Correlation Effects in Stock Market Volatility

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Abstract:

Understanding the correlation between stocks may give us new insight into modeling the behavior of financial markets. Multivariate stochastic volatility models can be used to describe the time-varying correlation between asset returns. We propose a new multivariate generalized autoregressive conditional heteroskedasticity (MGARCH) model which incorporates correlation effects between stocks. We model each stock series as a linear combination of underlying GARCH Processes, and correlate them using principal component analysis of the Cholesky decomposition of their covariance matrix. Additionally, we investigate the performance in option pricing of the double Heston model in comparison to the Heston model. We propose a possible generalization for the double Heston into a multivariate Heston model for future exploration.

GARCH models proved to be effective predictors of option prices, but little improvement was gained by the introduction of correlation effects in the multivariate model.

1 Introduction

It is well-known that stock prices for companies in the same sector (e.g. manufacturing, hardware, energy) tend to move up and down as a group. A change in one stock's price often correlates with changes in other stock prices. The correlation is often too strong to be dismissed as a coincidence. The goal of this project is to see if incorporating correlation effects into financial models improves option pricing.

The logarithmic **return** of a stock is defined as $r_{log}(t) = \log\left(\frac{S(t)}{S(t-1)}\right)$ where S is asset price at time t. Most of the financial literature models the logarithmic return series of the stocks. We use logarithmic returns mathematically convenient and time-additive ($\log\left(\frac{S2}{S0}\right) = \log\left(\frac{S1}{S0}\right) + \log\left(\frac{S2}{S1}\right)$). The **volatility** of a stock is simply the standard deviation of its return series.

An **option** is a contract that allows its holders the right to buy or sell some underlying asset. There are different types of options; however we will be only concerned with a specific type of option, the European option. A European option is an option that can only be exercised at the maturity date. We compare stochastic models for the stock market by using them to price options and calculate how close the computed option prices are to the actual option prices. The buyer of a **call option** has the right to buy an agreed quantity of a particular underlying asset from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). The buyer of a **put option** acquires the right to sell the underlying to the seller of the option for a strike price at a certain time. These options are then bought and sold, with the prices determined by the market. The price of an option should correspond roughly to the expected value of the option at time of expiration.

In this work, assets returns are modeled using the GARCH [1] and Heston [2] models. Both the Heston and the double Heston models have closed form solutions which we used to price options. Monte Carlo methods were used to price options using the GARCH model.

2 Data

The data we used was taken from Yahoo! Finance and Wharton Research Data Services [3,4]. We used daily data for 452 stocks and option pricing for 90 stocks over a 4 month period, about 150,000 separate quotes. Intraday proved more troublesome and less regular than daily data, and option prices were only available for daily data, so we did not pursue intraday data.

3 Models

3.1 Black-Scholes

The **efficient market hypothesis** states that markets have an immediate respond to any new information about an asset. Hence, asset prices must move randomly. In financial models it is common to model the return on an asset using a stochastic differential equation—a differential equation that contains terms with random variables—for instance $d\boldsymbol{S} = \mu(\boldsymbol{S},t)dt + \sigma(\boldsymbol{S},t)d\boldsymbol{B}$, where S is the price of an asset at time t, μ is the measure of the average rate of growth of an asset price (the **drift**), σ is the volatility, and dB is a **Weiner process** [5]. dB captures the randomness of the asset prices. It is a random variable drawn from a normal distribution such that the mean of dB is zero and the variance is dt. **Brownian motion** is the case when the drift and the volatility are constants. **Geometric Brownian motion** is a stochastic process S where log(S) follows Brownian motion. The Black-Scholes model assumes asset prices follow geometric Brownian motion. It is widely used for option valuation. The Black-Scholes formula has a closed-form solution which can be used to price call and put options.

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

This is a closed form equation for the price of a call option ($C(S,t)$), based on the interest rate r, time until maturity T, and stock and strike prices S, K. Additionally, the d_i are more complicated functions of these variables as well as the volatility σ of the underlying asset. The ease with which a close form can be computed is advantageous. However, the model makes simplifying assumptions about the market, such as constant volatility in asset returns. However, empirical evidence shows this is an incorrect assumption [6].

