Introduction to Character Theory and the McKay Conjecture

Ben West

MSRI, July 11-22, 2016

The following are my notes of lectures delivered by Professor Gabriel Navarro at the July 2016 MSRI Summer Graduate School on Character Theory and the McKay Conjecture. Any misprints, mathematical missteps, or general mistakes are mine alone.

Contents

1 July 11, 2016 1
2 July 12, 2016 3
3 July 13, 2016 4
4 July 14, 2016 8
5 July 18, 2016 10
6 July 20, 2016 13
7 July 22, 2016 17
8 Exercises 20

1 July 11, 2016

We use the notation $x^g = g^{-1}xg$.

**Theorem 1.1.** Let $G$ be a finite group, and $p$ be a prime. Suppose further that $P$ is a $p$-group that acts by automorphisms on $G$, and $p \nmid |G|$. Consider the $P$-invariant classes of $G$:

$$cl_P(G) = \{K : K \text{ is a conjugacy class of } G \text{ and } K^x = K \text{ for all } x \in P\}$$

Let $C = C_G(P)$ be the fixed point subgroup of $G$ under the action of $P$. Then the map

$$cl_P(G) \rightarrow cl(C) : K \mapsto K \cap C$$

is a well-defined bijection.

First, a small lemma.

**Lemma 1.2.** Suppose $G = N \rtimes H$. Then $N_N(H) = C_N(H)$.

**Proof.** Clearly $C_N(H) \subseteq N_N(H)$. Conversely, take $n \in N_N(H)$, and $h \in H$. Then since $n^{-1}h^{-1}n \in H$ since $n$ normalizes $H$, and since $h^{-1}nh \in N$ since $N \trianglelefteq G$, we see

$$[n, h] = n^{-1}h^{-1}nh \in N \cap H = 1$$
so $n$ and $h$ commute. Thus $n \in C_N(H)$.

Proof. (Of Theorem) We first show that $K \cap C$ is never empty, and that $K \cap C$ is a conjugacy class in $C$. Since $P$ acts on $G$ by automorphisms, let $\Gamma = GP$ be their semidirect product. Observe

$$[\Gamma : P] = \frac{|GP|}{|P|} = \frac{|G||P|}{|G \cap P||P|} = |G|.$$ 

Since $|G|$ is coprime to $p$, it follows that $P$ is a $p$-Sylow subgroup of $\Gamma$.

Now let $K$ be a $P$-invariant class, say $K = cl_G(x)$ for some $x$. If $y \in P$, then

$$K^y = cl_G(x^y) = K = cl_G(x).$$

Hence $x$ and $x^y$ are conjugate in $G$, so there exists $g \in G$ such that $x^y = x^g$. Thus $x^{yg^{-1}} = x$, which implies $z := yg^{-1} \in C_\Gamma(x)$, or $y = zg \in C_\Gamma(x)G$. Note that $C_\Gamma(x)G$ is a (sub)group of $\Gamma$ since $G \subseteq \Gamma$. As $y \in P$ is arbitrary, we have $P \subseteq C_\Gamma(x)G \subseteq \Gamma = GP$. But clearly $G \subseteq C_\Gamma(x)G$, thus $\Gamma = GP \subseteq C_\Gamma(x)G \subseteq \Gamma$. Hence we have $\Gamma = C_\Gamma(x)G$.

As before, we compute

$$[\Gamma : C_\Gamma(x)] = \frac{|G|}{|C_\Gamma(x) \cap G|}$$

which is coprime to $p$. It follows that if $Q \in Syl_p(C_\Gamma(x))$, then $Q \in Syl_p(\Gamma)$, since a maximal $p$-group in $C_\Gamma(x)$ is still a maximal $p$-group of $\Gamma$.

Since $P \in Syl_p(\Gamma)$, we find that $P$ and $Q$ are conjugate, so there exists $z \in \Gamma$ such that $P^z = Q \subset C_\Gamma(x)$. As $\Gamma = GP = PG$, we may write $z = z_1g$ for $z_1 \in P$, and $g \in G$. Thus

$$P^z = P^{z_1g} = P^g.$$ 

Since $P^g \subseteq C_\Gamma(x)$, we have $P \subseteq C_\Gamma(x^{g^{-1}})$, since direct computation yields, that $x^{yg} = x$ implies $(x^{g^{-1}})^y = x^{g^{-1}}$ for $y \in P$. So $x^{g^{-1}}$ is centralized by $P$, and is clearly conjugate to $x$, thus $x^{g^{-1}} \in K$. Hence $K \cap C \neq \emptyset$, as it contains $x^{g^{-1}}$ for instance.

We have that $K = cl_G(x) = cl_G(x^{g^{-1}})$, and $x^{g^{-1}} \in C_G(P)$, so without loss of generality, we may replace $x$ by $x^{g^{-1}}$ to assume that $K = cl_G(x)$, where $x \in C_G(P)$. Lastly, we must show $K \cap C$ is a conjugacy class in $C$. Since $x \in K \cap C$, any other element has form $x^g$ for some $g \in G$. We must show that $x^g = x^c$ for some $c \in C$, so that $x$ and $x^g$ are actually conjugate in $C$, not merely in $G$.

Take $x^g \in K \cap C$, so $x^g \in C$, thus $P \subset C_\Gamma(x^g)$, and in turn $P^{g^{-1}} \subset C_\Gamma(x)$. But $x \in C_G(P)$, so $P \subset C_\Gamma(x)$. Thus $P$ and $P^{g^{-1}}$ are both Sylow subgroups of $C_\Gamma(x)$, and thus must be conjugate in $C_\Gamma(x)$. Take $y \in C_\Gamma(x)$ such that $P^{g^{-1}} = P^y$.

From $\Gamma = GP$, and $C_\Gamma(x) \cap G = C_G(x)$, we see $C_\Gamma(x) = PC_G(x)$. So we may write $y = uv$, for $u \in P$, $v \in C_G(x)$, and so

$$P^{g^{-1}} = P^y = P^{uv} = P^v.$$ 

It follows that $P = P^{uv}$, so $vg \in N_G(P) = C_G(P)$, this equality following from the lemma. So $c := vg \in C_G(P)$. So finally,

$$x^c = x^{vg} = (x^v)^g = x^g$$

the last equality following since $v \in C_G(x)$. So $K \cap C$ is actually a $C$-conjugacy class, as was to be shown.

Lastly, we show the map is a bijection. Suppose $K$ and $L$ are classes in $cl_P(G)$. If $K \cap C = L \cap C$, these are nonempty, so $K \cap L \neq \emptyset$. As the are classes, we must have $K = L$. For surjectivity, suppose $X \in cl(C)$, say $X = cl_c(C)$. Let $K = cl_G(c)$. Since $K$ is $P$-invariant, $c \in K \cap C$, so $K \cap C = X$. 

\[ \square \]
Remark 1.3. Professor Isaacs pointed out a quicker route to showing $K \cap C$ is nonempty. It’s well known that if a $p$-group acts on a set $Ω$, and if $Ω_0$ is the set of fixed points, then $|Ω| \equiv |Ω_0| \pmod{p}$. Since $P$ is acting on $K$ in this case, 

$$|K| \equiv |K_0| = |K \cap C| \pmod{p}.$$ 

If $K = cl_G(x)$, then $|K| = [G : C_G(x)]$, so $p \nmid |K|$, since $p \nmid |G|$. So $p \nmid |K \cap C|$, so certainly $K \cap C \neq \emptyset$.

2 July 12, 2016

Suppose a group $H$ acts on a group $N$. If $X : N \rightarrow GL_n(\mathbb{C})$ is a representation, then for $h \in H$, we get an induced representation

$$X^h : N \rightarrow GL_n(\mathbb{C}) : n \mapsto X(n^{h^{-1}}).$$

If $X$ affords $χ$, we denote the character afforded by $X^h$ to be $χ^h$. One can compute that if $α$ and $β$ are characters, then

$$(α, β) = (α^h, β^h)$$

so that $α$ is irreducible iff $α^h$ is irreducible.

This raises the question, when is $|Irr_H(N)| = |cl_H(N)|$? The former set is the set of irreducible $H$-invariant characters, and the latter the $H$-invariant classes of $N$.

Recall that if $x, y \in G$, then $[H, K] = \langle [h, k] : h \in H, k \in K \rangle \subseteq ⟨H, K⟩$. Suppose that $H$ acts on $N$, and $M \triangleleft N$ is an $H$-invariant normal subgroup. We get a natural induced action of $H$ on $N/M$. We see that $H$ acts trivially on $N/M$ iff $(Mn)^h = Mn$, iff $Mn^h = Mn$, which holds iff $n^{h^{-1}} \in M$. Abusing notation in the semidirect product, since $[N, H] = \langle n^{-1}n^h : n \in N, h \in H \rangle$, we see that $H$ acts trivially on $N/M$ if $[N, H] \subseteq M$.

Let us consider the case where $N$ is abelian. In this case, $cl_N(n) = \langle n \rangle$, so $cl_N(n)$ is $H$-invariant iff $n^h = n$ for all $h \in H$, iff $n \in C_N(H)$. Hence it follows $|cl_H(N)| = |C_N(H)|$. Moreover, let $λ \in Irr(N)$, so $λ$ is linear as $N$ is abelian. We have

$$\lambda$ is $H$-invariant $\iff \lambda^h(n) = λ(n)$$

$$\iff λ(n^{h^{-1}}) = λ(n)$$

$$\iff λ(n^{h^{-1}}n^{-1}) = 1$$

$$\iff [N, H] \subseteq ker λ$$

$$\iff λ \in Irr(N/[N, H]).$$

That is,

$$|Irr_H(N)| = |Irr(N/[N, H])| = |N/[N, H]|,$$

the last equality following since $N/[N, H]$ is abelian, so there is an irreducible character for every conjugacy class, i.e., every singleton.

To recount, if $N$ is abelian, asking if $|Irr_H(N)| = |cl_H(N)|$ is the same as asking if $|N/[N, H]| = |C_N(H)|$. It turns out this is not generally true. For a counterexample, take $N = \mathbb{Z}/(3) \times \mathbb{Z}/(3)$, and let

$$H = \langle h, k \rangle \cong S_3 \leq GL_2(\mathbb{Z}/(3))$$

where

$$h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

If $(a, b) \in N$ is a fixed point under $H$, then $(a, b) = (a, b)^h = (-a, b)$, so that $a = 0$, and $(a, b) = (a, b)^k = (a, a + b)$, so $b$ can be arbitrary. Then $|C_N(H)| = 3$, but one computes $N = [N, H]$, so that $|N/[N, H]| = 1$.

The equality does hold if $H$ is cyclic.
Theorem 2.1. (Brauer) If a cyclic group $H$ acts as automorphisms on $N$, then

$$|\text{Irr}_H(N)| = |\text{cl}_N(H)|.$$  

Proof. Suppose $H = \langle h \rangle$, $\text{Irr}(N) = \{\chi_1, \ldots, \chi_k\}$, and $\text{cl}(N) = \{K_1, \ldots, K_n\}$, where $K_i = \text{cl}_N(n_i)$. Then put $X = (\chi_i(n_j))$ as the character table, known to be invertible when viewed as a matrix.

Define two $k \times k$ matrices $P = (p_{ij})$ and $Q = (q_{ij})$ by the following:

$$p_{ij} = \begin{cases} 0 & \chi^h_i = \chi^h_j, \\ 1 & \chi^h_i = \chi^h_i, \end{cases} \quad q_{ij} = \begin{cases} 0 & K^h_i = K^h_j, \\ 1 & K^h_i = K^h_i. \end{cases}$$

We claim $PX = XQ$. The $(i,j)$-entry of $PX$ is

$$\sum_{\ell=1}^k p_{i\ell} \chi^h_{\ell}(n_j) = \chi^h_i(n_j)$$

since $p_{i\ell} = 0$ unless $\chi^h_i = \chi^h_{\ell}$, in which case it is 1.

Likewise, the $(i,j)$-entry of $XQ$ is

$$\sum_{\ell=1}^k \chi_i(n_{\ell}) q_{\ell j} = \chi_i(n_{j}^{h^{-1}})$$

since $q_{\ell j}$ vanishes unless $K^h_i = K^h_j$, in which case $K^h_j = K^h_i \supset n_{j}^{h^{-1}}$, so we can evaluate $\chi_i$ on this representation. Since $\chi_i(n_{j}^{h^{-1}}) = \chi_i(n_{j}^h)$ by definition, the claim follows.

Since $X$ is invertible $X^{-1}PX = Q$, so that $\text{tr}(P) = \text{tr}(Q)$. However, by the definition of $P$ and $Q$, $\text{tr}(P)$ is the number of characters fixed by $h$, hence by $H$, and $\text{tr}(Q)$ is the number of classes fixed by $h$, hence by $H$. So $|\text{Irr}_H(N)| = |\text{cl}_H(N)|$. \hfill \qed

Proposition 2.2. Let $A$ be a finite-dimensional $F$-algebra, for $F$ a field. If $\lambda_1, \ldots, \lambda_k \in \text{Hom}(A,F) =: A^*$ are distinct, then they are linearly independent over $F$. In particular, $\dim_F(A^*) = \dim_F(A) \geq k$.

Proof. The proof essentially follows as Artin’s Lemma in Galois Theory. \hfill \qed

3 July 13, 2016

Corollary 3.1. Let $G$ be a finite group, $F$ a field. Then the number of distinct algebra homomorphisms $\text{Z}(F[G]) \to F$ is at most $|\text{Irr}(G)|$.

Proof. Since the conjugacy class sums form a basis of $\text{Z}(F[G])$, we have

$$\dim(\text{Z}(F[G])) = |\text{cl}(G)| = |\text{Irr}(G)|.$$  

Since distinct morphisms are linearly independent, there can be at most $\dim(\text{Z}(F[G])) = |\text{Irr}(G)|$ of them. \hfill \qed

Suppose $X: G \to \text{GL}_n(\mathbb{C})$ is an irreducible representation affording the irreducible character $\chi$. We can extend $X$ linearly to an algebra morphism $X: \mathbb{C}[G] \to M_n(\mathbb{C})$, also denoted by $X$. Since $X$ is irreducible, the extension is surjective onto $M_n(\mathbb{C})$. 
Suppose $z \in Z(\mathbb{C}[G])$. Then $X(z)$ is central in $\text{Im}(X) = M_n(\mathbb{C})$, so necessarily $X(z)$ is a scalar matrix. Thus we can write $X(z) = \omega(z)I_n$ for some $\omega(z) \in \mathbb{C}$. This value $\omega(z)$ depends only on $\chi$, for suppose $Y$ is another representation affording $\chi$. Then $Y$ and $X$ are similar representations, so for some $M \in GL_n(\mathbb{C})$,

$$Y(z) = M^{-1}X(z)M = M^{-1}\omega(z)I_nM = \omega(z)I_n.$$ 

Furthermore, since $X$ is an algebra morphism, we see the above map $\omega: Z(F[G]) \to \mathbb{C}$. For instance, we have for $z, w \in Z(F[G])$,

$$\omega(zw)I_n = X(zw) = X(z)X(w) = \omega(z)I_n\omega(w)I_n = \omega(z)\omega(w)I_n$$

so that $\omega(zw) = \omega(z)\omega(n)$. Other morphism properties follow similarly.

Since $\omega$ depends only on $\chi$, rename it $\omega_{\chi}$. Thus for $\chi \in \text{Irr}(G)$, we get an associated algebra morphism $\omega_{\chi}: Z(\mathbb{C}[G]) \to \mathbb{C}$.

These $\omega_{\chi}$ are called the central characters. As observed in Isaacs’ notes, if $K \in \text{cl}(G)$, say $K = \text{cl}_G(x)$, and we denote $\hat{K} = \sum_{k \in K} k$, then

$$\omega_{\chi}(\hat{K}) = \frac{|K| \chi(x)}{\chi(1)} \in \mathcal{R},$$

where $\mathcal{R}$ denotes the ring of algebraic integers.

Now fix a prime $p$. Let $p\mathcal{R}$ be the generated ideal, and pick a maximal ideal $\mathfrak{m}$ contains $p\mathcal{R}$. Set $F = \mathcal{R}/\mathfrak{m}$. Then the image of $p$ in $F$ is zero, so $\text{char}(F) = p$. We will denote the canonical map $\mathcal{R} \to F$ by $^*$. Observe that if $n \in \mathbb{Z}$, then $n^* = 0$ iff $p \mid n$ in $\mathbb{Z}$. For if $p \mid n$, then $up + vn = 1$ for some integers $u$ and $v$. Taking the projection into $F$, the $up + vn$ vanishes, so that $1 \in M$, contrary to $M$ being a maximal, hence proper, ideal.

Take $\chi \in \text{Irr}(G)$. Define an algebra morphism by

$$\lambda_{\chi}: Z(F[G]) \to F: \hat{K} \to \omega_{\chi}(\hat{K})^*$$

and extending linearly. This makes sense as the class sums $\hat{K}$ are a basis of $Z(F[G])$, and $\omega_{\chi}(\hat{K}) \in \mathcal{R}$.

