Crime Modeling with Lévy Flights

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Figure: Crime Hot-Spot Pattern, Long Beach, LA. Short et al. 2010 [2]
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  • Considerable empirical evidence behind them.

• We focus on burglaries for simplicity.
Hotspot Modeling: Theory

- What causes hotspots?
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  - Repeat/ Near Repeat Effects: increased knowledge of location after successful crime.

*Figure: Repeat Effects. (Short et al. 2009, [1])*
Hotspot Modeling: Theory

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- Broken Windows Theory: crime causes sense of lawlessness.
The Short et al Model (2008)

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  - Criminals are agents on a lattice that has “attractiveness” field

- Every time step, criminals move to neighboring lattice spaces based on attractiveness.
- Then decide to burgle or not based off attractiveness.
- Crimes are self-exciting; cause attractiveness at lattice point and neighbors to increase.
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  • Then decide to burgle or not based off attractiveness.
  • Crimes are self-exciting; cause attractiveness at lattice point and neighbors to increase.
Then, taking limits as grid spacing and time steps go to zero, we get system of PDEs

The Model

\[ A_t = \eta \Delta A - A + \rho A + A_0 \quad (1a) \]
\[ \rho_t = \left( A \Delta \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta A \right) - \rho A + \bar{A} - A_0 \quad (1b) \]

\( A = \) attractiveness at a point
\( \rho = \) criminal density at a location
\( A_0 = \) a constant, background level of attractiveness
\( \bar{A} = \) the spatially homogeneous equilibrium solution for \( A \).
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Lévy Flights

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  - Power law distribution of step sizes
    \[ P(k) \sim k^{-(2s+1)}, 0 < s \leq 1 \]
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  - Power law distribution of step sizes
    \[ P(k) \sim k^{-(2s+1)}, 0 < s \leq 1 \]
  - Fractal-like motion; reflects many scales of human movement.
• We change the transition probability as follows:

\[
p_{i \rightarrow i+1}(t) = \frac{A_{i+1}(t)}{A_{i+1}(t) + A_{i-1}(t)} \quad \text{(Brownian)}
\]

\[
p_{i \rightarrow j}(t) = \frac{A_j(t)|i-j|^{-(2s+1)}}{\sum_{k \in \mathbb{Z}, k \neq i} A_k(t)|i-k|^{-(2s+1)}} \quad \text{(Lévy)}
\]
• From before we had:

\[ A_t = \eta \Delta A - A + \rho A + A_0, \]

\[ \rho_t = \left( A \Delta \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta A \right) - \rho A + \bar{A} - A_0. \]
Implementation: Continuous

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\]

- Using our new transition probabilities and taking limits as before we get:

\[
A_t = \eta \Delta A - A + \rho A + A_0 \quad \text{(No change)} \\
\rho_t = \left( A \Delta^s \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta^s A \right) - \rho A + \bar{A} - A_0,
\]

where \( \Delta^s A = c_s \int_{-\infty}^{\infty} A(y) - A(x) |y-x|^{s+1} dy \).
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where

\[ \Delta^s A = c_s \int_{-\infty}^{\infty} \frac{A(y) - A(x)}{|y - x|^{2s+1}} dy, \quad (2) \]

\[ c_s \text{ is a constant and } 0 < s \leq 1. \]
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• The operator $\Delta^s$ is the Riesz Derivative,
  • Generalization of the second derivative such that $\lim_{s \to 1} \Delta^s A = \frac{\partial^2 A}{\partial x^2}$.

But note that $\Delta^1 \frac{\partial A}{\partial x} \neq \frac{\partial A}{\partial x}$.

Is a non-local operator. Leads to super-diffusion. Degree of non-locality controlled by $s$.

Fourier Transform has nice property: $\mathcal{F}\{\Delta^s A\} = -|q|^{2s} \hat{A}$.
Fractional Calculus

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- Fourier Transform has nice property: 
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• Discretize in space and the calculate derivatives in Fourier space. This turns into ODE in time.
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• We make use of the fact that $\mathcal{F}_{x \to q}\{\Delta^s A\} = -|q|^{2s} \hat{A}$.
• Discretize in space and calculate derivatives in Fourier space. This turns into ODE in time.

• We make use of the fact that $F_{x \rightarrow q} \{ \Delta^s A \} = -|q|^{2s} \hat{A}$.

• Then use MATLAB’s stiff ODE solver.
• Discretize in space and calculate derivatives in Fourier space. This turns into ODE in time.

• We make use of the fact that $\mathcal{F}_{x \rightarrow q}\{\Delta^s A\} = -|q|^{2s} \hat{A}$.