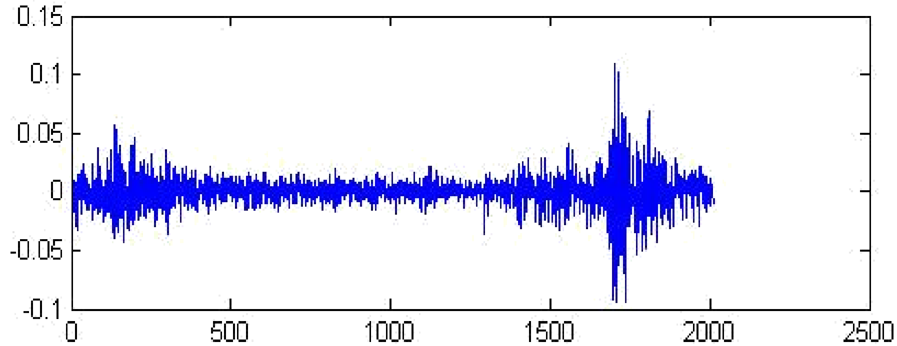

Figure 1: The daily returns of eBay from 2002 to 2010.[4]

Figure 1 shows how the return series for eBay's stock, which experiences clustered periods of high and low volatility. This time-varying property of the volatility (known as heteroskedasticity) led to the development of stochastic volatility models, such as the GARCH and the Heston. These models describe the volatility as a separate stochastic differential equation.

3.2 GARCH

In addressing the issue of the heteroskedasticity of stock returns, one of the most successful models has been the GARCH (generalized autoregressive conditional heteroskedasticity) model. An extension of the ARCH model first proposed by Engel [1], a general GARCH(p,q) model is of the form

$$S_t = C + \sigma_t \epsilon_t,$$
$$\sigma_t^2 = k + \sum_{i=1}^{p} G_i\, \sigma_{t-i}^2 + \sum_{j=1}^{q} A_i \sigma_{t-j}^2 \epsilon_{t-j}^2,$$

where p and q are chosen to give the order of the model [6] C denotes the expected return of the series, and the model thus presumes that the expected return for a single time period (usually a day or a weak), will be normally distributed with some standard deviation σ_t, with ϵ_t being a standard normal variable. However, the time dependent variance σ_t^2 is expected to follow the autoregressive process. k gives a minimum on the variance, while G_i and A_i denote the weights on the previous variances and squared innovations. Thus, in a GARCH process, large jumps up or down will tend to be grouped together, as they will increase the expected volatility of future days.

When selecting the values for p and q, it is typical to set both parameters equal to 1, and use a GARCH(1,1) model. While increasing p and q will invariably increase the fit to in-sample data, as the various models are nested, the advantages gained are minimal and can lead to over specification. Additionally, note that GARCH models have geometric Brownian motion nested within them, as GBM results when $G_i, A_i = 0$.

Due to the discrete and highly stochastic nature of GARCH processes, it is unreasonable to look for a closed form description of future price distribution. Thus, when attempting to price options, we turn to Monte Carlo methods to arrive at estimates of the desired distribution. When we performed such simulations, we observed predictions which corrected some of the historical issues with constant volatility models. A probability distribution is said to have “fat tails” if the probability of events well outside the standard deviation The most obvious of these was the “fat tails” observed in actual return series where evident in GARCH projections. Fat tails are when a probability distribution that appears to be roughly normal has a much higher probability of events well outside the standard deviation range. They are commonly found in empirical data and GARCH models capture this fact.