**Theorem 3.2.** Suppose $G$ is a finite group, and $p \mid |G|$. Then

$$\{\lambda_{\chi} : \chi \in \text{Irr}(G)\} = \text{Hom}_{\text{Alg}}(Z(F[G]), F).$$

**Proof.** It is sufficient to show that if $\chi$ and $\psi$ are distinct irreducible characters, then $\lambda_{\chi}$ and $\lambda_{\psi}$ are distinct. This will show that $\{\lambda_{\chi} : \chi \in \text{Irr}(G)\}$ has cardinality $|\text{Irr}(G)|$, which is the maximum number of algebra morphisms $Z(F[G]) \to F$ by the corollary above.

Let

$$f_{\psi} = \sum_{K \in \text{cl}(G)} \psi(x_k^{-1})\hat{K}$$

where $x_k$ is a representative of $K$. Note that $\psi(x_k^{-1}) \in \mathcal{R}$ since as the value of a character, it is a sum of roots of unity, which are algebraic integers themselves.

Thus we set

$$f_{\psi}^* = \sum_{K \in \text{cl}(G)} \psi^*(x_k^{-1})\hat{K}$$

which is in $Z(F[G])$ as the $\hat{K}$ are central, and $^*$ sends the coefficients of $f_{\psi}$ into $F$.

By the first orthogonality relation, we have

$$|G| \delta_{\chi,\psi} = \sum_{x \in G} \chi(x)\psi(x^{-1})$$

$$= \sum_{K \in \text{cl}(G)} |K| \chi(x_k)\psi(x_k^{-1})$$

$$= \chi(1) \sum_{K \in \text{cl}(G)} \omega_{\chi}(\hat{K})\psi(x_k^{-1}).$$

5
Since \(|G|\) and \(\chi(1)\) are integers, and \(\omega(\hat{K})\) and \(\psi(x_k^{-1})\) are algebraic integers, we can apply \(^*\), yielding

\[|G|^* \delta_{\chi, \psi}^* = \chi(1)^* \sum_{K \in cl(G)} \omega_{\chi}(\hat{K})^* \psi(x_k^{-1})^*.\]

By the definition of the \(\lambda_{\chi}\) above, the right side can be rewritten as

\[\chi(1)^* \sum_{K \in cl(G)} \lambda_{\chi}(\hat{K}) \psi(x_k^{-1})^* = \chi(1)^* \lambda_{\chi}(f_\psi^*),\]

where the equality follows by the \(F\)-linearity of \(\lambda_{\chi}\).

Recall \(p \nmid |G|\). So \(|G|^* \neq 0\), by our previous observation. Moreover, since \(\chi(1) \mid |G|\), certainly \(p \nmid \chi(1)\), so \(\chi(1)^* \neq 0\).

Observe that

\[\chi(1)^* \lambda_{\chi}(f_\chi^*) = |G|^* \neq 0\]

so that \(\lambda_{\chi}(f_\psi^*) \neq 0\). Suppose \(\lambda_{\chi} = \lambda_{\psi}\), but \(\chi \neq \psi\). Then since \(\delta_{\chi, \psi}^* = 0\),

\[0 = \lambda_{\chi}(f_\psi^*) = \lambda_{\psi}(f_\psi^*) \neq 0,
\]

a contradiction. \(\square\)

Now suppose \(P\) is a \(p\)-group that acts as automorphisms on a group \(G\), where \(p \nmid |G|\). Let \(C := C_G(P)\). We wish to find a bijection between \(\text{Irr}_P(G)\) and \(\text{Irr}(C)\). Suppose \(\chi \in \text{Irr}_P(G)\), so \(\chi^z = \chi\) for all \(z \in P\). We wish to construct a corresponding irreducible character of \(C\), which we have seen above are in bijection with the morphisms \(\lambda_{\chi}\) on \(Z(F[C])\), defined in terms of central characters. So in this case, we must produce a corresponding morphism \(\delta_{\chi} : Z(F[C]) \rightarrow F\).

Going back to the first section, we have a bijection \(cl_P(G) \rightarrow cl(C) : K \mapsto K \cap C\). So an arbitrary class sum in \(Z(F[C])\) has form \(\hat{K} \cap \hat{C}\) for \(K \in cl_P(G)\). Thus define

\[\delta_{\chi} : Z(F[C]) \rightarrow F : \hat{K} \cap \hat{C} \mapsto \lambda_{\chi}(\hat{K})\]

by extending linearly.

**Theorem 3.3.** The map \(\delta_{\chi}\) is an algebra morphism.

**Proof.** Now the action of \(P\) on \(G\) induces an action of \(P\) on \(cl(G)\), so we have

\[cl(G) = cl_P(G) \cup \mathcal{O}(K_1) \cup \cdots \cup \mathcal{O}(K_k)\]

where \(\mathcal{O}(K_i) = \{K_i^z : z \in P\}\). Since the \(K_i\) are the classes which are not \(P\)-invariant, \(|\mathcal{O}(K_i)| = p^{e_i} > 1\).

The orders are \(p\)-powers, since they must divide the order of \(P\) by the Orbit-Stabilizer Theorem.

We need to show \(\delta_{\chi}\) is multiplicative on a basis, as it is linear by construction. That is, we want

\[\delta_{\chi}(\hat{K} \cap CL \cap C) = \delta_{\chi}(\hat{K} \cap C) \delta_{\chi}(\hat{L} \cap C)\]

for \(K, L \in cl_P(G)\).

First, we have

\[\delta_{\chi}(\hat{K} \cap \hat{C}) \delta_{\chi}(\hat{L} \cap \hat{C}) = \lambda_{\chi}(\hat{K}) \lambda_{\chi}(\hat{L}) = \lambda_{\chi}(\hat{K} \hat{L})\]

by definition of \(\delta_{\chi}\), and the fact that \(\lambda_{\chi}\) is an algebra morphism.

Note that

\[\hat{K} \hat{L} = \sum_{M \in cl(G)} a_{KLM} M\]
where \( a_{KLM} \), is the number of pairs \((x, y) \in K \times L\) such that \( xy = x_M \), a representative of the class \( M \). Certainly \( \hat{K} \hat{L} \) is an positive integral linear combination of class sums, and we can compute the coefficients of \( \hat{M} \) by counting the pairs whose product if a fixed representative, because the number of summand of any element in \( M \) must be the same, else we would not have an integral coefficient of \( \hat{M} \).

Note for \( z \in P \), we have a bijection
\[
\{(x, y) \in K \times L : xy = x_M \} \longrightarrow \{(u, v) \in K^2 \times L^2 : uv = x_M^* \} : (x, y) \mapsto (x^*, y^*)
\]
Hence \( a_{KLM} = a_{K^*L^*M^*} \).

We can further decompose this sum. Taking smaller sums over the class sums in the various orbits listed above, we have
\[
\hat{K} \hat{L} = \sum_{M \in cl_P(G)} a_{KLM} \hat{M} + \sum_{X \in \mathcal{O}(K_i)} a_{KLX} \hat{X} + \cdots + \sum_{X \in \mathcal{O}(K_s)} a_{KLX} \hat{X}.
\]

However, the classes in \( \mathcal{O}(K_i) \) are of form \( K_i^z \) for \( z \in P \). By the above observation, we have
\[
a_{KLK_i^z} = a_{K^*L^*K_i^z} = a_{KLK_i} =: b_i,
\]
where the first equality follows since \( K = K^2 \) and \( L = L^2 \) as they are \( P \)-invariant. So the coefficient for each class sum in the orbit of a given non-\( P \)-invariant class is the same, so we can rewrite the product as
\[
\hat{K} \hat{L} = \sum_{M \in cl_P(G)} a_{KLM} \hat{M} + \sum_{X \in \mathcal{O}(K_i)} b_1 \hat{X} + \cdots + \sum_{X \in \mathcal{O}(K_s)} b_s \hat{X} = \sum_{M \in cl_P(G)} a_{KLM} \hat{M} + \sum_{j=1}^{s} b_j \left( \sum_{X \in \mathcal{O}(K_j)} \hat{X} \right).
\]

If \( M \in cl_P(G) \), then \( M \cap C \in cl(C) \) is nonempty, so pick \( x_M \in M \cap C \), so that \( x_M \) is fixed by \( P \). Set \( \Omega = \{(x, y) \in K \times L : xy = x_M \} \). Then \( P \) acts on \( \Omega \), since for \( z \in P \), \( (x^*, y^*) \in K^2 \times L^2 = K \times L \), and if \( xy = x_M \), then \( (xy)^z = x^2 y^2 = x_M^* = x_M \).

First recall that if \( P \) acts on a set \( \Omega \), we have the congruence \(|\Omega| \equiv |\Omega_0| \pmod{p} \), that is \( |\Omega| = |\Omega_0|^* \) in \( F = \mathbb{R}/m \). Note \( |\Omega| = a_{KLM}^* \), and the \( P \)-fixed points of \( K \) and \( L \) are \( K \cap C \) and \( L \cap C \), respectively, so
\[
\Omega_0 = \{(x, y) \in K \cap C \times L \cap C : xy = x_M \} = a_{K \cap C \cap L \cap C} \cap \mathcal{O}(K_i) \cap \mathcal{O}(M \cap C)
\]
since \( x_M \in M \cap C \), and \( K \cap C \) and \( L \cap C \) are classes in \( C \). Thus \( a_{KLM}^* = a_{K \cap C \cap L \cap C}^* \).

By hypothesis, \( \chi \in \text{Irr}_P(G) \), so if \( z \in P \), \( \chi = \chi^z \), so
\[
\lambda_\chi(\hat{X}^z) = \lambda_\chi(\hat{X}^z) = \lambda_\chi(\hat{X}^{z^{-1}}) = \lambda_\chi(\hat{X}).
\]

We now compute
\[
\lambda_\chi(\hat{K} \hat{L}) = \lambda_\chi \left( \sum_{M \in cl_P(G)} a_{KLM} \hat{M} \right) + \sum_{j=1}^{s} b_j \lambda_\chi \left( \sum_{X \in \mathcal{O}(K_j)} \hat{X} \right).
\]

But since \( \lambda_\chi \) is constant on classes,
\[
\lambda_\chi \left( \sum_{X \in \mathcal{O}(K_j)} \hat{X} \right) = |\mathcal{O}(K_j)| \lambda_\chi(\hat{X}) = p^{\rho_j} \lambda_\chi(\hat{X}) = 0
\]
since \( \text{char}(F) = p \), and \( p^{\rho_j} > 1 \).
Thus we simply have
\[
\lambda_\chi(\hat{K} \hat{L}) = \lambda_\chi \left( \sum_{\lambda \in \text{Irr}(\hat{G})} a_{\lambda \chi}(\hat{M}) \right) \\
= \lambda_\chi \left( \sum_{\lambda \in \text{Irr}(\hat{G})} a_{\lambda \chi}(\hat{M}) \right) \\
= \sum_{\lambda \in \text{Irr}(\hat{G})} a_{\lambda \chi}(\hat{M}) \\
= \sum_{\lambda \in \text{Irr}(\hat{G})} a_{\lambda \chi}(\hat{M}) \\
= \delta_\chi \left( \sum_{\lambda \in \text{Irr}(\hat{G})} a_{\lambda \chi}(\hat{M}) \right) = \delta_\chi(\hat{K} \hat{L}) \\
\]
which was to be shown.

\[\square\]

4 July 14, 2016

Suppose \( P \) is a \( p \)-group acting by automorphisms on a group \( G \), where \( p \nmid |G| \). In the previous section, we saw that if \( \chi \in \text{Irr}_P(G) \), there is an algebra morphism \( \delta_\chi : Z(F[G]) \to F : \hat{K} \hat{C} \to \lambda_\chi(\hat{K}) \). Since \( C \) has order coprime to \( p \), and we have determined all possible algebra morphisms \( Z(F[C]) \to F \) be a previous theorem, necessarily \( \delta_\chi = \lambda_\chi \) for a unique \( \hat{\chi} \in \text{Irr}(C) \).

Suppose \( c \in C \), \( K = cl_C(c) \), so \( K \cap C = cl_C(c) \). From their definitions, we have
\[
\left( \frac{|K|}{\chi(1)} \right)^* = \omega_\chi(K) = \lambda_\chi(K) = \delta_\chi(\hat{K} \hat{C}) = \lambda_\chi(\hat{K} \hat{C}) = \omega_\chi(\hat{K} \hat{C})^* = \left( \frac{|K \cap C| \cdot \hat{\chi}(c)}{\hat{\chi}(1)} \right)^*.
\]
Multiplying through by \( \chi(1)^* \cdot \hat{\chi}(1)^* \), and using the fact that \( ^* \) is a homomorphism, inspecting the leftmost and rightmost terms above yields
\[
\chi(1)^* \cdot |K|^* \cdot \hat{\chi}(c)^* = \chi(1)^* \cdot |K \cap C|^* \cdot \hat{\chi}(c)^*.
\]
However, \( |K| \equiv |K \cap C| \pmod{p} \), so that \( |K|^* \equiv |K \cap C|^* \neq 0 \pmod{p} \), so cancelling and pulling back to \( R \) under \( ^* \), we conclude
\[
\hat{\chi}(1) \cdot \chi(c) \equiv (\hat{\chi}(c) \mod{m}).
\]

**Theorem 4.1.** (Glauberman) Suppose \( P \) is a \( p \)-group acting on a group \( G \) by automorphisms, and \( p \nmid |G| \). Then the map
\[
\mathcal{G} : \text{Irr}_P(G) \to \text{Irr}(C) : \chi \mapsto \hat{\chi}
\]
is a bijection. (The character \( \hat{\chi} \) is determined by the algebra morphism \( \delta_\chi \) in the prior discussion.) Also, \( \chi_C = e\hat{\chi} + p\Delta \), where \( e \equiv \pm 1 \pmod{p} \), and where \( \Delta \) is a linear combination of irreducible characters. Note that \( e \) is not necessarily the multiplicity of \( \hat{\chi} \).

**Proof.** Let \( \chi \in \text{Irr}_P(G) \), and \( \theta \in \text{Irr}(C) \). Using the equation just derived above, we see
\[
\hat{\chi}(1) |C| (\chi_C, \theta) = \sum_{c \in \mathcal{C}} \hat{\chi}(1) \chi_C(c) \theta(c^{-1}) \equiv \sum_{c \in \mathcal{C}} \chi(1) \hat{\chi}(c) \theta(c^{-1}) = \chi(1) (\hat{\chi}, \theta).
\]

8
Thus
\[ \hat{\chi}(1) |C| (\chi_C, \theta) \equiv \chi(1) |C| (\hat{\chi}, \theta) \pmod{p} \]
and since \( p \nmid |C| \),
\[ \hat{\chi}(1)(\chi_C, \theta) \equiv \chi(1)(\hat{\chi}, \theta) \pmod{p}. \]

We consider two cases. If \( \hat{\chi} = \theta \),
\[ \hat{\chi}(1)(\chi_C, \hat{\chi}) \equiv \chi(1)(\hat{\chi}, \hat{\chi}) = \chi(1) \pmod{p}, \]
and so \( p \nmid (\chi_C, \hat{\chi}) \), since \( p \nmid \chi(1) \mid |G| \). On the other hand, if \( \hat{\chi} \neq \theta \), then \( (\hat{\chi}, \theta) = 0 \), so that \( (\chi_C, \theta) = 0 \pmod{p} \), as \( p \nmid \hat{\chi}(1) \). With this information of the coefficients, we have \( \chi_C = e\hat{\chi} + p\Psi \), where \( p \nmid e \), and \( \Psi \) is some linear combination of irreducible characters.

Now suppose \( \chi, \varphi \in \text{Irr}_P(G) \). We can write \( cl(G) = cl_P(G) \cup O(K_1) \cup \cdots \cup O(K_s) \), where the \( K_i \) are non-\( P \)-invariant classes. Observe that if \( K_i = cl_G(y_i) \), and \( z \in P \), then \( K_i^z = cl_G(y_i^z) \), and
\[ \chi(y_i^z) = \chi^{-1}(y_i) = \chi(y_i) \]
since \( \chi \) is \( P \)-invariant. Thus \( \chi \) takes the same value on classes in the same orbit. Since \( |O(K_i)| \) is a \( p \)-power greater than 1, it vanishes in \( F \), similar to a previous argument. So we have
\[
\hat{\chi}(1) |G| (\chi, \varphi) = \hat{\chi}(1) \sum_{g \in G} \chi(g) \varphi(g^{-1}) = \hat{\chi}(1) \sum_{K \in cl(G)} |K| \chi(x_K) \varphi(x_K^{-1}) = \hat{\chi}(1) \sum_{K \in cl_P(G)} |K| \chi(x_K) \varphi(x_K^{-1}) = \hat{\chi}(1) \sum_{K \cap C \in cl(C)} |K \cap C| \chi(x_K) \varphi(x_K^{-1}) = \hat{\chi}(1) |C| (\chi_C, \varphi_C)
\]
where \( x_K \in K \) is a class rep, and we choose \( x_K \in K \cap C \), so that \( x_K \) is \( P \)-invariant. The first congruence that appears follows since \( |K| \) vanishes modulo \( p \) when \( K \notin cl_P(G) \) as noted above.