• Then use MATLAB’s stiff ODE solver.

• We used to use a forward Euler spectral method, but that could not handle much of the parameter space.
Examples

Figure: Single Hot-Spot.
Examples

Figure: Four Hot-Spots, $D = 1, s = 1, \varepsilon = 0.05$. 
Examples

Figure: No Hot-Spot, $D = 1, s = 0.5, \varepsilon = 0.05$. 
Figure: Oscillating Hot-Spots, $s = 0.7$, $\eta = 0.1$, $\bar{\rho} = 0.4$, $\bar{A} = 0.12$. 
The homogeneous equilibrium does not change from Short et al. We get

$$\bar{A} = A^0 + \bar{B} \quad \text{and} \quad \bar{\rho} = \frac{\bar{B}}{A^0 + \bar{B}}.$$  \hspace{1cm} (3)
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(3)

• Perturb from the homogeneous equilibrium as follows and plug into linearized equations:

\[ A(x, t) = \bar{A} + \delta_A e^{\sigma t} e^{ik \cdot x}, \]  

(4a)

\[ \rho(x, t) = \bar{\rho} + \delta \rho e^{\sigma t} e^{ik \cdot x}. \]  

(4b)
Linear Stability of Homogeneous Equilibrium

- The homogeneous equilibrium does not change from Short et al. We get

\[
\bar{A} = A^0 + B \quad \text{and} \quad \bar{\rho} = \frac{B}{A^0 + B}. \tag{3}
\]

- Perturb from the homogeneous equilibrium as follows and plug into linearized equations:

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A(x, t) = \bar{A} + \delta_A e^{\sigma t} e^{ik \cdot x}, \quad \tag{4a}
\]

\[
\rho(x, t) = \bar{\rho} + \delta_{\rho} e^{\sigma t} e^{ik \cdot x}. \quad \tag{4b}
\]

- Solving the resulting eigenvalue problem results in this condition for instability: there exists \( |k| \) such that

\[
\eta |k|^{2s+2} - |k|^{2s}(3\bar{\rho} - 1) + \eta \bar{A} |k|^2 + \bar{A} < 0. \tag{5}
\]
The first attempt to solve Eq. (5): the condition for instability is equivalent to, there exists $|k|$ such that

$$\bar{\rho} > \frac{1}{3} (1 + \eta|k|^2) \left(1 + \frac{\bar{A}}{|k|^{2s}}\right).$$

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The first attempt to solve Eq. (5): the condition for instability is equivalent to,

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The first attempt to solve Eq. (5): the condition for instability is equivalent to,

\[ \bar{\rho} > \frac{1}{3} \left( 1 + \eta |k_*|^2 \right) \left( 1 + \frac{\bar{A}}{|k_*|^{2s}} \right), \tag{6} \]

where \( |k_*| \) is a root of

\[ |k|^{2s+2} + \bar{A}(1 - s)|k|^2 - \frac{\bar{A}s}{\eta} = 0. \tag{7} \]
The second attempt to solve Eq. (5): the condition for instability is equivalent to, there exists $|k|$ such that

$$\bar{A} < \frac{3\bar{\rho}|k|^{2s}}{1 + \eta|k|^2} - |k|^{2s}. \quad (8)$$
The second attempt to solve Eq. (5): the condition for instability is equivalent to,

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where \( |k_*| \) is a root of the equation

\[ \eta^2 s|k|^4 + \eta(3\bar{\rho}(1 - s) + 2s)|k|^2 + s(1 - 3\bar{\rho}) = 0. \]  

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(9)
• For $\bar{\rho} > \frac{1}{3}$, this generates condition for linear instability for the system:

$$\bar{A} < \bar{A}_*(\bar{\rho}, \eta, s) \equiv \left( \frac{-3\bar{\rho}(1 - s) - 2s + \sqrt{W}}{2\eta s} \right)^s \left( \begin{array}{c}
\frac{3\bar{\rho}(1 + s) - \sqrt{W}}{-3\bar{\rho}(1 - s) + \sqrt{W}}
\end{array} \right),$$

where $W = 3\bar{\rho}(3\bar{\rho}(1 - s)^2 + 4s)$. 


Stability (cont.)

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- When $s = 1$, the above inequality reduces to

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  \sqrt{A_\eta} + 1 < \sqrt{3\rho},
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  which agrees with the result from Short et al.
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which agrees with the result from Short et al.

• Has bifurcations in $s$, so changing degree of Lévy Flight alters stability.
Figure: Different possibilities of the effect of fractional diffusion.
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Changing Stability with Varying Parameter

Figure: Different possibilities of the effect of fractional diffusion.