3.3 Multivariate GARCH

While GARCH provides a better description of how a single stock moves, extending this model to multiple price series is nontrivial and can be done in a number of different ways. The most general of these is to model the daily innovations as a multinormal distribution. Once the means are subtracted, this distribution is defined by the covariance matrix Σ, where Σ_{ij} is the covariance of variables i and j. By necessity, this matrix will be a symmetric and positive definite matrix. Thus, it can be decomposed into M such that

$$\Sigma = MM^T.$$

Then, if Z is a vector of uncorrelated standard normals, then MZ will produce a multinormal distribution with correlation matrix Σ. Thus, this would suggest a model

$$S_t = C + M_t Z_t$$

Where we abuse notation slightly to let S_t and C now be vector analogs of their previous meanings. Then, to describe the evolution of the volatility of the system, it suffices to describe how M_t changes over time. It is in doing this that many divergent possibilities arise. All attempts to describe such a process must balance fidelity to the full dynamics of the system and computational efficiency. For example, the most complete model, which captures all possible relations, has approximately $O(n^4)$ parameters to be fitted for. Since the likelihood functions are highly nonlinear, solving for that many parameters quickly proves infeasible. Even restrictions [7]which reduce the number of parameters to $O(n^2)$ proved to be too slow to take into account a large (20+) number of stock prices. Thus, we sought an $O(n)$ solution that could still proved descriptive.

Our solution was to model our correlated return series as each being the result of taking linear combinations of *independent* univariate GARCH(1,1) processes. Thus, if S_t was the original series, then

$$S_t = AP_t$$

where P_t is the vector of uncorrelated processes. Note that, assuming A is invertible, this can be rewritten as

$$A^{-1}S_t = P_t$$

so that each GARCH process can be viewed as an uncorrelated portfolio of our original stocks. Thus, once we have settled on an A, we can solve for the portfolio series, use Matlab to perform a GARCH fit. From there, we can simulate the portfolio series using Monte Carlo techniques, then recover the original series by multiplication by A. Moreover, this novel (we believe) technique provides great computational speed, as we are effectively splitting the problem into a number of univariate GARCH fittings, which can be done with great speed.

However, first you must select a value for A. There is, however, no clear choice, and the choice will highly affect the dynamics of the system. Choosing the identity matrix, for instance, will result in an uncorrelated system and the same result as if each stock were processed independently. When choosing our A, we sought a decomposition of the correlation matrix B in the manner discussed previously, where $AA^T = B$. This will result in innovations with a correlation matrix similar to B so long as the volatilities of the various portfolios are close to each other. This will also result in correlations among individual stocks increasing or decreasing together for related stocks. This is the type of behavior we would desire, and see in real data. For instance in this figure

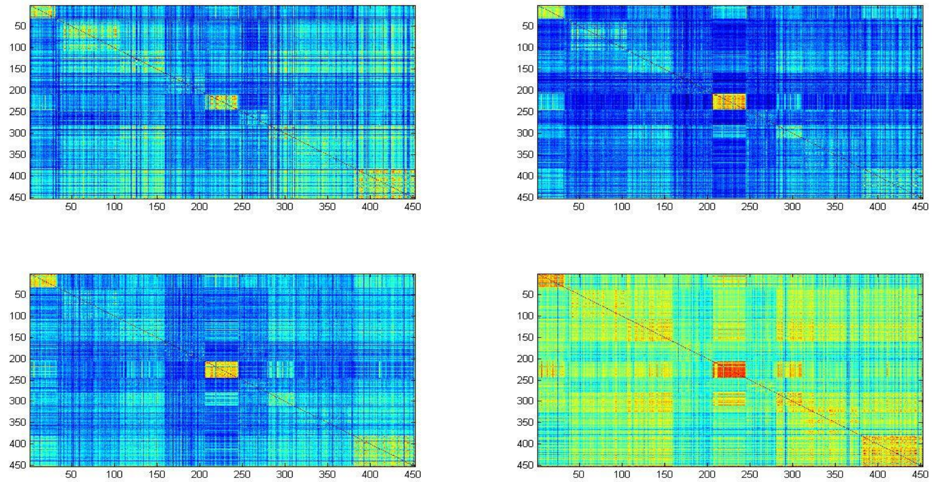

Figure 2: Correlation Matrices for Actual Data over 500 Day Periods

we see the correlation matrices in image form for 4 adjacent 500 day periods of 452 stocks. Red and yellow correspond to heightened correlation, while dark blue denotes uncorrelated pairs. The stock series were preordered by industry, which is the cause of the blocks seen. The correlation between blocks tends to move together between the four images, which indicates that stocks in each block are governed by related processes. When our multivariable GARCH is implemented, four 500 day simulations produced the following four plots.