To understand the final two equalities above, recall that from our bijection \( cl_P(G) \to cl(C) : K \mapsto K \cap C \), we have that \( K \cap C \) ranges over \( cl(C) \) as \( K \) ranges over \( cl_P(G) \), and since \( x_K \in C \), we can replace \( \chi \) and \( \varphi \) with their restrictions \( \chi_C \) and \( \varphi_C \) in this sum to get
\[ \hat{\chi}(1) |G| (\chi, \varphi) \equiv \hat{\chi}(1) |C| (\chi_C, \varphi_C) \pmod{p}. \]
Cancelling the \( \hat{\chi}(1) \) and using the fact that \( |G| \equiv |C| \not\equiv 0 \pmod{p} \), since \( C \) is the fixed point subgroup, we can justify the first equivalence below:
\[ (\chi, \varphi) \equiv (\chi_C, \varphi_C) = (e\hat{\chi} + p\Delta, d\hat{\varphi} + p\Psi) \equiv ed(\hat{\chi}, \hat{\varphi}) \pmod{p}, \]
where \( e, d \not\equiv 0 \pmod{p} \).

We claim \( \mathcal{G} \) is injective. If \( \hat{\chi} = \hat{\varphi} \), then \( (\chi, \varphi) \equiv ed \not\equiv 0 \pmod{p} \), so since \( (\chi, \varphi) = 0, 1 \) as they are irreducible, necessarily \( (\chi, \varphi) = 1 \), forcing \( \chi = \varphi \).

For surjectivity, take \( \alpha \in \text{Irr}(C) \), and suppose towards a contradiction that \( \alpha \neq \hat{\chi} \) for any \( \chi \in \text{Irr}_P(G) \). Since \( \chi_C = e\hat{\chi} + p\Delta \), then \( \alpha \) is a constituent of \( \Delta \) instead, so \( (\chi_C, \alpha) \equiv 0 \pmod{p} \). By Frobenius reciprocity, we also find \( (\chi_C, \alpha^c) \equiv 0 \pmod{p} \).
Write $\alpha^G = \sum_{\chi \in \text{Irr}(G)} (\alpha^G, \chi) \chi$, and as usual, decompose $\text{Irr}(G) = \text{Irr}_P(G) \cup \mathcal{O}(\chi_1) \cup \cdots \cup \mathcal{O}(\chi_t)$. Since induction of characters commutes with conjugation, if $z \in P$, we have $(\alpha^G)^z = (\alpha^G)_{\chi^z}$ as $\alpha$ is $P$-invariant, so that

$$(\alpha^G, \chi_i) = ((\alpha^G)^z, \chi_i^z) = (\alpha^G, \chi_i^z)$$

implying that the multiplicity of any character in $\mathcal{O}(\chi_i)$ as an irreducible constituent of $\alpha^G$ is the same.

We can write

$$\alpha^G = \sum_{\chi \in \text{Irr}_P(G)} (\alpha^G, \chi) \chi + \sum_{j=1}^t (\alpha^G, \chi_i) \left( \sum_{\psi \in \mathcal{O}(\chi_j)} \psi \right).$$

Evaluating at 1, and using the fact that $(\alpha^G, \chi) \equiv 0 \pmod{p}$ for $\chi \in \text{Irr}_P(G)$, and $|\mathcal{O}(\chi_j)| \equiv 0 \pmod{p}$ for the remaining characters, we find

$$\alpha^G(1) \equiv \sum_{j=1}^t (\alpha^G, \chi_i) |\mathcal{O}(\chi_j)| \chi_j(1) \equiv 0 \pmod{p}.$$

So $p \mid \alpha^G(1)$. However, from character theory, we know the degree of an induced character is equal to the product of the index of the group from which its induced, and the degree of the original character. In this case, $\alpha^G(1) = [G : C]\alpha(1)$. But $p \not| \alpha^G(1)$, since $p \not| \alpha(1)$, as $\alpha(1) \mid |C|$ and $p \mid [G : C]$, a contradiction. So $\mathcal{G}$ is surjective.

We extend this to a more general case.

**Theorem 4.2.** Suppose $A$ is a solvable group, acting by automorphisms on a group $G$, and $(|A|, |G|) = 1$. Then $|\text{Irr}_A(G)| = |\text{Irr}(CG(A))|$. 

**Proof.** We induct on $|A|$. Choose $B < A$ a proper, normal subgroup of order $p^e > 1$. (Such a $B$ does exist, for example, every minimal normal subgroup of a finite solvable group is elementary abelian, in particular it must be a $p$-group for some $p$.)

Then $B$ is a $p$-group of lesser order acting by automorphisms on $G$, so by the Glauberman correspondence, we have a bijection $\text{Irr}_B(G) \rightarrow \text{Irr}(CG(B))$. If $\chi \in \text{Irr}_B(G)$, then $\chi_{CG(B)} = c\chi + p\Delta$. We have that $CG(B)$ is $A$ invariant, since if $a \in A$, so $CG(B)^a = CG(B^a) = CG(B)$, as $B$ is normal in $A$. Also, if $g \in CG(B)$, then

$$\chi\big|_{CG(B)}(g) = \chi^a(g) = \chi(g^{a^{-1}}) = \chi_{CG(B)}(g^{a^{-1}}) = (\chi_{CG(B)})^a(g)$$

from which it follows that $\chi^a = \chi_{\chi^a}^a$. So $A$-invariant characters are mapped to $A$-invariant characters under the Glauberman correspondence, so $|\text{Irr}_A(G)| = |\text{Irr}_A(CG(B))|$. But $B$ acts trivially on $CG(B)$, and we get an induced action of $A/B$ on $CG(B)$. By the induction hypothesis, we get the first equality below:

$$|\text{Irr}_{A/B}(CG(B))| = |\text{Irr}(CG(B)(A/B))| = |\text{Irr}(CG(A))|.$$

The second equality follows since the points of $CG(B)$ fixed by $A/B$ are the points of $CG(B)$ fixed by $A$, since the $B$-action is trivial, which is to say $CG(B)(A/B) = CG(B) \cap CG(A) = CG(A)$ since $CG(A) \subseteq CG(B)$. Combining these two above equations proves the original claim.

## 5 July 18, 2016

Suppose $H$ is a subgroup of $G$, and $\psi \in \text{Irr}(H)$. We say $\psi$ extends to $G$ if there exists $\chi \in \text{Irr}(G)$ such that $\chi_H = \psi$. If $H \leq G$, then necessarily $\psi$ must be $G$-invariant. For if $\chi_H = \psi$, then as $\chi^g = \chi$ as a class function, we see

$$\psi = \chi_H = (\chi^g)_H = (\chi_H)^g = \psi^g.$$ 

**Theorem 5.1.** Suppose $N \leq G$, $[G : N] = p$, and $\theta \in \text{Irr}_G(N)$. Then $\theta$ extends to $G$. 

Proof. Consider the induced character \( \theta^G \), and pick \( \chi \) to be an irreducible constituent. Then
\[
(\chi_N, \theta) = (\chi, \theta^G) \neq 0
\]
so \( \theta \) is an irreducible constituent of \( \chi_N \). From Clifford Theory (cf. Isaacs, the Theorem following Clifford’s theorem in the slides), we know in this case that \( \chi_N \) is either irreducible, or the sum of \( p \) distinct irreducible characters in the same orbit. Since \( \theta \) is a \( G \)-invariant constituent, it has trivial orbit, so the latter case cannot happen, hence \( \chi_N \) is irreducible, and \( \chi_N = \theta \).

We denote by \( \text{Lin}(G) = \{ \alpha \in \text{Irr}(G) : \alpha(1) = 1 \} \) to be the multiplicative group of linear characters on \( G \). Observe that \( |\text{Lin}(G)| = |G : G'| \). For \( \chi \in \text{Irr}(G) \), denote by \( o(\chi) \) to be the order of \( \det(\chi) \) in \( \text{Lin}(G) \).

**Lemma 5.2.** Suppose that \( N \trianglelefteq G \), \( \theta \in \text{Irr}_G(N) \), \( \theta(1) = 1 \), \( o(\theta) \) is a \( p \)-power, and \( P/N \in \text{Syl}_p(G/N) \). Then if \( \theta \) extends to \( P \), \( \theta \) further extends to \( G \).

**Proof.** Since \( \theta \) extends to \( P \), we can find \( \mu \in \text{Irr}(P) \) such that \( \mu_N = \theta \). Thus \( \mu(1) = \mu_N(1) = \theta(1) = 1 \).

Since \( [G/N : P/N] = [G : P] \) is coprime to \( p \), we see \( P \) is a Sylow subgroup of \( G \), and thus
\[
\mu^G(1) = [G : P] \mu(1) = [G : P] \neq 0 \pmod{p}.
\]
Writing \( \mu^G = \sum_{\chi \in \text{Irr}(G)} (\mu^G, \chi) \chi \), evaluating at 1 shows \( p \nmid \chi(1) \) for some \( \chi \), lest \( p \mid \mu^G(1) \). For this \( \chi \),
\[
0 \neq (\chi, \mu^G) = (\chi_P, \mu),
\]
so \( \chi_P = e\mu + \cdots \), so that \( \chi_N = e\mu_N + \cdots \). However, \( \chi_N = e\theta \) as \( \chi_N \) is a sum of \( G \)-conjugates, but \( \theta \) is \( G \)-invariant. Then \( \chi_N(1) = e\theta(1) = e \), so that \( e = \chi_N(1) = \chi(1) \neq 0 \pmod{p} \). The corresponding representation \( X_N \) is thus similar to a direct sum of \( e \) copies of a linear representation, each affording \( \theta \). It follows that \( (\det \chi)_N = \theta^{\chi(1)} \). By hypothesis \( o(\theta) \) is a \( p \)-power, so \( (\chi(1), o(\theta)) = 1 \). We may find \( v \) such that \( v\chi(1) \equiv 1 \pmod{o(\theta)} \). Thus
\[
((d\chi)^v)_N = ((d\chi)N)^v = (d\chi(1))^v \theta^{\chi(1)v} = \theta.
\]

**Lemma 5.3.** Suppose \( N \trianglelefteq G \), \( \theta(1) = 1 \), and \( \theta \in \text{Irr}_G(N) \). Assume \( o(\theta), [G : N] = 1 \). Then \( \theta \) extends to \( G \), and there exists a unique \( \mu \in \text{Irr}(G) \) such that \( \mu_N = \theta \) such that \( o(\mu) = o(\theta) \).

**Proof.** Recall from group theory that if \( g \in G \) is of order \( n = n_1n_2 \) where \( (n_1, n_2) = 1 \), then we can decompose \( g = g_1g_2 \), where \( o(g_1) = n_1 \), \( o(g_2) = n_2 \), and \( g_1 \) and \( g_2 \) commute, and are the unique such elements to satisfy these properties. In this case, we can write \( g = g_1g_2 \), where \( o(g_1) = p^f \), and \( p \nmid o(g_2) \), and \( g_1g_2 = g_2g_1 \).

Now take \( \theta \in \text{Lin}(N) \). Applying the above, we can write \( \theta = \prod_p \theta_p \), where \( o(\theta_p) \) is a \( p \)-power. Assume \( \theta_p \neq 1 \), so that \( p \nmid [G : N] = 1 \), since \( p \nmid o(\theta) \). By the uniqueness of the decomposition, \( \theta_p \) is \( G \)-invariant, since \( \theta = \theta^G = \prod_p \theta_p^G \).

Since \( p \nmid [G : N] \), there is no proper \( P \supseteq N \) such that \( P/N \in \text{Syl}_p(G/N) \), for if there were \( P \) a Sylow subgroup of \( N \) containing \( N \), implying \( [G : N] = [G : P][P : N] \) so that \( p \nmid [P : N] \), a contradiction. So by Lemma 5.2, \( \theta_p \) extends to \( G \), so to \( \mu_p \). Thus \( (\mu_p)_N = \theta \), and \( \mu_p(1) = \theta(1) = 1 \), hence \( \mu_p \) is linear. Set \( \mu = \prod_p \mu_p \), and thus
\[
\mu_N = \prod_p (\mu_p)_N = \prod_p \theta_p = \theta.
\]
For uniqueness, since \( (o(\theta), [G : N]) = 1 \), we can find \( b \) such that \( b[G : N] \equiv 1 \pmod{o(\theta)} \). Set \( \nu = \mu^{b[G : N]} \). Since \( \mu_N = \theta \), we find
\[
\nu_N = (\mu^{b[G : N]})_N = (\mu_N)^{b[G : N]} = \theta^{b[G : N]} = \theta.
\]
We claim that \( o(\nu) = o(\mu) \), and that \( \nu \) is the unique such extension.
First we prove the equality of orders. Note
\[ \theta^{o(\nu)} = (\nu_N)^{o(\nu)} = (\nu^{o(\nu)})_N = 1_N \]
so that \( o(\theta) | o(\nu) \).

Conversely,
\[ \nu^{o(\theta)} = (\mu^{[G:N]}_\beta)^{o(\theta)} = (\mu^{[G:N]}_\beta)^{o(\theta)} = 1. \]

To see the last equality, observe that
\[ (\mu^{o(\theta)})_N = (\mu_N)^{o(\theta)} = \theta^{o(\theta)} = 1_N \]
whence \( \mu \in \text{Lin}(G/N) \). By Lagrange, \( \mu^{[G:N]} = 1_N \). Hence \( o(\nu) | o(\theta) \), and so \( o(\nu) = o(\theta) \).

For uniqueness, suppose we have two characters \( \nu \) and \( \epsilon \) such that \( \nu_N = \theta = \epsilon_N \), with \( o(\nu) = o(\theta) = o(\epsilon) =: m \). Then
\[ (\nu \epsilon)_N = \nu_N \epsilon_N \theta = 1_N \]
as these are all linear characters. So \( \nu \epsilon \in \text{Lin}(G/N) \), and \( (\nu \epsilon)^m = \nu^m \epsilon^m = 1_N \). However, \( (m, [G:N]) = 1 \) by hypothesis, so necessarily \( \nu \epsilon = 1_N \). Indeed, writing \( 1 = sm + t[G:N] \), we find \( \nu \epsilon = (\nu \epsilon)^{sm + t[G:N]} = 1_N \). □

**Lemma 5.4.** Suppose \( N \triangleleft G \), \( \theta \in \text{Irr}_G(N) \) extends to \( G \), and \( (\theta(1), [G:N]) = 1 \). Then the map
\[ \text{Ext}(G|\theta) \to \text{Ext}(G|\det \theta) : \chi \mapsto \det \chi \]
is a bijection.

**Proof.** Let \( \mu \in \text{Irr}(G) \) such that \( \mu_N = \theta \). By the Gallagher correspondence, the map
\[ \text{Irr}(G/N) \to \text{Irr}(G|\theta) : \beta \mapsto \beta \mu \]
is a bijection. Also, \( (\beta \mu)_N = \beta_N \mu_N = \beta(1) \theta \) since \( N \subseteq \ker \beta \), so that \( \beta_N = \beta(1) \). Hence we find that
\[ (\beta \mu)_N = \theta \iff \beta(1) \theta = \theta \iff \beta(1) = 1. \]
It follows that \( |\text{Ext}(G|\theta)| = |\text{Lin}(G/N)| \), since every extension of \( \theta \) to \( G \) is an irreducible character lying over \( \theta \) which corresponds to a character of \( G/N \) by the Gallagher correspondence, which is necessarily linear by the above string of equivalences. Observe that if \( \mu_N = \theta \), then \( (\det \mu)_N = \det \theta \), so that if \( \mu \) extends \( \theta \), then \( \det \mu \) extends \( \det \theta \). The above argument does not depend on \( \theta \), so by the same argument we have
\[ |\text{Ext}(G|\theta)| = |\text{Lin}(G/N)| = |\text{Ext}(G|\det \theta)|, \]
so it suffices to show the map in question is injective to get a bijection.

Take \( \beta_1 \mu \) and \( \beta_2 \mu \) with \( \beta_i \in \text{Lin}(G/N) \) such that \( \det(\beta_1 \mu) = \det(\beta_2 \mu) \). Since \( \beta_i \) simply scales the \( \mu(1) \) rows of the image of the affording representation, the equation states \( \beta_1^{o(\mu)} = \beta_2^{o(\mu)} \). Cancelling \( \det \mu \) yields \( \beta_1^{o(\mu)} = \beta_2^{o(\mu)} \). But \( \mu(1) = \mu_N(1) = \theta(1) \), so \( \beta_1^{o(\mu)} = \beta_2^{o(\mu)} \). Since \( \beta_1 \in \text{Lin}(G/N) \) and since in general, \( |\text{Lin}(H)| = |H : H'| | |H| \), we see \( |\text{Lin}(G/N)| = |G : N| \). Since \( (\theta(1), [G : N]) = 1 \), we have that
\[ (\beta_1 \beta_2)^{o(\mu)} = 1, \]
so that \( \beta_1 = \beta_2 \).

**Lemma 5.5.** Suppose \( N \triangleleft G \), \( G/N \) is solvable, \( \theta \in \text{Irr}_G(N) \), and \( (\theta(1), [G : N]) = 1 \). Then \( \theta \) extends to \( G \) iff \( \det \theta \) extends to \( G \).

