (B2). □

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