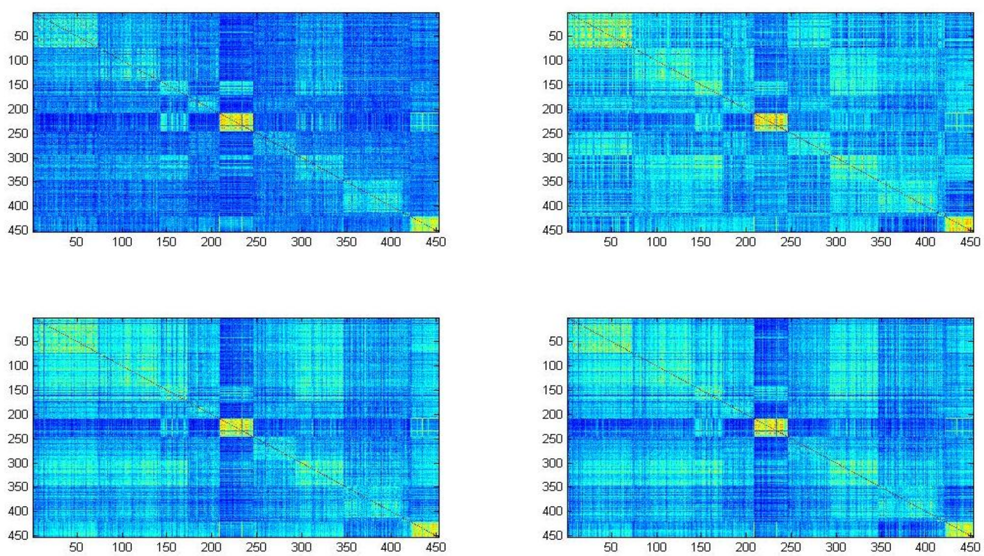

Figure 3: Correlation Matrices for Simulated Data over 500 Day Periods

While we don't see as drastic of a change between them, the changes in correlation are still governed by the block structure (note that our algorithm is not given any information to pre-cluster the stocks into these blocks).

We are thus comfortable with our choice of A which is the decomposition of the correlation matrix into a lower trianglular matrix L such that $B = LL^T$, multiplied by an orthogonal matrix O produced by a PCA (principle component analysis), so that $A = LO$. The PCA transforms the matrix A so that most of the variation between the stocks is isolated into a few of the portfolios. After having constructed the model, we were able to run Monte Carlo simulations and price options based on these prices (See evaluation).

3.4 Heston

The Heston model

In comparison to the Black-Scholes model, the Heston model allows for arbitrary correlation between a given stock price and its volatility. Heston's solution has the flexibility to be applied to problems other than financial modeling as it is based on the mathematics of characteristic functions. [8]

The model is given by

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S$$

$$d\nu_t = \kappa(\theta - \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^\nu$$

For specifics on the Heston model, the reader is referred to Prof. Heston's paper [8]. The Heston model gives the price of a call option as

$$C(S, v, t) = SP_1 - KP(t, T)P_2,$$

where

$$P_j(x, v, T; \ln[K]) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln[K]} f_j(x, v, T; \phi)}{i\phi}\right] d\phi.$$

For information on the characteristic function, *f*, as well its derivation, the reader is referred to the appendix of Heston's paper [8].