**Proof.** We have already seen the forward direction. So suppose \( \det \theta \) extends to \( G \). We induct on \( [G : N] \). If \( [G : N] = 1 \), \( G = N \), so the characters serve as their own extensions. So suppose \( [G : N] > 1 \), and take \( \lambda \in \text{Irr}(G) \) which extends \( \det \theta \). Pick \( M \) to be a maximal normal subgroup, so \( [G : M] = p \) for some prime \( p \). (To find such an \( M \), suppose \( G_1/N \) is the first term in some subnormal series witnessing \( G/N \) being solvable. Then \( (G/N)/(G_1/N) \rhd G/G_1 \) is cyclic of prime order. Then there exists a maximal normal subgroup of index \( p \) in \( G/G_1 \) as \( p \) is the smallest divisor of the order of \( [G/G_1] \), and by the correspondence theorem this
subgroup pulls back to a normal subgroup of $G$ with index $p$. Now $\det \theta$ extends to $M$ as $\lambda_M$ lies over it, so by the induction hypothesis $\theta$ extends to $M$ as well, and we may find $\psi \in \Irr(M)$ such that $\psi_N = \theta$. From Lemma 5.4, the map

$$\text{Ext}(M|\theta) \rightarrow \text{Ext}(M|\det \theta) : \eta \mapsto \det \eta$$

is a bijection. So there exists a unique $\eta \in \text{Ext}(M|\theta)$ such that $\det \eta = \lambda_M$. But $\lambda_M$ is $G$-invariant as the restriction of a character of $G$, and thus for $g \in G$,

$$\det \eta = \lambda_M = (\lambda_M)^g = (\det \eta)^g = \det(\eta^g),$$

so that $\eta = \eta^g$, and $\eta$ is $G$-invariant. So $\eta$ is a $G$-invariant character of $M$ lying over $\theta$, and $[G : M] = p$. By Theorem 5.1, $\eta$ extends to a character $\chi$ of $G$.

With these pieces, we finally have the following.

**Theorem 5.6.** Suppose $N \trianglelefteq G$, $G/N$ is solvable, $\theta \in \Irr_G(N)$, and $(o(\theta), \theta(1), [G : N]) = 1$. Then $\theta$ extends to $G$, and it has a unique extension $\hat{\theta} \in \Irr(G)$ such that $o(\hat{\theta}) = o(\theta)$.

**Proof.** By Lemma 5.2, $\det \theta$ extends to $G$. Note $\det(\theta)(1) = 1$ as a linear character, and $o(\det \theta)$ is coprime to $[G : N]$ since $o(\theta)$ is, and $\det \theta$ is $G$-invariant as $\theta$ is $G$-invariant. By Lemma 5.3, $\theta$ extends to $G$. Then by Lemma 5.4, there is a unique extension of $\det \theta$ with the same order. Then by Lemma 5.4, from the bijection $\text{Ext}(G|\theta) \leftrightarrow \text{Ext}(G|\det \theta)$, we see there is a unique extension of $\theta$ with the same order as $\theta$.

**Corollary 5.7.** Suppose $N \trianglelefteq G$, $G/N$ is solvable, and $N$ is a Hall subgroup. If $\theta \in \Irr_G(N)$, then $\theta$ extends to $G$.

**Proof.** We have $\theta(1) \mid |N|$, so that $(\theta(1), [G : N]) = 1$. Also, $o(\theta) \mid |\text{Lin}(N)| = [N : N'] \mid |N|$, so that $(o(\theta), [G : N]) = 1$. The above theorem now applies.

## 6 July 20, 2016

The **McKay Conjecture** states that if $G$ is a group, $p$ a prime, and $P \in \text{Syl}_p(G)$, then

$$|\Irr_p(G)| = |\Irr_p(N_G(P))|$$

where $\Irr_p(G) = \{ \chi \in \Irr(G) : p \nmid \chi(1) \}$.

Recall that if $K \trianglelefteq P$ is a normal Hall subgroup, and $G/K$ is a $p$-group, then $\theta \in \Irr(K)$ extends to $G$ if and only if $\theta$ is $G$-invariant. In this situation, the $G$-invariant characters are precisely the $P$-invariant characters for $P \in \text{Syl}_p(G)$. For then $G = K P$, since $|KP| = |K||P| = |K||G : K| = |G|$, since $p \nmid |K|$, so $K$ has index the maximal $p$-power of $|G|$. Taking $g = ky$, for $k \in K$, $y \in Y$, then if $\theta$ is $P$-invariant, and automatically $K$-invariant,

$$\theta^g = (\theta^k)^y = \theta^y = \theta$$

so $\theta$ is $G$-invariant.

**Theorem 6.1.** Suppose there exists $K \trianglelefteq G$ such that $p \nmid |K|$, and $G/K$ is a $p$-group. Let $P$ be a complement of $K$. Then the McKay Conjecture holds, i.e.,

$$|\Irr_p(G)| = |\Irr_p(N_G(P))|.$$ 

**Proof.** Let $\theta \in \Irr_P(K)$. As $\theta$ is $P$-invariant, it has an extension $\hat{\theta} \in \Irr(G)$. By the Gallagher correspondence, $\text{Ext}(G|\theta) = \{ \beta \hat{\theta} : \beta \in \text{Lin}(G/K) \}$. We claim

$$\Irr_p(G) = \bigsqcup_{\theta \in \Irr_P(K)} \{ \beta \hat{\theta} : \beta \in \text{Lin}(G/K) \}$$

for all $\beta \in \text{Lin}(G/K)$.
We have \( \hat{\theta}_K = \theta \), so \( \hat{\theta}(1) = \theta(1) \) divides \( |K| \), has is not divisible by \( p \). So
\[
(\beta \hat{\theta})(1) = \beta(1)\hat{\theta}(1) = \hat{\theta}(1) \neq 0 \pmod{p}
\]
so that \( \beta \hat{\theta} \in \text{Irr}_p(G) \).

Conversely, take \( \chi \in \text{Irr}_p(G) \). Let \( \theta \) be an irreducible constituent of \( \chi_K \). We know \( \chi(1)/\theta(1) \mid [G : K] \), a \( p \)-power. Since \( p \mid \chi(1) \), as \( \chi(1) \mid |K| \), necessarily \( \chi(1)/\theta(1) = 1 \), Thus \( \chi_K = \theta \) by comparing degrees. So \( \chi_K \in \text{Irr}_p(G) \), and is \( P \)-invariant as the restriction of a \( G \)-invariant character. By the Gallagher correspondence, \( \chi = \beta \hat{\theta} \) for some \( \beta \in \text{Lin}(G/K) \), giving the equality.

To see the union is disjoint, suppose \( \beta \hat{\theta} = \gamma \hat{\eta} \). Then
\[
\beta_K \hat{\theta}_K = (\beta \hat{\theta})_K = (\gamma \hat{\eta})_K = \gamma_K \hat{\eta}_K.
\]
This implies \( \beta(1)\theta = \gamma(1)\eta \), or that \( \eta = \theta \). Hence
\[
|\text{Irr}_p'(G)| = \sum_{\theta \in \text{Irr}_p(K)} \left| \{ \beta \hat{\theta} : \beta \in \text{Lin}(G/K) \} \right| = |\text{Irr}_p(K)| |\text{Lin}(G/K)|.
\]

However, \( P \) is a Sylow subgroup of \( N_G(P) \), with complement \( K \cap N_G(P) = N_K(P) = C_K(P) \), this last equality following from an earlier lemma. Applying the above equation to this group,
\[
|\text{Irr}_p'(N_G(P))| = |\text{Irr}_p(N_K(P))||\text{Lin}(N_G(P)/N_K(P))|
= |\text{Irr}_p(C_K(P))||\text{Lin}(G/K)|
= |\text{Irr}(C_K(P))||\text{Lin}(G/K)|
= |\text{Irr}(C_K(P))||\text{Lin}(G/K)|
= |\text{Irr}_p(K)||\text{Lin}(G/K)|
\]
where the second equality follows since
\[
\frac{N_G(P)}{N_K(P)} = \frac{N_G(P)}{N_G(P) \cap K} \simeq \frac{N_G(P)K}{K} = G/K,
\]
the third equality following since any character on \( C \) is automatically \( P \)-invariant, and the last equality follows by the Glauberman correspondence.

Consider the case where \( P \) is self-normalizing, that is, \( P = N_G(P) \). As the \( p' \)-degree characters of \( P \) are precisely the linear ones, McKay posits that
\[
|\text{Irr}_p'(G)| = |\text{Irr}_p'(N_G(P))| = |\text{Irr}_p'(P)| = |\text{Lin}(P)| = [P : P'].
\]

**Lemma 6.2.** Suppose \( N \trianglelefteq G \), \( P \in \text{Syl}_p(G) \), and \( \chi \in \text{Irr}_p'(G) \). Then \( \chi_N \) has a \( P \)-invariant irreducible constituent \( \theta \), and any two \( P \)-invariant irreducible constituents are conjugate in \( N_G(P) \).

**Proof.** Suppose \( \eta \text{Irr}(N) \) is a constituent of \( \chi_N \). Put \( T = \text{I}_G(\eta) \) to be the stabilizer group. By the Clifford correspondence, we find \( \psi \text{Irr}(T/\eta) \) such that \( \psi^G = \chi \). Observe that \( \chi(1) = \psi^G(1) = |G : T|\psi(1) \), so \( p \nmid |G : T| \) since \( p \nmid \chi(1) \). Thus any \( p \)-Sylow subgroup of \( T \) is a \( p \)-Sylow subgroup of \( G \). Thus \( P^g \subseteq T \) for some \( g \in G \). Hence
\[
P \subseteq T^{g^{-1}} = \text{I}_G(\eta)^{-1} = \text{I}_G(\eta^{-1}).
\]
Let \( \theta = \eta^{-1} \). As \( \eta \) is a constituent of \( \chi_N \), Clifford’s Theorem implies any \( G \)-conjugate of \( \eta \) is a constituent as well, so \( \theta \) is a constituent of \( \chi_N \), which is \( P \)-invariant.

Now suppose \( \gamma \) is another \( P \)-invariant irreducible constituent of \( \chi_N \). By Clifford’s Theorem, there exists \( x \in G \) such that \( \gamma^x = \theta \), so
\[
T^{g^{-1}} = I_G(\theta) = I_G(\gamma^x) = I_G(\gamma)^x.
\]
Since $\gamma$ is $P$-invariant, $P \subseteq I_G(\gamma)$, so $P^x \subseteq T^{g^{-1}}$. Thus $P$ and $P^x$ are both $p$-Sylow subgroups of $T^{g^{-1}}$, thus there is $t \in T^{g^{-1}}$ such that $P^xt = P$, so that $xt \in N_G(P)$. Then $\gamma^{xt} = \theta^t = \theta$, since $t \in T^{g^{-1}} = I_G(\theta)$. So $\gamma$ and $\theta$ are conjugate in $N_G(P)$.

**Lemma 6.3.** Suppose $N \trianglelefteq G$, $N$ is a $p$-group, and $P \in \text{Syl}_p(G)$ is Sylow subgroup containing $N$. Let $\Omega = \{ \lambda \in \text{Lin}(N) : \lambda \text{ extends to } P \}$. Note that $N_G(P)$ acts on $\Omega$ by conjugation, that is, if $n \in N_G(P)$, then $\lambda^n \in \text{Lin}(N^n) = \text{Lin}(N)$, and $\lambda^n$ extends to $P^n = P$. Decompose $\Omega$ into orbits as $\Omega = \bigsqcup_{j=1}^k O(\lambda_k)$. Set $T_j = I_G(\lambda_j)$. We claim

$$|\text{Irr}_{p'}(G)| = \sum_{j=1}^k |\text{Irr}_{p'}(T_j/N)|.$$  

**Proof.** We show

$$\text{Irr}_{p'}(G) = \bigsqcup_{j=1}^k \text{Irr}_{p'}(G|\lambda_j).$$

Take $\chi \in \text{Irr}_{p'}(G)$. The restriction $\chi_P$ must have an irreducible constituent of degree coprime to $p$, else evaluating $\chi(1) = \chi(P)$ shows $p \mid |\chi(1)|$, a contradiction. Let $\mu$ be such a constituent. Since $\mu(1) \mid |P|$, necessarily $\mu$ is linear. So $\mu_N \in \text{Irr}(N)$, and in fact $\mu_N \in \Omega$, so $\mu_N$ is $N_G(P)$-conjugate to some $\lambda_i$. As the irreducible constituents of $\chi_N$ form an orbit, $\lambda_i$ is also a constituent of $\chi_N$, so $\chi$ lies over $\lambda_i$. Hence $\chi \in \text{Irr}_{p'}(G|\lambda_i)$.

The reverse containment is obvious, so it remains to show the union is disjoint. Suppose $\chi \in \text{Irr}_{p'}(G|\lambda_i) \cap \text{Irr}_{p'}(G|\lambda_j)$ for $i \neq j$. Then $\lambda_i$ and $\lambda_j$ are irreducible constituents of $\chi_N$, hence $N_G(P)$ conjugate by the previous lemma, a contradiction as representatives of distinct orbits. Hence we conclude

$$|\text{Irr}_{p'}(G)| = \sum_{j=1}^k |\text{Irr}_{p'}(G|\lambda_j)|.$$  

By the Clifford correspondence, we have a bijection

$$\text{Irr}(T_j|\lambda_j) \rightarrow \text{Irr}(G|\lambda_j) : \psi \mapsto \psi^G.$$  

Since $\lambda_j$ extends to $P$, it is $P$-invariant, so $P \subseteq T_j$, hence $p \nmid |G : T|$. From $\psi^G(1) = [G : T]|(\psi(1))$, we see that $p \nmid \psi(1)$ iff $p \nmid \psi^G(1)$, so that in fact

$$\text{Irr}_{p'}(T_j|\lambda_j) \rightarrow \text{Irr}_{p'}(G|\lambda_j).$$  

Thus

$$|\text{Irr}_{p'}(G)| = \sum_{j=1}^k |\text{Irr}_{p'}(G|\lambda_j)| = \sum_{j=1}^k |\text{Irr}_{p'}(T_j|\lambda_j)|.$$  

By assumption, $\lambda_j$ extends to $P$, is $T_j$-invariant by definition, and since $o(\lambda_j)$ and $\lambda_j(1)$ divide $|N|$ hence are $p$-powers, we see $(o(\lambda_j), [T_j : P]) = 1 = (\lambda(j), [T_j : P])$. By the lemma above, $\lambda_j$ extends to $T_j$, denote this extension $\nu_j \in \text{Irr}(T_j)$. By Gallagher’s correspondence, we have an isomorphism

$$\text{Irr}(T_j/N) \rightarrow \text{Irr}(T_j|\lambda_j) : \beta \mapsto \beta \nu_j.$$  

Note $\nu_j(1) = 1$ as the extension of a linear character, so $\beta$ is $p'$-degree iff $\beta \nu_j$ is $p'$-degree. Thus $|\text{Irr}_{p'}(T_j/N)| = |\text{Irr}_{p'}(T_j|\lambda_j)|$, and we get the formula.

**Corollary 6.4.** A minimal counterexample to the McKay conjecture must have $O_p(G) = 1$. (Here $O_p(G)$ denotes the largest, normal $p$-subgroup of $G$.) Setting $T_j = I_{G}(P_j(\lambda_j))$, the Clifford correspondence gives a bijection $\text{Irr}_{p'}(T_j|\lambda_j) \rightarrow \text{Irr}_{p'}(T_j|\lambda)$, so the formula applied to $G$ and $N_G(P)$ gives the result. □
Theorem 6.5. Suppose $G$ is $p$-solvable, $P \in \text{Syl}_p(G)$, and $P = N_G(P)$. If $\chi \in \text{Irr}_p(G)$, then $\chi_P = \lambda + \Delta$, where $\lambda \in \text{Lin}(P)$, and $\Delta = 0$, or the constituents of $\Delta$ have degree divisible by $p$. Furthermore, the map

$$\text{Irr}_p'(G) \to \text{Irr}(P/P') : \chi \mapsto \lambda$$

is a bijection.

Proof. We consider several cases. First suppose there exists $K \subseteq G$ such that $p \nmid |K|$, and $K > 1$. Pick $\chi \in \text{Irr}_p(G)$. As we have seen before, there exists $\theta \in \text{Irr}_p(K)$ a constituent of $\chi$, and since $p \nmid |K|$ and $K \leq G$, $P$ acts on $K$ by conjugation, so the Glauberman correspondence applies and $|\text{Irr}_p(K)| = |\text{Irr}(C_K(P))|$. But $C_K(P) \subseteq N_G(P) = P$, so $C_K(P) \subseteq P \cap K = 1$. Thus $|\text{Irr}_p(K)| = 1$, and necessarily $\theta = 1_K$. So $\chi_K$ has the principal character $1_K$ as a constituent, and any other constituent is a conjugate of $1_K$. Since $1_K$ is invariant, $1_K$ is the only constituent, so $K \subseteq \ker \chi$. So $\chi \in \text{Irr}_p(G/K)$.

