Crime Modeling with Lévy Flights

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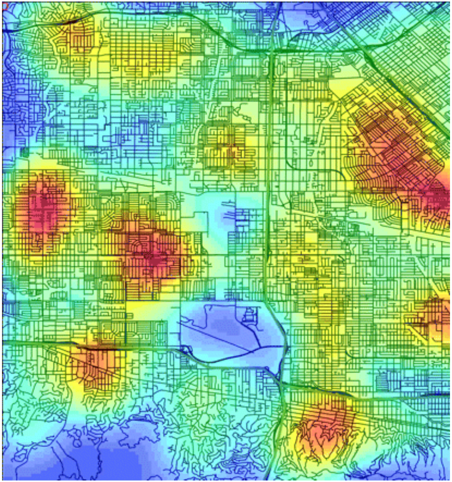

Figure: Crime Hot-Spot Pattern, Long Beach, LA. Short et al. 2010 [2]

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  - Crime hotspots are when several crimes occur in a short period of time and in a small area.
  - Considerable empirical evidence behind them.
- We focus on burglaries for simplicity.

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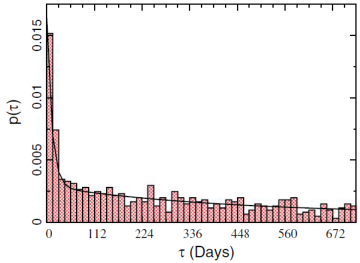
  - Repeat/ Near Repeat Effects: increased knowledge of location after successful crime.



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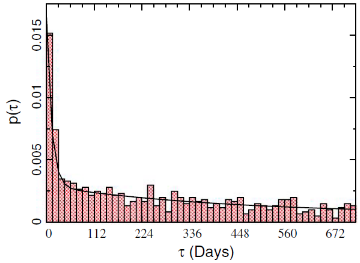

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  - Broken Windows Theory: crime causes sense of lawlessness.

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  - Every time step, criminals move to neighboring lattice spaces based on attractiveness.
  - Then decide to burgle or not based off attractiveness.
  - Crimes are self-exciting; cause attractiveness at lattice point and neighbors to increase.

The Short et al Model (2008) (cont.)

Then, taking limits as grid spacing and time steps go to zero, we get system of PDEs

The Model

$$A_t = \eta \Delta A - A + \rho A + A_0 \tag{1a}$$

$$\rho_t = \left(A \Delta \left(\frac{\rho}{A} \right) - \frac{\rho}{A} \Delta A \right) - \rho A + \overline{A} - A_0 \tag{1b}$$

$A =$ attractiveness at a point
$\rho =$ criminal density at a location
$A_0 =$ a constant, background level of attractiveness
$\overline{A} =$ the spatially homogeneous equilibrium solution for A.

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- Short et al model assumes Brownian motion (and thus distance traveled normally distributed)
- We use Lévy Flights
    - Power law distribution of step sizes ($P(k) \sim k^{-(2s+1)}, 0 < s \leq 1$)
    - Fractal-like motion; reflects many scales of human movement.

Implementation: Discrete

- We change the transition probability as follows:

$$p_{i\to i+1}(t) = \frac{A_{i+1}(t)}{A_{i+1}(t) + A_{i-1}(t)} \qquad \text{(Brownian)}$$

$$p_{i\to j}(t) = \frac{A_j(t)|i-j|^{-(2s+1)}}{\sum_{k\in\mathbb{Z},k\neq i} A_k(t)|i-k|^{-(2s+1)}} \qquad \text{(Lévy)}$$

Implementation: Continuous

- From before we had:

$$
\begin{aligned}
A_t &= \eta \Delta A - A + \rho A + A_0, \\
\rho_t &= \left(A \Delta \left(\frac{\rho}{A} \right) - \frac{\rho}{A} \Delta A \right) - \rho A + \overline{A} - A_0.
\end{aligned}
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- Using our new transition probabilities and taking limits as before we get:

$$A_t = \eta \Delta A - A + \rho A + A_0 \qquad \text{(No change)}$$
$$\rho_t = \left(A\Delta^s \left(\frac{\rho}{A} \right) - \frac{\rho}{A} \Delta^s A \right) - \rho A + \overline{A} - A_0,$$

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where

$$\Delta^s A = c_s \int_{-\infty}^{\infty} \frac{A(y) - A(x)}{|y - x|^{2s+1}} \mathrm{d}y, \qquad (2)$$

c_s is a constant and $0 < s \leq 1$.