The Double Heston model

In 2009, Christoffersen et al. presented an extension of the Heston model which, instead of one equation of stochastic variance as in the Heston model, makes use of two stochastic variance terms. Christoffersen et al. claim an average improvement of 24% in-sample and 23% out-of sample in comparison to the original Heston model. The additional SDE in the Double Heston model enables correlation between the total volatility and the asset price to be modeled stochastically. Perhaps a more intuitive interpretation for this increased flexibility is that the Double Heston model is able to capture independent movements in options prices. [9] the Double Heston model is described by the following stochastic differential equations.

$$\begin{aligned} dS &= rSdt + \sqrt{V_1}Sdz_1 + \sqrt{V_2}Sdz_2 \\ dV_1 &= (a_1 - b_1V_1)dt + \sigma_1\sqrt{V_1}dz_3 \\ dV_2 &= (a_2 - b_2V_2)dt + \sigma_2\sqrt{V_2}dz_4 \end{aligned}$$

The Double Heston model by Christoffersen et al. gives the price of a European call option as

$$C(t) = \frac{1}{2}\left(S(t) - Xe^{-r\tau}\right) + \frac{e^{-r\tau}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi\ln(S(t)/X)}f(\phi+1)}{i\phi} - \frac{Xe^{i\phi\ln(S(t)/X)}f(\phi)}{i\phi}\right]d\phi.$$

where *f*, as before represents the characteristic function. For information on the characteristic function of the Double Heston model and its derivation, the reader is referred to [9].

Calibration of the Heston Models

Moodley provides a MatLab code for calibrating the one-factor Heston model using a least squares approach. Moodley also acknowledges the benefit of using the ASA method to arrive at initial conditions to feed into the least-squares method of calibration [10]. For the Double Heston model, Christoffersen presents in his paper a method for calibration [9]. Fig. 4 shows the flexibility of the Double Heston model in comparison to the Heston model. Long mean variance is 0.04 for the Heston model, and 0.02 for each of the two variance processes for the Double Heston models. Initial variance values are given in the figure.

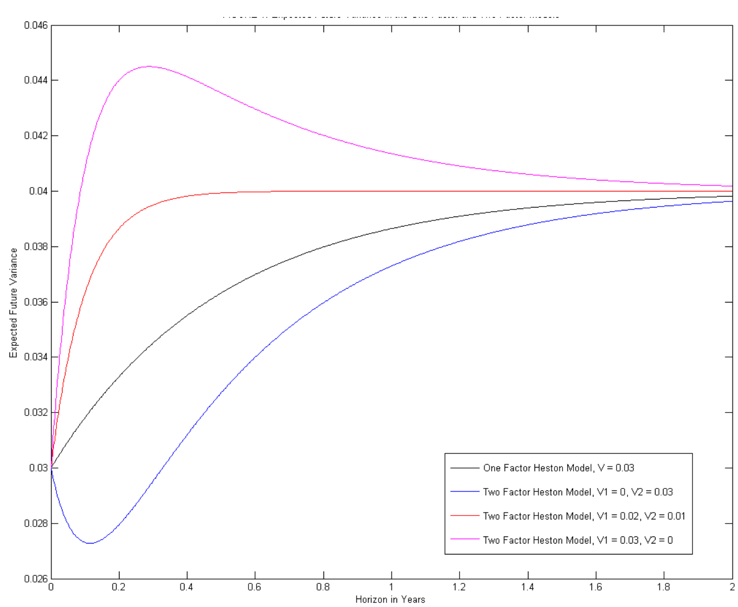

Figure 4: Expected Future Variance in the One-Factor and Two Factor Heston Models

Once the Heston model has been calibrated, you can calculate the option prices for a range of strike prices and expiration times. This yields a surface which describes the structure of the option prices. Figure 5a and 5b show these surfaces for the One-Factor and Two-Factor models, while Figure 5c shows the differences between the two models.