Observe that $|(G/K)/(PK/K)| = |G/PK|$, and $|G/PK| = |G|/|P| |K|$, so that $PK/K$ is a $p$-Sylow subgroup of $G/K$, and $N_{G/K}(PK/K) = N_G(PK)/K = PK/K$. This show that if $P$ is self-normalizing in $G$, then $PK/K$ is self-normalizing in $G/K$. By the induction hypothesis, $\chi_{PK/K} = \lambda + \Delta$ where $\lambda \in \text{Lin}(PK/K)$ and the constituents of $\Delta$ have degree divisible by $p$. Since $K \subseteq \ker \chi$, and $K \subseteq \ker \lambda$, we can view this as a character on $P$, so $\chi_P = \lambda_P + \Delta_P$, and we get the result.

Now suppose no such $K$ exists. Let $N = O_p(G)$. As $N$ is a $p$-group, it is contained in some Sylow subgroup $P$, and let $\theta \in \text{Irr}_P(N)$ lie under $\chi$. If $T = I_G(\theta)$ is the inertia group, by the Clifford correspondence, there exists $\psi \in \text{Irr}_p(T/\theta)$ such that $\psi^G = \chi$. Also by the Clifford correspondence, $\chi_T = \psi + \Xi$, where no irreducible constituent of $\Xi$ lies over $\theta$. Then $\chi_P = \psi_P + \Xi_P$. Suppose $T < G$ is proper. Then $N_T(P) = N_G(P) \cap T = P \cap T = P$, as $P \subseteq T$, as $\theta$ is $P$-invariant. By the induction hypothesis, considering the tower of group $T/P/N$, we have $\psi_P = \lambda + \Delta$, where $\lambda \in \text{Lin}(P)$, and the degrees of the constituents of $\Delta$ are divisible by $p$. Thus

$$\chi_P = \psi_P + \Xi_P = \lambda + \Delta + \Xi_P$$

and it remains to show that $\Xi_P$ has all constituents with degrees divisible by $p$. Towards a contradiction, suppose $\Xi_P$ has a constituent $\mu \in \text{Irr}(P)$ such that $1 \nmid |\mu|$. Since $|\mu(1)|$ necessarily $\mu(1) = 1$. Note $\mu$ is $P$-invariant as it is a class function on $P$, so $\eta := \mu_N$ is $P$-invariant and lies under $\chi$. But $P$-invariant characters of normal subgroups are $N_G(P)$-conjugate, So $\eta$ and $\theta$ are $N_G(P)$ conjugate, but since $N_G(P) = P$ and they are $P$-invariant, $\eta = \theta$. This is a contradiction since no constituent of $\Xi$ lies over $\theta$.

Lastly, suppose $T = G$. Then $\chi_P$ has $p'$-degree since $\chi$ does, but every irreducible constituent of $\chi_P$ must have degree dividing $|P|$. Hence there must exist a constituent $\epsilon$ which is linear. So $\epsilon_P$ and $\theta$ are $P$-invariant constituents of $\chi_N$, so as we have seen before $\epsilon_P = \theta$, and thus $\theta$ is linear. Thus $\theta$ extends to $P$, and $\sigma(\theta)$ is a $p$-power since $\theta \in \text{Lin}(N)$, and $|\text{Lin}(N)| = [N : N']$ is a $p$-power. To reiterate, we have a $P$-invariant linear character of $p$-power order which extends to $P$, hence to $G$, by a previous theorem above. Call this extension $\sigma \in \text{Irr}(G)$.

Consider $\chi \bar{\nu} \in \text{Irr}_p'(G)$, since $\chi$ is irreducible, and $\bar{\nu}$ is linear. Then $\chi_N = \chi(1)\theta$, since $\theta$ is the only constituent of $\chi_N$, and so

$$(\chi \bar{\nu})_N = \chi(1)\theta \bar{\nu}_N = \chi(1)\sigma(\theta) = \chi(1)\epsilon_N.$$

So $N \subseteq \ker \chi \bar{\nu}$, and thus $\chi \bar{\nu} \in \text{Irr}(G/N)$.

By the induction hypothesis, $(\chi \bar{\nu})_P = \delta + \Psi$ where $\delta$ is linear, and the constituents of $\psi$ have degrees divisible by $p$. So

$$\chi_P = \delta \nu_P + \Psi \nu_P = \delta \epsilon + \Psi \epsilon.$$

Note that $\delta \epsilon$ is still linear, and the degrees of the constituents of $\Psi \epsilon$ are unchanged as $\epsilon$ is linear.

It remains to show the map is a bijection. Both sets in question have the same cardinality. If $K$ exists as above, then we make take the quotient by $K$ as it is in the kernel of every character involved to reduce the problem to a smaller group. Otherwise, suppose we are in the case where nontrivial $N = O_p(G)$ exists. In this case, the $T_j/N$ are smaller groups with self-normalizing $p$-Sylow subgroup $P/N$, so the McKay conjecture holds. Also,

$$N_{T_j/N}(P/N) = N_{T_j}(P)/N = N_G(P) \cap T_j/N = I_{N_G(P)}(\lambda_j)/N$$
so that
\[
|\text{Irr}_p(G)| = \sum |\text{Irr}_p(T_j/N)| = \sum |\text{Irr}_p(N_{T_j/N}(P/N))| = \sum |\text{Irr}_p(I_{N_G(P)}(\lambda_j)/N)| = |\text{Irr}_p(N_G(P))| = |\text{Irr}(P/P')|
\]
where the second equality follows by the induction hypothesis, the third by the preceding observation, the fourth by the lemma, and the last since \(N_G(P) = P\), so the \(p^i\)-degree characters are precisely the linear ones of \(P\).

Take \(\lambda \in \text{Irr}(P/P')\). Then \(\lambda^G\) has \(p^j\)-degree since \(\lambda^G(1) = [G : P]\lambda(1) = [G : P]\). So some constituent of \(\chi^G\) has \(p^j\)-degree, which is to say \((\lambda^G, \chi) \neq 0\) for some \(\chi \in \text{Irr}_p(G)\). By reciprocity, \((\lambda, \chi_P) \neq 0\). Then \(\lambda\) is a constituent of \(\chi_P = \epsilon + \Delta\), where \(\epsilon\) is linear and \(\Delta\) has constituents with degrees divisible by \(p\). So necessarily \(\epsilon = \Delta\). This shows the map is surjective, hence bijective for cardinality reasons.

\[\Box\]

7 July 22, 2016

Recall that if \(A\) is an algebra over a field \(F\), then the Jacobson radical is \(J(A) = \bigcap \text{ann}_A(V)\), where the intersection is taken over the simple right \(A\)-modules \(V\). The algebra \(A\) is said to be semisimple if \(J(A) = 0\), and it follows that \(A/J(A)\) is semisimple.

**Remark 7.1.** Suppose \(A\) is a semisimple algebra. We have the following facts:

- There exist a finite number \(B_1, \ldots, B_n\) of distinct minimal ideals of \(A\), and \(A = \bigoplus_{i=1}^n B_i\).
- Let \(I_j\) be a minimal right ideal in \(B_j\). Then the \(I_j\) are the simple \(A\)-modules up to isomorphism. Let \(X_i\) denote the corresponding irreducible representations.
- Let \(r_j: A \rightarrow \text{End}_F(I_j)\), \(a \mapsto r_j(a)\), where \(r_j(a)\) is the map acting by right multiplication by \(a\) on \(I_j\). Since \(B_iB_j \subseteq B_i \cap B_j = 0\), it follows \(\ker r_j = \bigoplus_{i \neq j} B_j\), and the restriction \(r_j|_{B_j}\) is an isomorphism of algebras onto its image.
- The \(B_i \cong M_{n_i}(F)\) for some \(n_i\), so \(A\) is a direct sum of matrix algebras. If \(\chi_i\) is the trace of \(X_i\), then \(\{\chi_1, \ldots, \chi_n\}\) is linearly independent.

**Proof:** This is standard. For the last part, by Wedderburn, the irreducible representations \(X_i\) are surjective, so we can choose \(x_i \in B\) such that \(X_i(x_i) = \text{diag}(1,0,\ldots,0)\). It follows that \(\chi_i(x_j) = \delta_{ij}\), and linear independence follows.

Now, suppose \(G\) is a finite abelian group, and \(F\) is a field of characteristic \(p\), and \(X: G \to GL_n(F)\) an irreducible representation. Then \(X(g)X(h) = X(h)X(g)\) for all \(g, h \in G\), and thus \(X(g)\) commutes with the image of \(X\), so by Schur, which still applies, \(X(g) = \lambda_g I_n\) for some \(\lambda_g\). Since \(X\) is irreducible, necessarily \(n = 1\).

Let \(R\) be the ring of algebraic integers, and \(p\) a prime. Pick a maximal ideal \(m\) such that \(pR \subseteq m \subseteq R\), and set \(F = R/m\). We then have a canonical projection
\[
*: R \rightarrow F: r \mapsto r* = r + m
\]
and \(\text{char}(F) = p\).

Denote by \(U = \{\zeta \in \mathbb{C} : \zeta^m = 1,\ p \nmid m\}\), which is a subset of \(R\).

**Lemma 7.2.** The map
\[
U \rightarrow F^\times : u \mapsto u^*
\]
is a group isomorphism, and \(F\) is the algebraic closure of \(\mathbb{Z}^* = \mathbb{Z}/p\mathbb{Z}\).
Proof. Suppose $\zeta \neq 1$ is an element of $U$, so that $o(\zeta) = m$ for some $p \nmid m$. We have the identity

$$1 + X + \cdots + X^{m-1} = \frac{X^{m-1} - 1}{X - 1} = \prod_{i=1}^{m-1} (X - \zeta^i).$$

Evaluating at $X = 1$ yields

$$m = (1 - \zeta) \prod_{i=2}^{m-1} (1 - \zeta^i),$$

and so $1 - \zeta | m$ in $R$. If $\zeta^* = 1$, then $m^* = 0$ in $F$, implying $p | m$, a contradiction. So the map has trivial kernel, and is injective.

Furthermore, the extension $F/\mathbb{Z}^*$ is algebraic. Take $f \in F$, so $f = r^*$ for some $r \in R$. As $r$ is an algebraic integer, it satisfies some relation

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

where $a_i \in \mathbb{Z}$. Applying the canonical surjection gives

$$f^n + a_{n-1}^*f^{n-1} + \cdots + a_1^* f + a_0^* = 0,$$

so that $f$ is algebraic over $\mathbb{Z}^*$.

Suppose $K/F$ is an algebraic extension. I claim $K^\times \subseteq \mathcal{U}^*$. Take $k \in K^\times$. Since $K/F$ and $F/\mathbb{Z}^*$ are algebraic, $k$ is algebraic over $\mathbb{Z}^*$. Thus $L := \mathbb{Z}^*(k) = \mathbb{Z}^*[k]$ is a finite field, as a finite extension of a finite field. Thus $k|L^\times| = 1$, so $o(k) | |L^\times| \equiv -1 \pmod p$, since $|L|$ is a $p$-power. Thus the order of $k$ is not divisible by $p$, and if $|L^\times| = m$, then $k$ is a root of $X^m - 1$, so $k = \zeta^*$ for some $\zeta$ a root of $X^m - 1$. Thus $k \in \mathcal{U}^*$. Since $K^\times \subseteq \mathcal{U}^*$, we see $F^\times \subseteq \mathcal{U}^*$, so that $\mathcal{U} = F^\times$. This shows the map is surjective. It also implies $K \subseteq F$, so that $K = F$, and thus $F$ is algebraically closed.

Let $G^p = \{g \in G : p \nmid o(g)\}$ denote the set of $p$-regular elements. Let $X: G \to GL_n(F)$ be a representation. Since $F$ is algebraically closed, $X(g)$ has $n$-eigenvalues, denote them $\lambda_1, \ldots, \lambda_n$, all nonzero since $X(g)$ is invertible. By the above isomorphism, $\lambda_i = u_i^*$ for a unique $u_i \in \mathcal{U}$. Define a function

$$\varphi: G^p \to R : g \mapsto \sum_{i=1}^{n} u_i.$$

This $\varphi$ depends on the choice of $X$, and is the Brauer character afforded by $X$. This is a close analogue to the trace function, as

$$\varphi(g)^* = \sum u_i^* = \sum \lambda = \text{tr}(X(g)).$$

Let $\text{IBr}(G)$ denote the set of irreducible Brauer characters, i.e., the characters corresponding to irreducible representations. This set is finite as there only finitely many irreducible representations by Wedderburn. As $F[G]/J(F[G])$ is semisimple, it decomposes as a sum of matrix algebras of sizes $n_i$, so

$$\dim(F[G]/J(F[G])) = \sum n_i = \sum_{\varphi \in \text{IBr}(G)} \varphi(1)^2.$$

Also, $\varphi(1)$ is the degree of the representation.

**Lemma 7.3.** Suppose $X$ is an $F$-representation of $G$ with trace $\psi$. If $g \in G$ decomposes as the unique commuting product $g = g_p g_{p'}$, where $o(g_p)$ is a $p$-power, $o(g_{p'})$ a $p'$-power, then $\psi(g) = \psi(g_{p'})$.

**Proof.** Without loss of generality, we may assume $G = \langle g \rangle$, as we only care about a property of $g$. Since $X(g)$ is similar to an upper triangular matrix with irreducible representations on the diagonal, it suffices to prove the claim for irreducible representations of an abelian group, i.e., linear representations. Suppose $X_i: G \to F^\times$ is such a representation. Since $g_p$ has order $p^l$, say, then

$$X_i(g_p)^{p^l} = X_i(g_p^{p^l}) = 1$$
so that \( X_i(g_p) \) is a root of the polynomial \( Z^{p^i} - 1 = (Z - 1)^{p^i} \) over \( F[Z] \), so that \( X_i(g_p) = 1 \). Thus 
\[
X_i(g) = X_i(g_p)X_i(g_{p'}) = X_i(g_{p'}),
\]
as required. 

From these two lemmas, it is immediate that if \( \varphi \in \text{IBr}(G) \), and we define \( \varphi^* \) by \( \varphi^*(g) = \varphi(g_{p'})^* \), then 
\[
\{ \varphi^* : \varphi \in \text{IBr}(G) \} = \{ \text{tr}(X) : X \in \text{Irr}(G) \}
\]

It turns out the irreducible Brauer characters are linearly independent over \( \mathbb{C} \). We will need a fact from algebraic number theory. 

**Lemma 7.4.** Suppose \( I \leq R \) is a proper ideal, and \( \alpha_1, \ldots, \alpha_n \) are algebraic complex numbers, not all 0. Then there exists \( \beta \in \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \) such that \( \beta \alpha_i \in R \) for all \( i \), but there exists at least one \( \beta \alpha_i \notin I \).

**Theorem 7.5.** The set \( \text{IBr}(G) \) is linearly independent over \( \mathbb{C} \), and \( \text{cf}(G^\circ) \) are a \( \mathbb{C} \)-space of dimension equal to the number of \( p \)-regular conjugacy classes of \( G \).

**Proof.** Since the irreducible Brauer characters take values in \( R \subseteq \overline{Q} \), it is enough to show they are \( \overline{Q} \)-linearly independent. Viewing the characters as elements in \( \overline{Q} \)-space, the matrix whose columns are the coordinate vectors has nonzero determinant, and this determinant does not change if we view it as a matrix in \( \mathbb{C} \).

So suppose 
\[
\sum_{\varphi \in \text{IBr}(G)} a_{\varphi} \varphi = 0
\]
where the \( a_{\varphi} \in \overline{Q} \) are not all 0. Since \( m \leq R \) is a proper ideal, by the above lemma, we can find \( \beta \) such that \( \beta \alpha_{\varphi} \in R \) for all \( \varphi \), but there exists \( \mu \) such that \( \beta \mu \notin m \). Scaling by \( \beta \) yields 
\[
\sum (\beta a_{\varphi}) \varphi = 0
\]
and since the coefficients are in \( R \), applying the canonical surjection gives 
\[
\sum (\beta a_{\varphi})^* \varphi^* = 0,
\]
but since the \( \varphi^* \) are the traces of the irreducible representations, hence \( F \)-linear independent, we must have \( (\beta a_{\varphi})^* = 0 \), or \( \beta \alpha_{\varphi} \in m \) for all \( \varphi \in \text{IBr}(G) \), a contradiction. 

We can extend \( * \) to the localization \( R_m \) by 
\[
R_m \to F : \alpha \beta \mapsto \alpha^*(\beta^*)^{-1}.
\]

Now suppose \( X : G \to \text{GL}_n(R_m) \) affords \( \chi \). The map \( R_m \to F \) induces a map \( \text{GL}_n(R_m) \to \text{GL}_n(F) \), so by postcomposition we get an induced representation \( X^* : G \to \text{GL}_n(F) \). Thus a \( \mathbb{C} \)-irreducible character defines an \( F \)-representation in characteristic \( p \), which thus has a Brauer character. Observe \( X^* \) affords \( \chi^\circ := \chi|_{G^\circ} \). Every Brauer character comes from an \( F \)-representation, which decomposes into irreducible representations, so 
\[
\chi^\circ = \sum_{\varphi \in \text{IBr}(G)} d_{\chi \varphi} \varphi
\]
where the \( d_{\chi \varphi} \) are nonnegative integers known as decomposition numbers.