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  - Is a non-local operator. Leads to super-diffusion. Degree of non-locality controlled by s.
  - Fourier Transform has nice property: $\mathcal{F}_{x \to q}\{\Delta^s A\} = -|q|^{2s} \hat{A}$

Numerical Solutions: Spectral Method of Lines

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- Then use MATLAB's stiff ODE solver.
- We used to use a forward Euler spectral method, but that could not handle much of the parameter space.

Examples

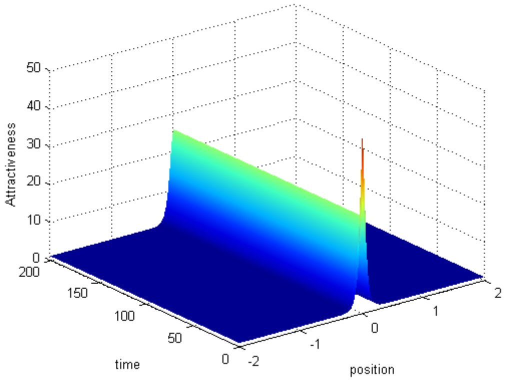

Figure: Single Hot-Spot.

Examples

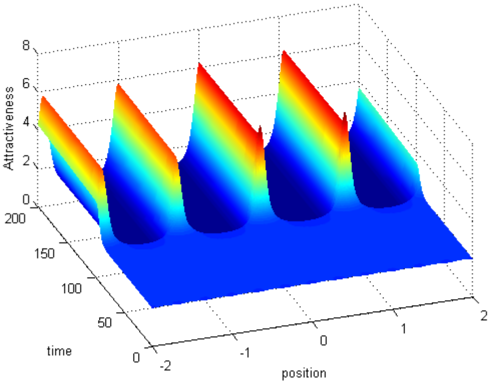

Figure: Four Hot-Spots, $D = 1, s = 1, \varepsilon = 0.05$.

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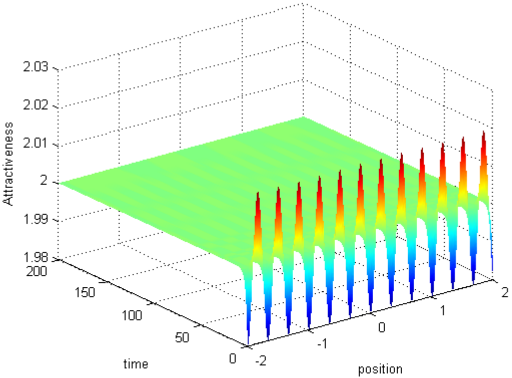

Figure: No Hot-Spot, $D = 1, s = 0.5, \varepsilon = 0.05$.

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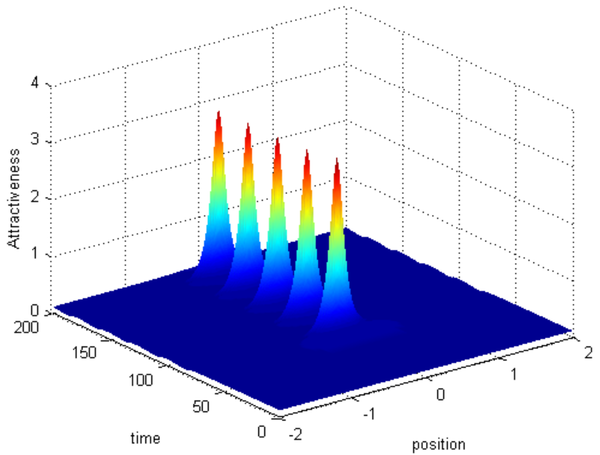

Figure: Oscillating Hot-Spots, $s = 0.7, \eta = 0.1, \overline{\rho} = 0.4, \overline{A} = 0.12$.