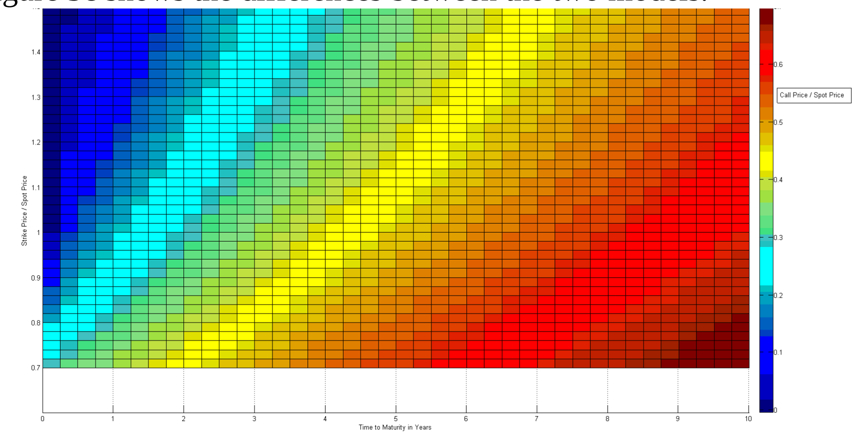

Figure 5a: Option Prices for One-Factor Model

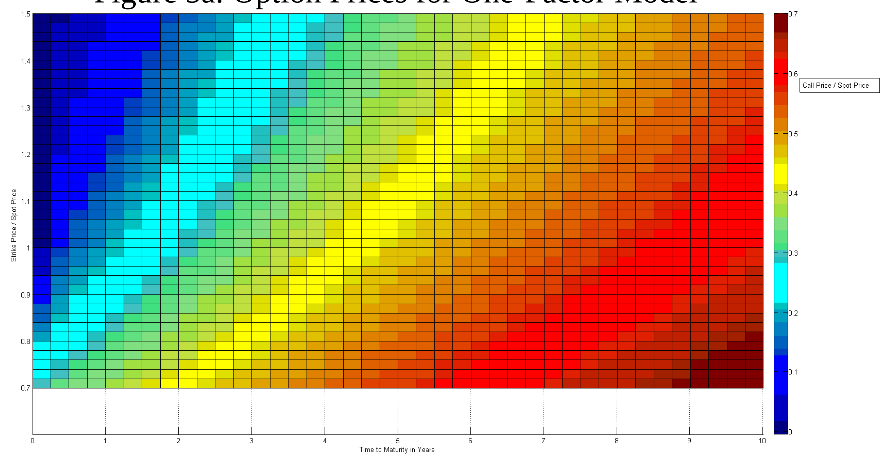

Figure 5b: Option Prices for Two-Factor Model

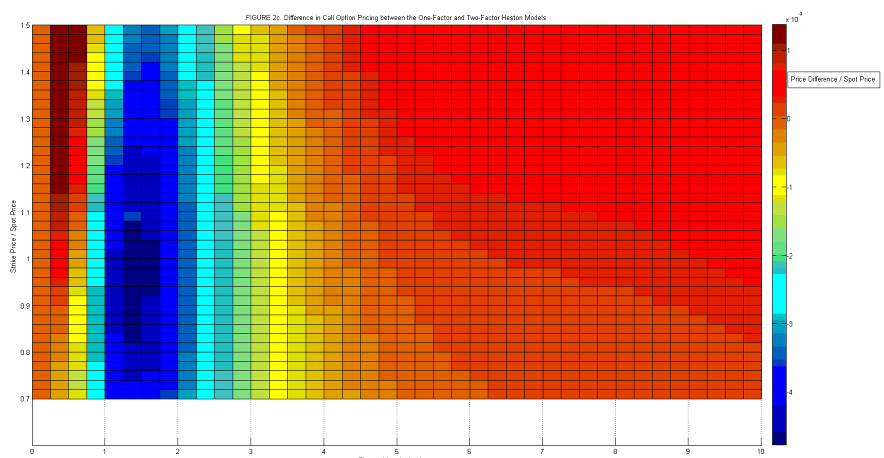

Figure 5c: Differences between One and Two Factor Models

As can be seen the differences between the two models are primarily in the near term option prices.