(a) Fractional diffusion leads to stability

(b) Fractional diffusion leads to stability
Changing Stability with Varying Parameter

(a) Fractional diffusion leads to unstability

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Figure: Different possibilities of the effect of fractional diffusion.
Changing Stability with Varying Parameter

(a) Fractional diffusion leads to stability and then instability

(b) Fractional diffusion leads to stability and then instability

Figure: Different possibilities of the effect of fractional diffusion.
Remark. The fixed points $|k|$ of $\sigma(|k|)$ with respect to $s$ are given by

$$|k_1| = 1, \quad |k_2| = \sqrt{\frac{6(\bar{\rho} - \frac{1}{3}) + 3\bar{A}}{2\eta}}.$$  (12)
Changing Stability with Varying Parameter

Figure: Different possibilities of the effect of fractional diffusion.

(a) A regime in which $|k_2| < |k_1| = 1$
Numerical Verification

We need to automate a hot-spot detection.

- The variance and its derivative.
- The difference of the maximum of the solution in the final frame from \( A \) as a percentage of \( A \).
- The difference between the maximum and the minimum.
- An ensemble method.
Numerical Verification

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![Graphs showing hot-spot detection](image)
Numerical Verification

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- The variance and its derivative.
- The difference of the maximum of the solution in the final frame from $\bar{A}$ as a percentage of $\bar{A}$. 

\[ \eta = 0.01, \bar{\rho} = 1, \bar{A} = 13, s = 0.8 \]

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Numerical Verification (cont.)

Figure: A parameter analysis with a bifurcation curve. Fix $\overline{\rho}$ and $\eta$. 
Figure: A hot-spot formation when $\bar{A} = 13, \ s = 0.8$. 

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Numerical Verification (cont.)

Figure: A parameter analysis with a bifurcation curve. Fix $\bar{\rho}$ and $\eta$. 
Figure: No hot-spot formation when $\overline{A} = 21, s = 0.6$
Numerical Verification (cont.)

Figure: A parameter analysis with a bifurcation curve. Fix $\eta = 0.01$, $s = 0.7$. 

- Green dots: No Hot-Spot
- Red pluses: Hot-Spot

Diagram: $\bar{A}$ vs. $\overline{\rho}$ with $A_\ast$
Hot-spot Shape

- Are the hotspots any different under the two regimes?

- Yes and no. Attractiveness hotspots do not change, and first order approximations from Kolokolnikov et al. still work very well.

- But distribution of criminals changes.
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Hotspot Shape (cont.)

Figure: The inner region and the outer region of a hotspot.

- Let $x = \varepsilon y$, and $v = \frac{\rho}{A^2}$.

- In the inner region ($|x| < \varepsilon$), $A \sim \varepsilon^{-1} v^{-1/2} 0 w(y)$, (12)

- $v \sim v_0$, (13)

where $v_0$ and $v_1$ are constants, and $w = \sqrt{2 \text{sech}(y)}$ (same as Kolokolnikov et al.).

- In the outer region ($\varepsilon \ll |x| \leq \lambda$), we have $A = \alpha + o(1)$, (14a)

  $v = h_0(x) + o(1)$, (14b)

where $\Delta s_h(x) = \zeta = \alpha - \gamma D_0 \alpha^2 < 0$, $0 < |x| \leq \lambda$, (15)
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$$\nu \sim \nu_0, \quad \quad (13)$$

where $\nu_0$ and $\nu_1$ are constants, and $w = \sqrt{2}\sech y$ (same as Kolokolnikov et al.).
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- In the outer region (\( \varepsilon \ll |x| \leq l \)), we have

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A = \alpha + o(1), \quad (14a)
\]
\[
v = h_0(x) + o(1), \quad (14b)
\]

where

\[
\Delta^s h_0(x) = \zeta = \frac{\alpha - \gamma}{D_0 \alpha^2} < 0, \quad 0 < |x| \leq l, \quad (h_0)_x(\pm l) = 0, \quad (15)
\]
• Analyze the perturbation near hot-spots.
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• Study the dynamics of $K$-hot-spots with Lévy Flights.

Brownian based model. Lévy based model.
Future Work

- Analyze the perturbation near hot-spots.
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- Improve the numerical simulation, especially hotspot detector.
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- Analyze the perturbation near hot-spots.
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Brownian based model.  Lévy based model.
- Add the police.
- Improve the numerical simulation, especially hotspot detector.
- Analyze weakly-nonlinear stability.
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