Linear Stability of Homogeneous Equilibrium

- The homogeneous equilibrium does not change from Short et al. We get

$$\overline{A} = A^0 + \overline{B} \quad \text{and} \quad \overline{\rho} = \frac{\overline{B}}{A^0 + \overline{B}}. \tag{3}$$

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- Perturb from the homogeneous equilibrium as follows and plug into linearized equations:

$$A(x,t) = \overline{A} + \delta_A e^{\sigma t} e^{ik\cdot x}, \tag{4a}$$

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- Solving the resulting eigenvalue problem results in this condition for instability: there *exists* $|k|$ such that

$$\eta|k|^{2s+2} - |k|^{2s}(3\overline{\rho} - 1) + \eta\overline{A}|k|^2 + \overline{A} < 0. \tag{5}$$

Stability (cont.)

The first attempt to solve Eq. (5): the condition for instability is equivalent to, there exists $|k|$ such that

$$\bar{\rho} > \frac{1}{3}\left(1+\eta|k|^2\right)\left(1+\frac{\bar{A}}{|k|^{2s}}\right). \tag{6}$$

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where $|k_*|$ is a root of

$$|k|^{2s+2} + \overline{A}(1-s)|k|^2 - \frac{\overline{A}s}{\eta} = 0. \tag{7}$$

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The second attempt to solve Eq. (5): the condition for instability is equivalent to, there exists $|k|$ such that

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$$\overline{A} < \overline{A}_*(\overline{\rho}, \eta, s) \equiv \left(\frac{-3\overline{\rho}(1-s) - 2s + \sqrt{W}}{2\eta s} \right)^s \left(\frac{3\overline{\rho}(1+s) - \sqrt{W}}{-3\overline{\rho}(1-s) + \sqrt{W}} \right), \tag{10}$$

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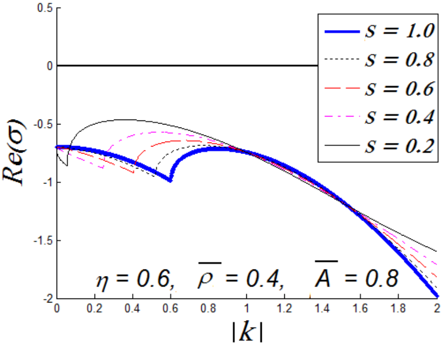
- Has bifurcations in s, so changing degree of Lévy Flight alters stability.

Changing Stability with Varying Parameter



(a) A stable regime

Figure: Different possibilities of the effect of fractional diffusion.

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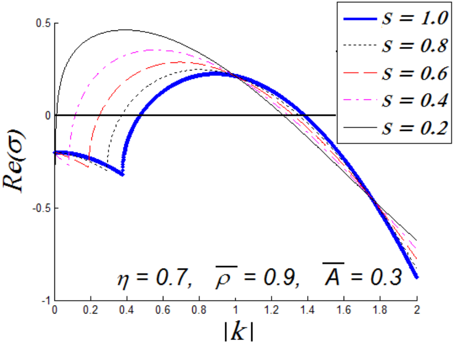

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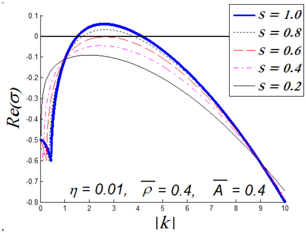
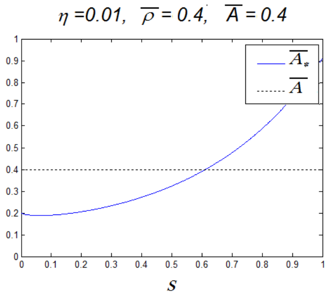