4 Evaluation

In evaluating our models we were able to draw upon approximately 150,000 option price quotes over a four month database. In doing so, we were able to test the pricing models provided by both univariate and multivariate GARCH models a sample of these pricings can be found in Figure 6. For each of these, we ran approximately 5000 Monte Carlo simulations to predict future price predictions. The Multivariate GARCH simulation was run as a single entity, while the univariate simulations could be run individually.

Strike	Ask	Bid	MGARCH	GARCH
24	0.55	0.5	0.6294	0.00092
10	13.2	12.9	13.12	13.05
15	8.2	7.9	8.12	7.98
20	3.2	3.1	3.21	3.02

Figure 6: Model and Market Option Prices for stock AXP with 12 days until expiration.

There were several different paradigms that we could adopt when evaluating the predictions of the models. First, we can presume that the market prices of the options are accurate representations of the true value of the option, and thus any deviation from these prices is an "error" in our model. Under this paradigm, our models perform relatively well. Both models had an average absolute error of approximately 0.50$, and about a third of the estimated prices were between the bid and ask prices of the quotes. That being said, there was no conclusive evidence that the multivariate model outperformed its counterpart.

A second paradigm of evaluation is to assume that the model prices represent model estimates of the actual value of the option, and determine if acting on this information benefits an agent. Thus, we form investing strategies based on our option prices, and then determine if the "advice" of the model is sound or not. This is the type of evaluation that might be of interest to potential investors. As an initial test, we used a fairly simple metric for constructing a portfolio to be tested. The portfolio P_c consists of equal investments in all options where the model's price is at least c times that of the market price. Under this scheme, we can get a curve for the percent increase in the portfolio for varying values of c. The next figure shows the performance of these

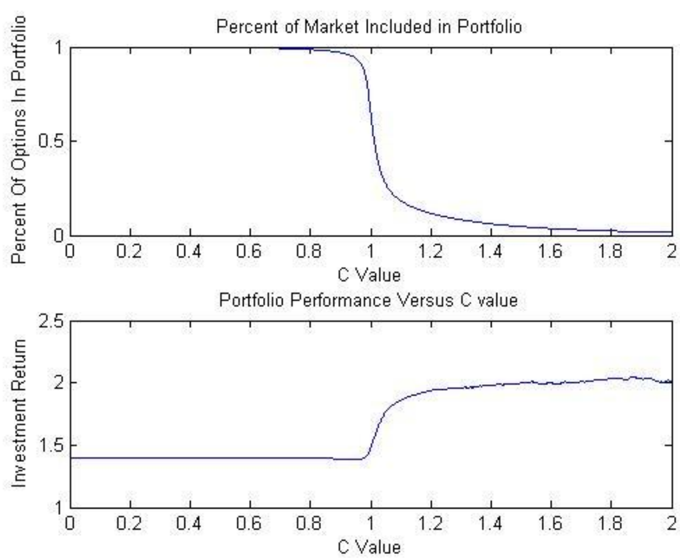

Figure 7: Performance and Composition of Portfolios for Varying C Values

portfolios for different values of c. As can be seen, the market as a whole performs very well, with over a 40% gain, but restricting the portfolio to those options whose model prices were even a little greater than their market prices vastly increased the return. While this is an exceptionally encouraging result, it must be taken with some caveats. The main issue is that the market as a whole experienced large growth (35%) over the four month period we sampled (July-October 2009). Thus, since we restricted our analysis to call options, most call options proved profitable. The next figure shows the relationship between the ratio of the model price and market price and the ratio of the strike price and current price. As can be seen, our model prices options much more accurately when the