**Theorem 7.6.** (Brauer) The set \( \text{IBr}(G) \) is a basis of \( \text{cf}(G^\circ) \). In particular, \( |\text{IBr}(G)| \) is the number of conjugacy classes of \( G \) of elements of \( p' \) order.

**Proof.** Take \( \delta \in \text{cf}(G^\circ) \), and extend it to \( \text{cf}(G) \) by zero on the \( p \)-singular classes. Call this extension \( \eta \). Then 
\[
\eta = \sum_{\chi \in \text{Irr}(G)} a_{\chi} \chi \implies \eta^\circ = \delta = \sum_{\chi \in \text{Irr}(G)} a_{\chi} \chi^\circ = \sum_{\chi \in \text{Irr}(G)} a_{\chi} \left( \sum_{\varphi \in \text{IBr}(G)} d_{\chi \varphi} \varphi \right).
\]
8 Exercises

Exercise 8.1. Let \( K \) be an arbitrary field, and let \( z \) be the sum of all elements of a finite group \( G \) in the group algebra \( K[G] \). Show that the one-dimensional subspace \( Kz \) of \( K[G] \) is an ideal. Also, prove that \( z \) is nilpotent iff the characteristic of \( K \) divides \(|G|\).

Proof. It is clear \( Kz \) is an additive subgroup. Observe that if \( g \in G \), then \( gz = gz = z \), since left or right multiplication simply permutes the summands of \( z \). If \( \sum c_i g_i \in K[G] \) is arbitrary, then

\[
\left( \sum c_i g_i \right) z = \sum c_i g_i z = \sum c_i z = \left( \sum c_i \right) z \in Kz.
\]

So \( Kz \) is a left ideal, and it follows easily that \( Kz \) is a two-sided ideal of \( K[G] \).

Observe \( z^2 = (\sum_{g_i \in G} g_i)z = \sum_{g_i \in G} gz = |G|z \), and inductively, we see \( z^n = |G|^{n-1}z \). So suppose \( z \) is nilpotent, say \( z^n = 0 \). Then \( 0 = z^n = |G|^{n-1}z \). It follows that \(|G|^{n-1} = 0 \), so that \( \text{char}(K) \mid |G|^{n-1} \). Since \( \text{char}(K) \) is necessarily prime in this case, Euclid’s theorem gives \( \text{char}(K) \mid |G| \). On the other hand, suppose \( \text{char}(K) \nmid |G| \). Then \( |G| = 0 \), so in particular \( z^2 = |G|z = 0 \). So \( z \) is nilpotent, and \( z^2 = 0 \) even.

Exercise 8.2. With the notation as above, show that \( K[G] \) has a proper ideal \( I \) such that \( Kz + I = K[G] \) iff the characteristic of \( K \) does not divide \(|G|\).

Proof. First, suppose there is a proper \( I \) such that \( Kz + I = K[G] \). Observe

\[
|G| = \dim(K[G]) = \dim(Kz + I) = \dim(Kz) + \dim(I) - \dim(Kz \cap I) = 1 + \dim(I)
\]

since \( Kz \cap I = 0 \), as \( Kz \) is one dimensional, and not in \( I \). So in fact \( Kz \oplus I = K[G] \). Towards a contradiction, suppose \( \text{char}(K) \nmid |G| \), so that \( z^2 = 0 \) by the previous exercise. Since \( 1 \in Kz \oplus I \), we can write \( 1 = \lambda z + x \) for \( x \in I \), necessarily \( x = 1 - \lambda z \). Since \( I \) is an ideal,

\[
(1 - \lambda z)z = z - \lambda z^2 = z \in I
\]

implying \( I = K[G] \), a contradiction.

Now suppose \( \text{char}(K) \mid |G| \). Then the claim follows immediately from Maschke’s Theorem. More directly, We have a surjective morphism

\[
\varphi: K[G] \rightarrow K: \sum a_i g_i = \sum a_i.
\]

Then \( I := \ker \varphi \) is a proper ideal of \( K[G] \), and by rank-nullity,

\[
\dim I = \dim(K[G]) - \dim K = |G| - 1.
\]

Since \( \varphi(z) = |G| \neq 0 \), it follows from dimension considerations that \( Kz \oplus I = K[G] \), so \( Kz \) has a complement.

Exercise 8.3. If \( \chi \) is a character of \( G \) and \( \lambda \) is a linear character, define \( \lambda \chi \) to be the function defined by \((\lambda \chi)(g) = \lambda(g) \chi(g)\). Show that \( \lambda \chi \) is a character, and that it is irreducible iff \( \chi \) is irreducible.

Proof. Suppose the representation \( \rho \) affords \( \chi \). Define a map \( \tilde{\rho}: G \rightarrow GL_n(K) \) by \( \tilde{\rho}(g) = \rho(g) \lambda(g) \). Since \( \lambda \) is linear, it takes values in \( \mathbb{C}^\times \), so it is easy to check \( \tilde{\rho} \) is a homomorphism. Furthermore,

\[
\text{tr}(\tilde{\rho}(g)) = \text{tr}(\rho(g) \lambda(g)) = \lambda(g) \text{tr}(\rho(g)) = \lambda(g) \chi(g) = (\lambda \chi)(g)
\]

so that \( \lambda \chi \) is a character.
Also, if \( \lambda \chi \) is irreducible, then \( \chi \) is irreducible, for if \( \chi = \alpha + \beta \), then \( \lambda \chi = \lambda \alpha + \lambda \beta \). Conversely, if \( \chi \) is irreducible, then \( (\chi, \chi) = 1 \). Since \( \lambda \) is linear, it takes roots of unity as its values. Then we find that

\[
(\lambda \chi, \lambda \chi) = \frac{1}{|G|} \sum_{g \in G} \lambda(g)\chi(g)\overline{\lambda(g)}\overline{\chi(g)} = \frac{1}{|G|} \sum_{g} |\lambda(g)|^2|\chi(g)|^2 = \frac{1}{|G|} \sum_{g} |\chi(g)|^2 = (\chi, \chi) = 1
\]

so that \( \lambda \chi \) is irreducible.

**Exercise 8.4.** Let \( N \trianglelefteq G \) and \( \psi \in \operatorname{Irr}(G/N) \). Define \( \chi : G \to \mathbb{C} \) by setting \( \chi(g) = \psi(Ng) \). Show that \( \chi \in \operatorname{Irr}(G) \) and that \( N \subseteq \ker \chi \). Also, so that the map \( \psi \mapsto \chi \) from \( \operatorname{Irr}(G/N) \) to \( \operatorname{Irr}(G) \) defines a bijection from \( \operatorname{Irr}(G/N) \) to the set \( \{ \chi \in \operatorname{Irr}(G) : N \subseteq \ker \chi \} \).

**Proof.** Let \( R : G/N \to \operatorname{Aut}(G/N) \) be the regular representation of \( G/N \) acting on itself by multiplication. If \( \chi_i \) is an irreducible character, put \( N_i = \ker \chi_i \). Then

\[
\ker R = \{ Ng \in G/N : Ng \text{ acts as the identity } \} = \{ Ng : g \in N \} = N/N.
\]

Since \( N \) acts trivially, we get an induced representation \( \tilde{R} : G \to \operatorname{Aut}(G/N) \) given by \( \tilde{R}(g) = R(Ng) \). Let \( \chi \) be the character afforded by \( \tilde{R} \), so

\[
\chi(g) = \operatorname{tr}(\tilde{R}(g)) = \operatorname{tr}(R(Ng)).
\]

I claim that \( N = \ker \chi \). Let \( n \in N \). Then

\[
\chi(n) = \operatorname{tr}(R(Nn)) = |G/N| = \operatorname{tr}(R(N))
\]

so \( N \subseteq \ker \chi \).

Conversely, suppose \( g \in \ker \chi \). So

\[
\chi(1) = \operatorname{tr}(\tilde{R}(1)) = \operatorname{tr}(R(N)) = |G/N|.
\]

Thus \( \chi(g) = |G/N| = \operatorname{tr}(R(Ng)) \), so that \( g \in N \).

Also, observe that if \( \chi \) and \( \psi \) are as in the problem statement,

\[
(\chi, \chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \frac{1}{|G|} \sum_{g \in G} \psi(Ng)\overline{\psi(Ng)} = \frac{|N|}{|G|} \sum_{Ng \in G/N} \psi(Ng)\overline{\psi(Ng)} = (\psi, \psi)
\]

from which we see irreducibility is preserved.

**Exercise 8.5.** Show that

\[
\bigcap \{ \ker \chi : \chi \in \operatorname{Irr}(G) \} = 1,
\]

the trivial subgroup of \( G \).

**Proof.** We can number the irreducible characters as \( \operatorname{Irr}(G) = \{ \chi_1, \ldots, \chi_k \} \). If \( \chi \) is a character, then we can write

\[
\chi = \sum_{i=1}^{k} m_i \chi_i
\]

for \( m_i \geq 0 \) integers. I claim \( \ker \chi = \bigcap_{i:m_i > 0} \ker \chi_i \). Recall that \( |\chi_i(g)| \leq \chi_i(1) \). If \( g \in \ker \chi \), then \( \chi(g) = \sum m_i \chi_i(g) = \sum m_i \chi_i(1) = \chi(1) \), so necessarily \( \chi_i(g) = \chi_i(1) \), and thus \( g \in \bigcap_{i:m_i > 0} \ker \chi_i \). Conversely, if \( g \in \bigcap_{i:m_i > 0} \ker \chi_i \), then

\[
\chi(g) = \sum m_i \chi_i(g) = \sum m_i \chi_i(1) = \chi(1)
\]
so \( g \in \ker \chi \).

From this we can conclude \( \bigcap_{\chi \in \text{Irr}(G)} \ker \chi = 1 \). Take the regular character \( \rho \). If \( g \neq 1 \), then \( \rho(g) = 0 \), while \( \rho(g) = |G| \), so assuming \( |G| \neq 0 \) in the field, \( \ker \rho = 1 \), and we get equality.

**Exercise 8.6.** Let \( N \trianglelefteq G \). Show that

\[
N = \bigcap_{\chi \in \text{Irr}(G)} \ker \chi.
\]

**Proof.** Let \( \rho \) be the regular representation of \( G/N \). We can view this as a representation of \( G \) with kernel \( N \). Explicitly, if \( \rho(gN) \) is the automorphism determined by multiplication by \( gN \), we can define a representation \( \rho' \) of \( G \) to be \( \rho'(g) = \rho(gN) \), which is well-defined. Let \( \chi \) be the character afforded by \( \rho' \). We see that

\[
\chi(g) = \text{tr}(\rho'(g)) = \text{tr}(\rho(gN)) = \begin{cases} |G/N| & \text{if } g \in N, \\ 0 & \text{if } g \not\in N. \end{cases}
\]

It follows that \( N = \ker \chi \). If \( n \in N \), then

\[
\chi(n) = \text{tr}(\rho(N)) = |G/N| = \chi(1)
\]

so \( N \subset \ker \chi \). Conversely, if \( g \in \ker \chi \), then

\[
\text{tr}(\rho'(g)) = \chi(g) = \chi(1) = \text{tr}(\rho(N)) = |G/N|
\]

so that \( g \in N \). Since we have a character whose kernel is precisely \( N \), it follows that

\[
N = \bigcap_{\chi \in \ker \chi} \ker \chi.
\]

**Exercise 8.7.** Suppose \( \chi \) is the unique member of \( \text{Irr}(G) \) having degree \( d \). Show that \( \chi(x) = 0 \) for all \( x \in G \setminus G' \).

**Proof.** If \( G = G' \), the claim is vacuous. Otherwise, pick \( x \in G \setminus G' \). Since \( \bigcap_{\chi_i \in \text{Irr}(G/G')} \ker \chi_i \), we can find \( \lambda \in \text{Irr}(G/G') \) such that \( \lambda(x) \neq 1 \), for if not, then \( x \in \bigcap_{\chi_i \in \text{Irr}(G/G')} \ker \chi_i = 1 \), that is, the class of \( x \) vanishes in \( G/G' \), contrary to \( x \not\in G' \). Since \( G/G' \) is abelian, \( \lambda \) is necessarily a linear character. Then the product of characters \( \lambda \chi \) is a character, and by the Kronecker product of matrices used in the proof of this fact, we see that the degree of \( \lambda \chi \) is the product of degrees. Hence \( \lambda \chi \) is a character of degree \( d \), so by uniqueness, \( \lambda \chi = \chi \). It follows that

\[
(\lambda(x) - 1)\chi(x) = \lambda(x)\chi(x) - \chi(x) = 0,
\]

whence \( \chi(x) = 0 \).

**Exercise 8.8.** Let \( \chi \) be a character of \( G \), and let \( \mathcal{X} \) afford \( \chi \). Define the function \( \lambda \) on \( G \) by \( \lambda(g) = \det(\mathcal{X}(g)) \). Show that \( \lambda \) is a linear character of \( G \) and that it does not depend on the choice of the representation \( \mathcal{X} \) affording \( \chi \).

**Proof.** We have \( \lambda : G \to \mathbb{C}^\times \), and \( \lambda \) is a homomorphism as the composition of two homomorphisms. It follows that \( \lambda \) is a linear character. Since the determinant is a similarity invariant, and different representations affording the same character \( \chi \) are similar, \( \lambda \) is independent of the choice of representation.

**Exercise 8.9.** Let \( \chi \) be a character of \( G \). Prove that \( Z(\chi)/\ker(\chi) \) is a cyclic subgroup of \( Z(G/\ker(\chi)) \).
Proof. Let $X$ be a representation affording $\chi$. Then for $g \in G$, $X(g) \sim \text{diag}(\epsilon_i)$, where each $\epsilon_i$ is a root of unity. Then

$$\chi(g) = \text{tr}(X(g)) = \sum \epsilon_i.$$ 

Now $g \in Z(\chi)$ iff $|\chi(g)| = |\sum \epsilon_i| = \chi(1)$. This equality is achieved iff all $\epsilon_i$ are equal. If we denote that common value as $\epsilon$, we have $X(g) \sim \epsilon I$, so that $X(g) = \epsilon I$. Thus we can characterize the center as

$$Z(\chi) = \{g \in G : X(g) = \epsilon I \text{ for some } \epsilon \in \mathbb{C}\}.$$ 

We can define a map $\lambda : G \to \mathbb{C}^\times$ by $X(g) = \lambda(g)I$, so that $\lambda$ is a homomorphism, thus a linear character of $G$. Then for $z \in Z(\chi)$, we have

$$\chi_{Z(\chi)}(z) = \chi(z) = \text{tr}(X(z)) = \text{tr}(\lambda(z)I) = \lambda(z)\chi(1)$$

so that $\chi_{Z(\chi)} = \chi(1)\lambda$.

From this, we find $\ker \chi = \ker \lambda$. Take $g \in \ker \chi$. Then $\chi(g) = \chi(1)$, which holds iff $X(g) = I$, hence iff $X(g) = \lambda(g)I = I$, iff $g \in \ker \lambda$. As $\lambda$ is a group homomorphism, we have

$$G/\ker \chi = G/\ker \lambda \simeq \text{Im}(\lambda),$$

which is cyclic as a finite subgroup of the multiplicative group of the field $\mathbb{C}$.

Lastly, observe that $X(Z(\chi)) \subseteq Z(X(G))$. For take $g \in Z(\chi)$, so that $X(g)$ is a scalar matrix, as observed above. Then $X(g)$ is central in the matrix algebra, hence in $X(G)$ as well. Now let $g \in \ker \chi$ and $h \in G$. Then

$$X(gh) = X(g)X(h) = X(h)X(g) = X(hg)$$

so that $gh(hg)^{-1} \in \ker X = \ker \chi$. Thus $gh \ker \chi = hg \ker \chi$, so $g \ker \chi \subseteq Z(G/\ker \chi)$, so that $Z(\chi)/\ker \chi \leq Z(G/\ker \chi)$. 

Exercise 8.10. If $\chi \in \text{Irr}(G)$, show that $Z(\chi)/\ker(\chi) = Z(G/\ker \chi)$. 

Proof. We need only show the reverse direction. So take $g \ker \chi \in Z(G/\ker \chi)$. If $X$ affords $\chi$, we see that $X$ induces a representation $\tilde{X}$ in $G/\ker X$. Then for $h \in G,$

$$X(g)X(h) = X(gh) = \tilde{X}(g \ker X h \ker X) = \tilde{X}(h \ker X g \ker X) = X(hg) = X(h)x(g),$$

the second equality following since $g \ker X = g \ker \chi$ is central. Since $\chi$ is irreducible $X(G)$ is surjective, so $X(g)$ is central in the codomain matrix group, hence is a scalar matrix, say $X(g) = zI$ for some $z \in \mathbb{C}$. So $g \in Z(\chi)$, and $g \ker \chi \subseteq Z(\chi)/\ker \chi$. 

Exercise 8.11. Let $P$ be a $p$-group for some prime $p$. Show that $P$ has a faithful irreducible character if and only if $Z(P)$ is cyclic.