(a) Fractional diffusion leads to stability



(b) Fractional diffusion leads to stability

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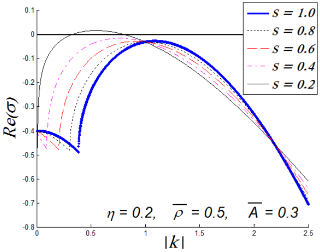

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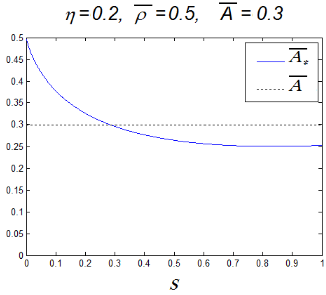

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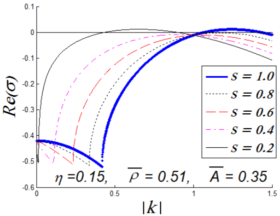
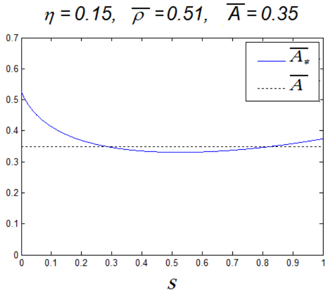


(a) Fractional diffusion leads to stability and then instability



(b) Fractional diffusion leads to stability and then instability

Figure: Different possibilities of the effect of fractional diffusion.

Remark. The fixed points $|k|$ of $\sigma(|k|)$ with respect s are given by

$$|k_1| = 1, \quad |k_2| = \sqrt{\frac{6(\overline{\rho} - \frac{1}{3}) + 3\overline{A}}{2\eta}}. \tag{12}$$

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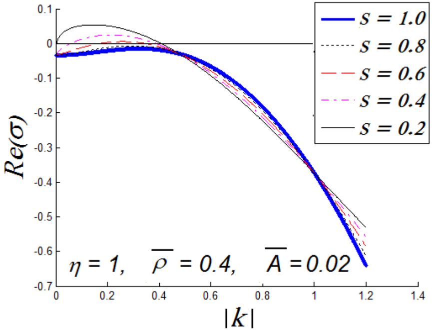

(a) A regime in which $|k_2| < |k_1| = 1$

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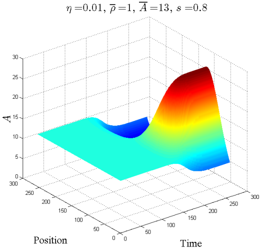
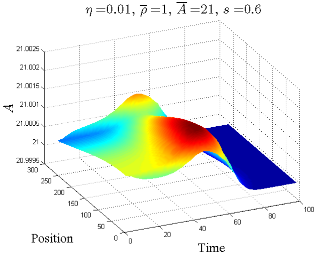
Numerical Verfication

We need to automate a hot-spot detection.





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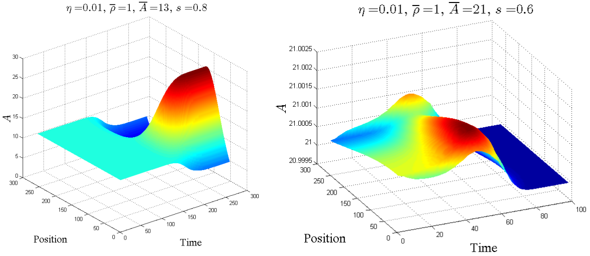

- The variance and its derivative.

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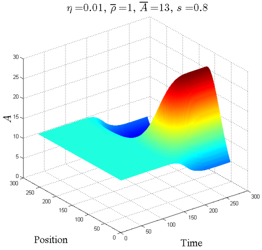

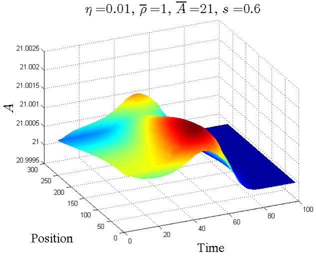

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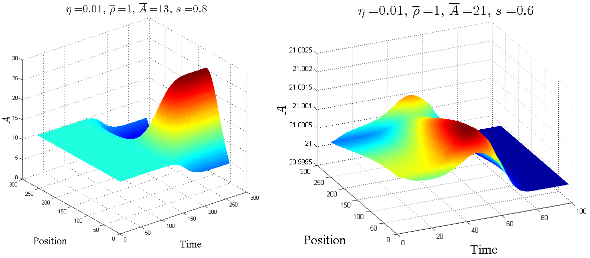