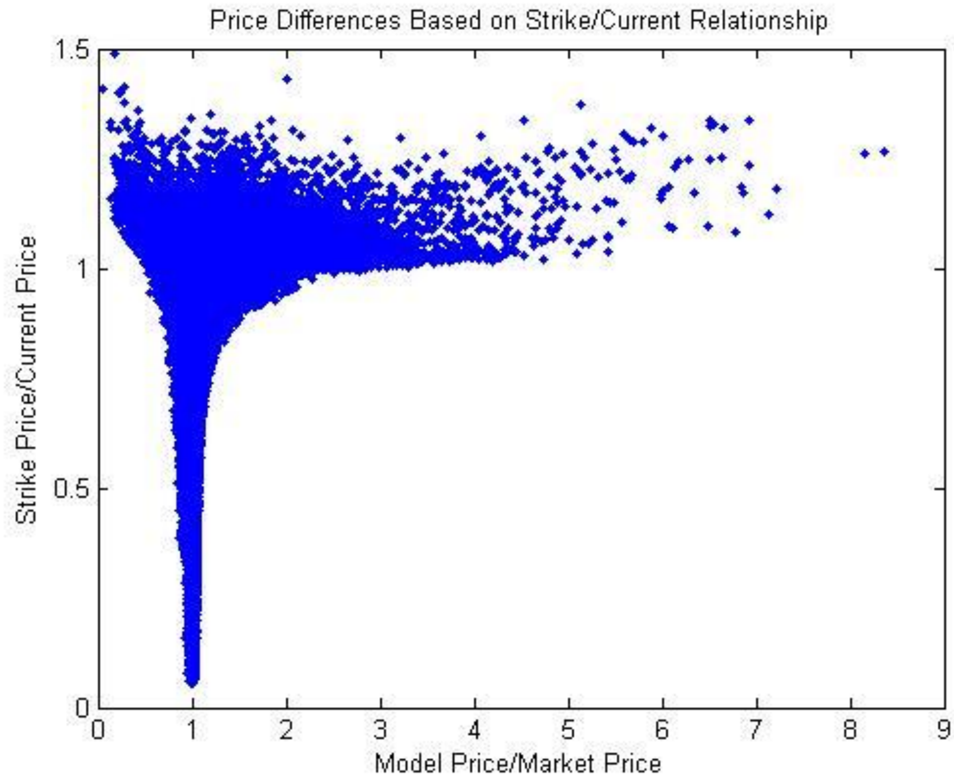

Figure 8: Relation between Strike/Current ratio and Model/Market ratio

strike price is much less than the current price. This is unsurprising, as options which are "in the money" (strike price less than current price) carry a certain underlying value, the bulk of which is unlikely to be changed based on small price fluctuations of the underlying. Options with the strike price greater than the current price, however, are much more effected by such fluctuations, and thus their price may be harder to calculate. Our model tended to overprice these options, which, because the market rose, turned out to be very profitable. Thus, we would need to test over more time periods in order to fully evaluate the efficacy of our models.

It may seem concerning that Multivariable GARCH didn't perform much better than single variable, but on reflection, this is unsurprising. If we had two perfect models, one single variable and one multivariable, we would suspect that they would give the same future distribution for any single asset. It is when you are concerned with the *joint* distributions that multivariate models would really show their strength. The hope was that the multivariable nature would lead to better estimates of volatility by picking up on marketwide effects, but this didn’t seem to have a measurable impact. More direct applications of the multivariate model would be in the pricing of “exotic options” whose value is based on multiple stocks, or in looking at potential arbitrage opportunities given the joint distributions.

5 Conclusion

We were able to construct an effective multivariate model of stock prices, and apply this to option pricing. However, this did not seem to provide a notable improvement over existing univariate models in the pricing of options.

Given the reliance of GARCH methods on Monte Carlo methods for pricing options, the Heston model provided a continuous model for pricing options off of a closed form, similar to the Black Scholes equation, but taking into account heteroskedasticity. By relating current conditions to option prices, the Heston model is able to be calibrated on the basis of current option prices. These prices may incorporate public knowledge about future events in ways that the stock prices alone fail to do, and thus their utilization could result in more accurate option prices.

The Heston model in its current state still requires improvements to make it workable on the same scale as the GARCH models. Due to the large number of parameters, some simplifying assumptions may need to be made to simplify calibration of the model. Additionally, there may be multiple possible extensions to incorporate the correlation between stocks.

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