Proof. Suppose $\chi \in \text{Irr}(P)$ is faithful, so $\ker \chi = 1$. By the previous exercise, we have

$$Z(P) \simeq Z(P/\ker \chi) = Z(\chi)/\ker \chi \simeq Z(\chi).$$

Since $Z(\chi)/\ker \chi$ is cyclic, necessarily $Z(P)$ is cyclic. Note that we need not assume $P$ is a $p$-group for this implication.

Conversely, suppose $Z(P)$ is cyclic. Let $N \subseteq P$ be a nontrivial normal subgroup. Then $N \cap Z(P) \neq 1$, since $N$ is a union of conjugacy classes, and the noncentral classes have $p$-power cardinality. Since $\{e\}$ is a class in $N$, it follows $N$ must have other singleton conjugacy classes, hence nontrivial central elements.

Let $H$ be the unique subgroup of order $p$ in $P$. Since $N \cap Z(P)$ is a nontrivial $p$-group in $Z(P)$, it must contain $H$. Suppose no irreducible character is faithful. Then their kernels are nontrivial normal subgroups containing $H$, implying

$$H \leq \bigcap_{\chi \in \text{Irr}(P)} \ker \chi = 1,$$

a contradiction. So $P$ has a faithful, irreducible character.
**Exercise 8.12.** Let $\chi \in \text{Irr}(G)$, where $G$ is a simple group of even order. Show that $\chi(1) \neq 2$.

**Proof.** By Cauchy’s theorem, pick $g \in G$ having order 2, and towards a contradiction, assume $\chi(1) = 2$. Let $X$ be the representation affording $\chi$. Then $X(g) \sim \text{diag}(\epsilon_1, \epsilon_2)$. Since $X(g)^2 = X(g^2) = I_2$, the $\epsilon_i$ are square roots of unity, hence $X(g) \sim \text{diag}(\pm 1, \pm 1)$, so necessarily $\chi(g) = \pm 2$. If $\chi(g) = \pm 2$, then $g \in Z(\chi)$, so $Z(\chi)$ is a nontrivial normal subgroup of $G$. Since the center is the preimage of the scalar matrices in $X(G)$, and $X(G)$ is surjective when irreducible as its image is a nonzero ideal of a simple ring, $Z(\chi)$ is proper, contrary to $G$ being simple.

So $\chi(g) = 0$. Then $X(g)$ is similar to a matrix with diagonal entries of opposite signs, so $\det(\chi)(g) = \det(X(g)) = -1$. However, $\det(\chi)$ is a linear character of $G$. Since $\chi$ is nonlinear by assumption, $G$ is not abelian. Thus $[G, G] \neq 1$ is a nontrivial normal subgroup of $G$, simplicity gives $[G, G] = G$, so $G$ has a unique linear character. Necessarily $\det(\chi) = 1_G$, implying $\det(\chi)(g) = 1\neq -1$, again a contradiction. So $\chi(1) \neq 2$. \qed

**Exercise 8.13.** Let $G$ be a simple group and suppose $\chi \in \text{Irr}(G)$ has degree 3. Compute $\chi(g)$ for an element $g \in G$ of order 2.

**Proof.** Let $X$ afford $\chi$. By the same reasoning as the previous exercise, $\chi(1) = 3$, and $X(g) \sim \text{diag}(\pm 1, \pm 1, \pm 1)$. And by inspection, the possible values are $\chi(g) = \pm 1, \pm 3$. So if $\chi(g) = \pm 3$, then $g \in \ker \chi$, so $Z(\chi)$ is a nontrivial normal subgroup as before, a contradiction.

So suppose $\chi(g) = \pm 1$. Then $X(g)$ is similar to a diagonal matrix with either two entries equal to 1, and one entry equal to $-1$, or two entries equal to 1 and one entry equal to $-1$. However, since $G$ is simple, but has a nonlinear character $\chi$, necessarily $G = [G, G]$, so the principal character is the unique linear character, as explained above. Since $\det(\chi)$ is linear, it is principal, so

$$1 = \det(\chi)(g) = \det(X(g))$$

so that $X(g)$ is similar to a diagonal matrix with two entries equal to $-1$, and one entry equal to 1. So $\chi(g) = 1 - 1 - 1 = -1$. \qed

**Exercise 8.14.** Repeat the above if $\chi$ has degree 4.

**Proof.** Now $\chi(1) = 4$, and $X(g) \sim \text{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$, giving the possibilities $\chi(g) = 0, \pm 2, \pm 4$. But if $\chi(g) = \pm 4$, then $Z(\chi)$ is a nontrivial normal subgroup, a contradiction. The only diagonal matrices with traces $\pm 2$, have either three entries 1 and one entry $-1$, or vice versa, and in either case, the determinant is $-1$. This is impossible since $\det(\chi)$ must be the principal character. So $X(g)$ is similar to a matrix with two entries 1 and two entries $-1$, when $\chi(g) = 1 + 1 - 1 - 1 = 0$. \qed

**Exercise 8.15.** If $H \subseteq G$ and $\chi$ is a character of $G$, it is easy to see that the restriction of $\chi$ to $H$, denoted $\chi_H$, is a character of $H$. Show that $H \subseteq Z(\chi)$ iff $\chi_H$ as the form $e\lambda$, where $e$ is a positive integer and $\lambda$ is a linear character of $H$.

**Proof.** Suppose $H \subseteq Z(\chi)$. We have characterized above that $Z(\chi) = \{g \in G : X(g) = eI\text{ for some }e \in \mathbb{C}\}$, and we can define a linear character $\lambda$ by $X(g) = \lambda(g)I$. Thus for $h \in H$,

$$\chi_H(h) = \chi(h) = \text{tr}(X(h)) = \text{tr}(\lambda(h)I) = \lambda(h)\chi(1)$$

so that $\chi_H = \chi(1)\lambda_H$. Since $\chi(1) = e$ is a positive integer, $\chi_H$ has the desired form.

Conversely, suppose $\chi_H = e\lambda$. Then for $h \in H$, $\chi_H(h) = e\lambda(h)$. Since $\lambda$ is a linear character, it takes values in the roots of unity, so

$$|\chi_H(h)| = |e\lambda(h)| = e \cdot |\lambda(h)| = e.$$

In particular, $\chi(1) = |\chi_H(1)| = e$, so $|\chi_H(h)| = \chi(1)$, so $H \subseteq Z(\chi)$. \qed

**Exercise 8.16.** Let $H \subseteq G$ and let $\chi$ be a character of $G$ such that the restriction $\chi_H$ is irreducible. Show that $C_G(H) \subseteq Z(\chi)$.
Proof. Let $X : G \to GL_n(K)$ be a representation affording $\chi$, and let $X_H$ be the restriction to $H$. Since $\chi_H$ is irreducible, $X_H$ is irreducible. Take $g \in C_G(H)$, so $gh = hg$ for all $h \in H$. Then $X(g)X(h) = X(h)X(g)$ for all $h \in H$, implying $X(g)$ centralizes $X(H) = X_H(H)$. Since $X_H$ is irreducible, Schur’s lemma implies that $X(g)$ is necessarily scalar, hence $g \in Z(\chi)$, by a previous characterization of $Z(\chi)$.

Exercise 8.17. Let $H \subseteq G$ and $\chi \in \text{Irr}(G)$, and recall that $\chi_H$ is the character of $H$ obtained by restricting $\chi$ to $H$. Show that $(\chi_H, \chi_H) \leq [G : H]$ with equality if and only if $\chi$ has the value 0 on all elements of $G \setminus H$.

Proof. Observe that

$$(\chi_H, \chi_H) = \frac{1}{|H|} \sum_{h \in H} |\chi_H(h)|^2$$

$$\leq \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2 + \frac{1}{|H|} \sum_{g \in G \setminus H} |\chi(g)|^2$$

$$= \frac{|G|}{|H|} (\chi, \chi) = [G : H] (\chi, \chi).$$

Assuming $\chi$ is irreducible, $(\chi, \chi) = 1$, and we get the desired inequality. Equality is achieved if and only if $\sum_{g \in G \setminus H} |\chi(g)|^2 = 0$, which holds if and only if $\chi$ is zero on $G \setminus H$. \hfill \Box

Exercise 8.18. If $G$ is abelian, show that $(\chi, \chi) \geq \chi(1)$ for all (not necessarily irreducible) characters $\chi$ of $G$.

Proof. Since $G$ is abelian, every irreducible character is linear. We can express a character $\chi$ as

$$\chi = \sum_{\psi \in \text{Irr}(G)} (\chi, \psi) \psi$$

where $(\chi, \psi)$ is a nonnegative integer. Then $\chi(1) = \sum_{\psi \in \text{Irr}(G)} (\chi, \psi)$ using the fact that $\psi(1) = 1$, and $(\chi, \chi) = \sum_{\psi \in \text{Irr}(G)} (\chi, \psi)^2$, so the statement reduces to showing

$$\sum_{\psi \in \text{Irr}(G)} (\chi, \psi) \leq \sum_{\psi \in \text{Irr}(G)} (\chi, \psi)^2.$$

This is immediate since the $(\chi, \psi)$ are nonnegative integers. \hfill \Box

Exercise 8.19. Let $H \subseteq G$, with $H$ abelian, and let $\chi \in \text{Irr}(G)$. Show that $\chi(1) \leq [G : H]$.

Proof. Using the previous exercise, we have $(\chi_H, \chi_H) \geq \chi_H(1)$, so that

$$\chi(1) = \chi_H(1) \leq (\chi_H, \chi_H) \leq [G : H] (\chi, \chi) = [G : H].$$

Exercise 8.20. Let $\chi \in \text{Irr}(G)$, where $G$ is a nontrivial group. Show that there is a nonidentity element $g \in G$ such that $\chi(g) \neq 0$.

Proof. Towards a contradiction, suppose that $\chi(g) = 0$ for all $g \neq 1$. We have

$$1 = (\chi, \chi) = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = \frac{1}{|G|} |\chi(1)|^2.$$

This implies

$$\chi(1)^2 = |G| \sum_{\psi \in \text{Irr}(G)} \psi(1)^2.$$

This implies that $\chi$ is the only irreducible character of $G$, so $G$ has only one conjugacy class, which occurs if $G$ is trivial, contrary to $G$ being a nontrivial group. \hfill \Box
Exercise 8.21. Let $H < G$ with $H$ abelian. If $\chi \in \text{Irr}(G)$ with $[G : H] = \chi(1)$, show that $H$ contains a nonidentity normal subgroup of $G$.

Proof. First observe that

$$\chi(1) = \chi|_H(1) \leq (\chi|_H, \chi|_G) \leq [G : H](\chi, \chi) = [G : H] = \chi(1)$$

giving equality throughout. Since $(\chi|_H, \chi|_H) = [G : H]$, by an above exercise, $\chi$ vanishes on $G \setminus H$. Let $N = \langle g \in G : \chi(g) \neq 0 \rangle$. This generated group is inside $H$, hence is abelian. By the previous exercise, $N$ is nontrivial. To see $N$ is normal, take $x \in N$ to be a generator, so $\chi(x) \neq 0$, and $g \in G$. Then $\chi(gxg^{-1}) = \chi(x) \neq 0$, so $gxg^{-1} \in N$, and it follows that $N$ is normal. \hfill \square

Exercise 8.22. Let $G$ be a $p$-group, where $G' = Z(G)$ has order $p$. Prove the following.

1. Each noncentral class of $G$ has size $p$.
2. $G$ has exactly $p - 1$ nonlinear irreducible characters.
3. The average of $\chi(1)^2$ for nonlinear $\chi \in \text{Irr}(G)$ is $|G|/p$.
4. $\chi(1)^2 \leq |G|/p$ for all $\chi \in \text{Irr}(G)$, and thus $\chi(1)^2 = |G|/p$ for nonlinear $\chi \in \text{Irr}(G)$.

Proof.

1. Let $K$ be a noncentral class. Thus $|K| \neq 1$, and since $|K| \mid |G|$, $|K| = p^j$ for some $j \geq 1$. Let $y = x^g$. Then $yx^{-1} = x^g x^{-1} = [g^{-1}, x] \in G' = Z(G)$, so $Z(G)x = Z(G)y$, or $y \in Z(G)x$. This shows the entire conjugacy class of $x$ is contained in a coset of $Z(G)$, so that $|K| \leq |Z(G)| = p$, and thus $|K| = p$.

2. Let $N$ be the number of noncentral classes. If $|G| = p^a$, we have

$$|G| = p^a = |Z(G)| + pN = p + pN.$$

Thus $N = p^{a-1} - 1$. So the total number of conjugacy classes is $p^{a-1} - 1 + p$, and this is the number of irreducible characters. The number of linear characters is $|G/G'| = p^{a-1}$, so the number of nonlinear irreducible characters is the difference $p - 1$.

3. Observe

$$p^a = |G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\chi(1) = 1} \chi(1)^2 + \sum_{\chi(1) \neq 1} \chi(1)^2 = p^{a-1} + \sum_{\chi(1) \neq 1} \chi(1)^2.$$

Hence

$$\sum_{\chi(1) \neq 1} \chi(1)^2 = p^a - p^{a-1}.$$

Thus the average is

$$\frac{p^a - p^{a-1}}{p - 1} = p^{a-1} = \frac{|G|}{p}.$$

4. Recall

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2.$$

Since there is more than one irreducible character of $G$, necessarily $\chi(1)^2 < |G|$. Since $\chi(1) \mid |G|$, $\chi(1)^2$ is a $p$-power, so necessarily $\chi(1)^2 \leq |G|/p$. By the previous part, since the average of the squared degrees of nonlinear irreducible characters is $|G|/p$, equality must be achieved for each of them. \hfill \square

Exercise 8.23. Suppose $G = HK$, where $H$ and $K$ are subgroups, and let $D = H \cap K$. If $\alpha$ is a class function on $H$, show that

$$(\alpha^G)_K = (\alpha_D)^K.$$
Proof. If $x \in K$,
\[
(\alpha_D)^K(x) = \frac{1}{|D|} \sum_{g \in K} \alpha_D^g(gxg^{-1})
\]
\[
= \frac{1}{|D|} \sum_{g \in K, gxg^{-1} \in D} \alpha(gxg^{-1})
\]
\[
= \frac{1}{|D|} \sum_{g \in K, gxg^{-1} \in H} \alpha(gxg^{-1})
\]
\[
= \frac{|G|}{|H||K|} \sum_{g \in K, gxg^{-1} \in H} \alpha(gxg^{-1})
\]
\[
= \frac{1}{|H|} [G : K] \sum_{g \in K, gxg^{-1} \in H} \alpha(gxg^{-1})
\]
\[
= \frac{1}{|H|} \sum_{g \in G, gxg^{-1} \in H} \alpha(gxg^{-1})
\]
\[
= (\alpha^G)(x) = (\alpha^G)_K(x).
\]

Exercise 8.24. Let $H \subseteq G$. Let $\alpha$ be a class function on $H$ and let $\beta$ be a class function on $G$. Show that
\[
(\beta_H \alpha)^G = \beta \alpha^G.
\]

Proof. If $x \in G$,
\[
(\beta_H \alpha)^G(x) = \frac{1}{|H|} \sum_{g \in G} (\beta_H \alpha)^g(gxg^{-1})
\]
\[
= \frac{1}{|H|} \sum_{g \in G, gxg^{-1} \in H} (\beta_H \alpha)(gxg^{-1})
\]
\[
= \frac{1}{|H|} \sum_{g \in G, gxg^{-1} \in H} \beta(gxg^{-1}) \alpha(gxg^{-1})
\]
\[
= \frac{1}{|H|} \sum_{g \in G, gxg^{-1} \in H} \beta(x) \alpha(gxg^{-1})
\]
\[
= \beta(x) \frac{1}{|H|} \sum_{g \in G} \alpha^g(gxg^{-1})
\]
\[
= (\beta \alpha^G)(x).
\]

Exercise 8.25. Let $H \subseteq K \subseteq G$ and let $\alpha$ be a class function on $H$. Show that
\[
(\alpha^K)^G = \alpha^G.
\]

Proof. First, we have for all $x \in G$,
\[
(\alpha^G)(x) = \frac{1}{|H|} \sum_{g \in G} \alpha^g(gxg^{-1}) = \frac{1}{|H|} \sum_{g \in G, gxg^{-1} \in H} \alpha(gxg^{-1}).
\]

On the other hand, for $x \in G$,
\[
(\alpha^K)^G(x) = \frac{1}{|K|} \sum_{g \in G, gxg^{-1} \in K} (\alpha^K)^g(gxg^{-1}) = \frac{1}{|K|} \sum_{g \in G, gxg^{-1} \in K} \frac{1}{|H|} \sum_{y \in K, yxg^{-1} \in H} \alpha(ygx^{-1}y^{-1}).
\]
Making the change of variables $g' = yg$, we have
\[
(\alpha^K)_G(x) = \frac{|K|}{|H||K|} \sum_{g' \in G \atop g'xg^{-1} \in H} \alpha(g'xg^{-1})
\]
and the conclusion follows. \(\square\)

**Exercise 8.26.** Suppose that every nonlinear irreducible character of $G$ has degree at least $n$. Show that if $H \subseteq G$ and $[G : H] \leq n$, then $G' \subseteq H$, and thus $H \leq G$.