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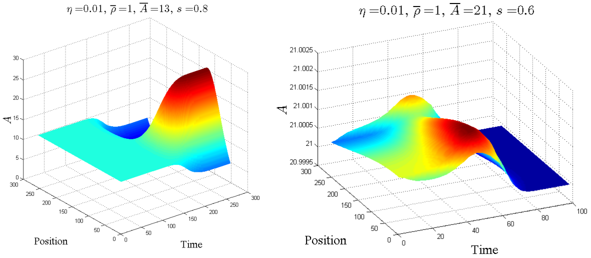

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- The difference of the maximum of the solution in the final frame from $\overline{A}$ as a percentage of $\overline{A}$.
- The difference between the maximum and the minimum.
- An ensemble method.

Numerical Verfication (cont.)

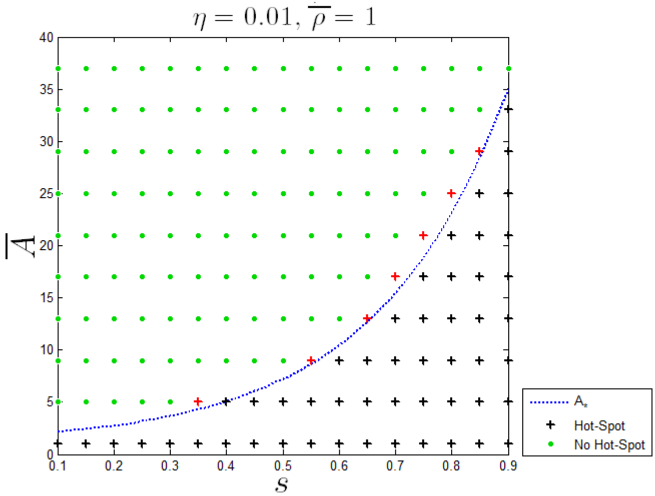

Figure: A parameter analysis with a bifurcation curve. Fix $\overline{\rho}$ and η.

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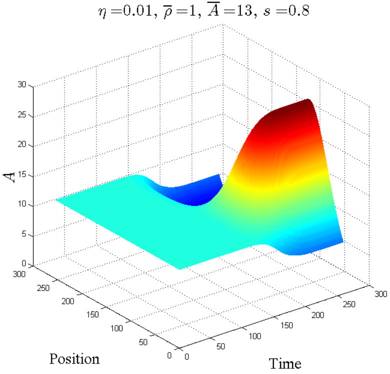

Figure: A hot-spot formation when $\overline{A} = 13$, $s = 0.8$.

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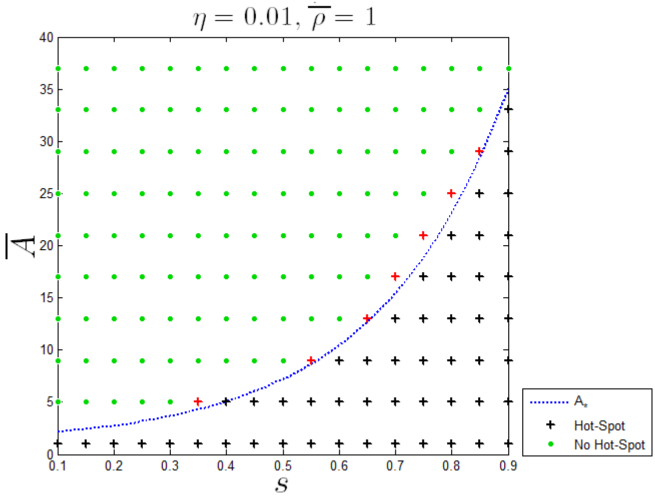

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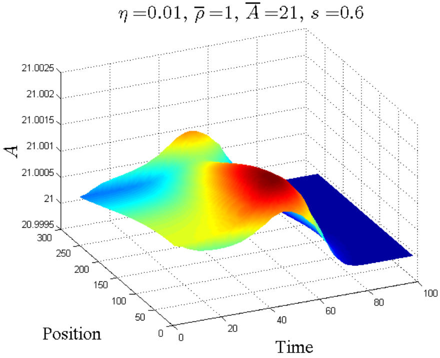
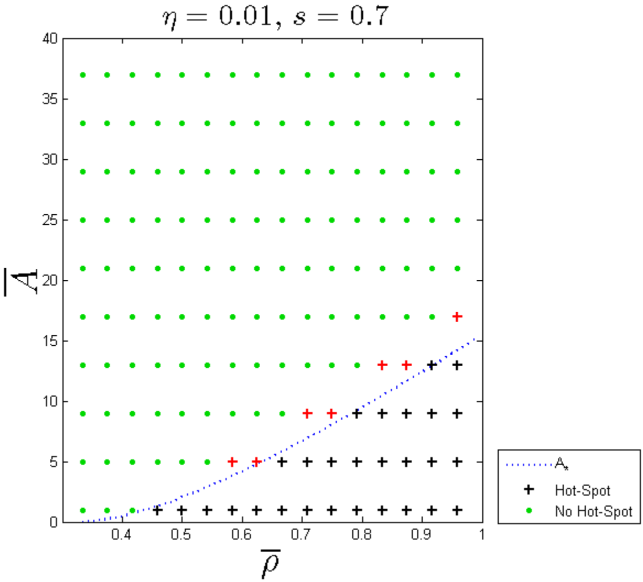


Figure: No hot-spot formation when $\overline{A} = 21, s = 0.6$

Numerical Verfication (cont.)



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Hotspot Shape (cont.)

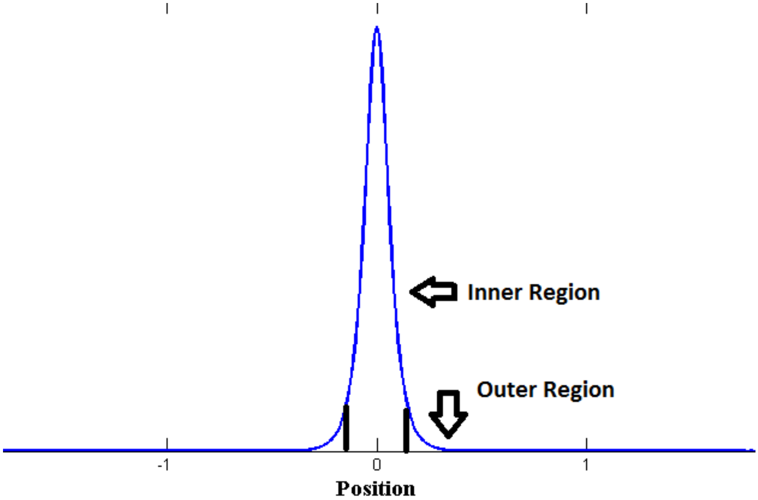

Figure: The inner region and the outer region of a hotspot.

Hotspot Shape (cont.)

- Let $x = \varepsilon y$, and $v = \rho / A^2$.

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$$v \sim v_0, \tag{13}$$

where v_0 and v_1 are constants, and $w = \sqrt{2}\,\text{sech}\, y$ (same as Kolokolnikov et al.).

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- In the outer region ($\varepsilon \ll |x| \leq l$), we have

$$A = \alpha + o(1), \tag{14a}$$

$$v = h_0(x) + o(1), \tag{14b}$$

where

$$\Delta^s h_0(x) = \zeta = \frac{\alpha - \gamma}{D_0 \alpha^2} < 0, \quad 0 < |x| \leq l, \quad (h_0)_x(\pm l) = 0, \tag{15}$$

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