**Proof.** First observe
\[
((1_H)^G, 1_G) = (1_H, (1_G)_H) = (1_H, 1_H) = 1
\]
so $(1_H)^G$ contains $1_G$ as a constituent.

Now summing over a complete set of coset representatives in $G/H$, we have
\[
(1_H)^G(1) = \sum 1^o_H(g1g^{-1}) = \sum 1_H(1) = [G : H] \leq n.
\]

We can write $(1_H)^G$ as a sum of irreducible characters, necessarily linear characters as the nonlinear characters have degree at least $n$. So write $(1_H)^G = \sum_{i=1}^t \chi_i$ where $\chi_i(1) = 1$ and $t = [G : H]$, and these characters are not necessarily distinct.

Suppose $x \in \ker(1_H)^G$, so
\[
(1_H)^G(x) = (1_H)^G(1) = [G : H] = \sum_{g \in G/H} 1^o_H(gxg^{-1}) = \sum_{g \in G/H \atop g \notin H} 1^o_H(gxg^{-1}).
\]
Necessarily every summand must have value 1, so $1^o_H(x) = 1$, hence $x \in H$. So $\ker(1_H)^G \subseteq H$. Since $G' \subseteq \ker \chi_i$ for all $i$ as linear characters, we have
\[
G' \subseteq \ker \left( \sum \chi_i \right) = \ker(1_H)^G \subseteq H.
\]

It now follows that $H$ is normal, for if $g \in G$, then $ghg^{-1}h^{-1} \in G' \subseteq H$, implying $ghg^{-1} \in H$. \(\square\)

**Exercise 8.27.** Let $G = NH$, where $N \trianglelefteq G$ and $N \cap H = 1$, and assume $1 < N < G$. Show that $H$ is a Frobenius complement in $G$ iff $C_N(h) = 1$ for all $h \in H$ with $h \neq 1$.

**Proof.** Suppose $H$ is a Frobenius complement. Suppose $h \in H$ is not the identity. If $x \in C_N(h)$, then $h \in H^x$, so $h \in H \cap H^x$, and thus $x \in H$ by definition of Frobenius complement. So $x \in N \cap H = 1$, and thus $C_N(h) = 1$.

Conversely, suppose $C_N(h) = 1$ for all $h \in H$ not equal to 1. Take $x \in G$, so $x = yn$ where $y \in H$, and $n \in N$. Suppose $H \cap H^y = H \cap H^y > 1$, so we can find a nontrivial $h \in H \cap H^y$. So $h = a^n$ for some $a \in H$. Thus $ha^{-1} = n^{-1}ana^{-1} \in H \cap N$. Since $H \cap N = 1$, we see $h = a$, so $n \in C_N(h) = 1$. Thus $x \in H$, implying $H$ is a Frobenius complement. \(\square\)

**Exercise 8.28.** If $H$ is a Frobenius complement in $G$ and $|H|$ is even, show that the Frobenius kernel $N$ is abelian.

**Proof.** First consider the case where $|H| = 2$, let $h \in H$ be the nonidentity element. Let $\varphi_h : N \to N$ be the inner automorphism induced by $h$. Since $hn \notin N$, which means $hn \in H^x$ for some $x \in G$. Thus $hn$ and $h$ have the same order, so $(hn)^2 = 1$, or $hnh = n^{-1}$. Thus $\varphi_h$ is just the inversion map on $N$, and as a homomorphism, $N$ must be abelian.

Now suppose $2 \mid |H|$. By Cauchy, we find $h \in H$ of order 2, and let $\hat{H} = \langle h \rangle$. Then $\hat{H}$ acts on $N$ by conjugation as before, so by the same argument, we find $N$ is abelian. \(\square\)
Exercise 8.29. Let \( \text{cd}(G) = \{ \chi(1) : \chi \in \text{Irr}(G) \} \). Suppose all members of \( \text{cd}(G) \) are powers of the prime \( p \).

1. If \( G \) is nonabelian, show that \( [G : G'] \) is divisible by \( p \).

2. Show that \( G \) has an abelian normal subgroup \( A \), where \( [G : A] \) is a power of \( p \) and \( p \) does not divide \( |A| \).

Proof.

1. We have
   \[
   |G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\chi \in \text{Irr}(G) \chi(1)=1} \chi(1)^2 + \sum_{\chi \in \text{Irr}(G) \chi(1)>1} \chi(1)^2 = [G : G'] + \sum_{\chi \in \text{Irr}(G) \chi(1)>1} \chi(1)^2.
   \]
   Since \( \chi(1) \mid |G| \), and \( G \) has nonlinear characters as it is nonabelian, \( p \mid |G| \), from which it follows \( p \mid [G : G'] \).

2. We proceed by induction on \( G \), the case \( |G| = 1 \) being trivial. Suppose \( G \) is abelian, hence a direct product of its Sylow subgroups. Let \( A \) be the complement of \( P \in \text{Syl}_p(G) \). Then \( A \) is abelian, normal, and \( G/A \cong P \) is a \( p \)-power, and \( p \nmid |A| \) as its order is the product of the orders of the other Sylow subgroups. In fact, \( A \) is characteristic, for if \( g \in A \), then \( p \nmid o(g) \). If \( \varphi \) is any automorphism of \( g \), then \( p \nmid o(\varphi(g)) \), so \( \varphi(g) \) has no component in the factor \( P \). So \( \varphi(g) \in A \).

So suppose \( G \) is nonabelian. Then \( G/G' \) is a group of smaller order. By the induction hypothesis, there exists an abelian, characteristic subgroup \( H/G' \) of \( G/G' \) of \( p \)-power index, such that \( p \mid [H : G'] \). Since \( H/G' \) is characteristic in \( G \), and \( G' \) is characteristic in \( G \), then \( H \) is characteristic in \( G \). As \( H \) is normal, Clifford’s Theorem implies that \( \psi(1) \mid \chi(1) \) for any \( \psi \in \text{Irr}(H) \) and \( \chi \in \text{Irr}(G/[\psi]) \). It follows that the character degrees of \( H \) are power of \( p \) as well. Again by the induction hypothesis, there exists an abelian characteristic subgroup \( A \) of \( H \) of \( p \)-power index such that \( p \mid |A| \). Since being a characteristic subgroup is transitive, \( A \) is characteristic in \( G \), hence normal, and \( [G : A] = [G : H][H : A] \) is a \( p \)-power. So this \( A \) is the desired subgroup.

Exercise 8.30. Let \( N \) be a normal Hall \( \pi \)-subgroup of \( G \), where \( \pi \) is a set of primes, such that \( G/N \) is solvable. Let \( \chi \in \text{Irr}(G) \). Show that the degrees of the irreducible constituents of \( \chi_N \) are equal to the \( \pi \)-part of \( \chi(1) \).

Proof. Suppose \( \pi = \{p_1, \ldots, p_k\} \) is the set of primes, and \( |G| = p_1^{a_1} \cdots p_k^{a_k} q \), where \( q \) is coprime to the \( p_i \). Thus \( |N| = p_1^{a_1} \cdots p_k^{a_k} \) and \( [G : N] = q \). Since \( G/N \) is solvable, the corollary to Clifford’s theorem implies that for any \( \theta \) any irreducible constituent of \( \chi_N \), we have
   \[
   \chi(1) = \chi_N(1) = et\theta(1)
   \]
   where \( e \) is the ramification, and \( t \) is the size of the orbit of \( \theta \). Thus \( et = \chi(1)/\theta(1) \) divides \( |G : N| = q \), and \( \theta(1) \mid |N| = p_1^{a_1} \cdots p_k^{a_k} \). Since \( et\theta(1) = |G| = |G| = p_1^{a_1} \cdots p_k^{a_k} q \), and \( \theta(1) \) is coprime to \( q \), and \( et \) is coprime to all \( p_i \), necessarily \( \theta(1) = p_1^{a_1} \cdots p_k^{a_k} \).

Exercise 8.31. Write \( b(G) \) for the largest irreducible character degree of \( G \).

1. If \( H \subseteq G \), show that \( b(H) \leq b(G) \).

2. If \( N \leq G \) and \( G/N \) is nonabelian, show that \( b(N) \leq b(G)/2 \).

Proof.

1. Take \( \theta \in \text{Irr}(H) \) such that \( \theta(1) = b(H) \). Let \( \chi \) be an irreducible constituent of \( \theta^G \). Then
   \[
   0 \neq (\chi, \theta^G) = (\chi_H, \theta)
   \]
   so that \( \theta \) is a constituent of \( \chi_H \). Then \( \chi(1) = \chi_H(1) \geq \theta(1) \), so that \( b(H) \leq b(G) \).
2. Let \( \theta \in \text{Irr}(N) \) have maximal character degree, and choose \( \chi \in \text{Irr}(G|\theta) \). By Clifford’s theorem,

\[
\chi_N = e \sum_{i=1}^{t} \theta_i
\]

for some \( e \) and for \( \theta_i \) the orbit of \( \theta \). Then \( \chi(1) = e\theta(1) \), since conjugate characters have the same degree. If either of \( e \) or \( t \) is greater than 1, then \( \chi(1) \geq 2\theta(1) \), and we are done. If \( e = t = 1 \), then \( \chi_N = \theta \). Since \( G/N \) is nonabelian, it has a nonlinear irreducible character, call it \( \beta \). By the Gallagher correspondence, \( \beta\chi \in \text{Irr}(G|\theta) \), and

\[
(\beta\chi)(1) \geq 2\chi(1) = 2\chi_N(1) = 2\theta(1),
\]

so \( G \) has an irreducible character of degree at least twice that of \( \theta \), and we conclude.

\[\square\]

**Exercise 3.33.** Let \( G \) be solvable. Show that the derived length \( dl(G) \) of \( G \) is at most \( 1 + 2\log_2(b) \), where \( b = b(G) \) is the largest irreducible character degree.

**Proof.** We consider two cases based on the parity of \( dl(G) \). Suppose \( dl(G) \) is odd. If \( dl(G) = 1 \), then \( G \) is abelian, so \( b = 1 \), and the claim is trivial. If \( dl(G) > 1 \), then since the derived series is a normal series, \( G^{(2)} \leq G \), but \( G/G^{(2)} \) is nonabelian as \( G^{(2)} \) is properly contained in the commutator of \( G \). By the previous exercise, \( b(G^{(2)}) \leq b(G)/2 \). Inductively, we see \( b(G^{(2k)}) \leq b(G)/2^k \). Hence

\[
1 \leq b(G^{(dl(G)-1)}) \leq b(G)/2^{dl(G)/2}.
\]

Solving this inequality yields \( dl(G) \leq 1 + 2\log_2(b) \). If \( dl(G) \) is even, then \( 1 \leq b(G^{dl(G)}/2) \leq b(G)/2^{dl(G)/2} \), and solving yields \( dl(G) \leq 2\log_2(b) \), which is an even stronger inequality.

\[\square\]

**Exercise 3.34.** Let \( \theta \in \text{Irr}(N) \), where \( N \trianglelefteq G \). Show that \( \theta^G \) is irreducible iff \( N \) is the full stabilizer of \( \theta \) in \( G \).

**Proof.** Suppose \( \theta^G \in \text{Irr}(G) \), so that \( \theta^T \in \text{Irr}(T) \). Let \( T \) denote the stabilizer of \( \theta \) in \( G \). By Clifford’s theorem,

\[
(\theta^T)_N = ((\theta^T)_N, \theta) \sum_{i=1}^{t} \theta_i
\]

where the sum ranges over \( T \)-conjugates of \( \theta \), of which \( \theta \) is the only one. So \( (\theta^T)_N = ((\theta^T)_N, \theta)\theta \). However,

\[
((\theta^T)_N, \theta) = (\theta^T, \theta^T) = 1,
\]

so \( (\theta^T)_N = \theta \). Then

\[
\theta(1) = \theta^T(1) = |T : N|\theta(1),
\]

implying \( [T : N] = 1 \), hence \( T = N \). The converse direction follows immediately by Clifford’s correspondence.

\[\square\]
2. By the previous part, \( \chi_A = e\lambda \), for \( \lambda \) a linear character of \( A \). So if \( a \in A \), \( \chi_A(a) = e\lambda(a) \), so \( |\chi(a)| = e \cdot |\lambda(a)| = e = \chi(1) \). Thus \( A \subseteq \mathbb{Z}(\chi) \).

\( \square \)

**Exercise 8.35.** Show that a primitive character of a \( p \)-group must be linear.

**Proof.** First, any \( p \)-group has a normal, self-centralizing subgroup. To see this, let \( A \) be a subgroup which is maximal among normal, abelian subgroups, and put \( C = C_G(A) \). As \( A \) is abelian, \( A \subseteq C \), so suppose \( A \neq C \). Observe that \( C \) is normal in \( G \), for if \( c \in C \), \( g \in G \) and \( a \in A \), then

\[
gcg^{-1}agc^{-1}g^{-1} = gg^{-1}agcc^{-1}g^{-1} = a,
\]

the first inequality holds since \( g^{-1}ag \in A \), so it commutes with \( c \). This \( gcg^{-1} \) centralizes \( a \), so \( gcg^{-1} \in C \).

It follows that \( C/A \trianglelefteq G/A \) is a nontrivial normal subgroup of a \( p \)-group, hence \( C/A \cap \mathbb{Z}(G/A) \) is nontrivial. Take \( cA \in C/A \cap \mathbb{Z}(G/A) \) to be nontrivial, and consider \( \langle A, c \rangle \). This group is abelian, since \( c \) commutes with all of \( A \), and is normal, for if \( g \in G \), then in \( G/A \),

\[
gcg^{-1}A = gAcAg^{-1}A = cA
\]
as \( cA \) is central, so \( gcg^{-1}c^{-1} \in A \), implying \( gcg^{-1} \in \langle A, c \rangle \). So \( A \subseteq \langle A, c \rangle \), a contradiction to maximality among normal, abelian subgroups. Hence \( A = C \).

Now let \( \chi \in \text{Irr}(G) \) be a primitive character. We can view it as an irreducible character on \( G/\ker\chi \), and on this quotient \( \chi \) is primitive and faithful. Let \( A \) be a self-centralizing normal subgroup of \( G/\ker\chi \). By the previous exercise, since \( \chi \) is primitive \( A \subseteq \mathbb{Z}(\chi) = \mathbb{Z}(G/\ker\chi) \), the latter equality holding since \( \chi \) is faithful. Since \( A \) is central and self-centralizing, necessarily \( A = G/\ker\chi \). So \( G/\ker\chi \) is abelian, hence \( \chi \) is linear, as an irreducible character of an abelian group.

\( \square \)

**Exercise 8.36.** Show that nilpotent groups are \( M \)-groups.

**Proof.** We proceed by induction on \( |G| \). Suppose the claim holds for all nilpotent groups \( N \) with \( |N| < |G| \). If \( G \) is abelian, the claim is trivial. Otherwise, pick \( \chi \in \text{Irr}(G) \) nonlinear. If \( \chi \) is not faithful, viewing it as a character on \( G/\ker\chi \), the claim follows by the induction hypothesis. So suppose \( \chi \) is faithful, and \( G \) is not abelian.

Consider the upper central series

\[
1 \trianglelefteq \mathbb{Z}(G) \trianglelefteq \mathbb{Z}_2(G) \trianglelefteq \cdots
\]

Then \( \mathbb{Z}_2(G)/\mathbb{Z}(G) = \mathbb{Z}(G/\mathbb{Z}(G)) \) is nontrivial as the center of a nontrivial nilpotent group, and let \( g \) be a representative of a nontrivial coset in \( \mathbb{Z}_2(G)/\mathbb{Z}(G) \). Then \( A = \langle \mathbb{Z}(G), g \rangle \) is abelian and normal in \( G \) by similar arguments as above. By Clifford’s Theorem, \( \chi_A = e \sum_{i=1}^{t} \theta_i \), where the \( \theta_i \) are linear characters of \( A \).

If \( t = 1 \), then \( \chi(1) = e \). Let \( X : G \to GL_n(k) \) be the representation affording \( \chi \), so \( X|_A \sim eY \), where \( Y \) is the representation affording \( \theta := \theta_1 \). If \( a \in A \), \( X(a) \sim eY(a) \), so \( X(a) \) is similar to a scalar matrix, hence central. So \( X(A) \) is central in \( X(G) \). But since \( X \) is faithful, as \( \chi \) is, \( X \) is an isomorphism onto its image, implying \( A \subseteq \mathbb{Z}(G) \), a contradiction.

If \( t \neq 1 \), let \( \theta \) be an irreducible constituent of \( \chi_A \), and \( T \) its inertia group. Since \( \theta \) is not \( G \)-invariant, as the other constituents form the \( G \)-orbit, we have \( T \) proper in \( G \). Then \( \chi = \psi^G \) for some \( \psi \in \text{Irr}(T) \), and since \( T \) is nilpotent, by the induction hypothesis, \( \psi \) is monomial. By transitivity of induction, \( \chi \) is monomial.

\( \square \)

**Exercise 8.37.** Suppose \( G' \) is abelian. Show that \( G \) is an \( M \)-group.

**Proof.** Suppose \( \chi \) is not a primitive character, so \( \chi = \theta^G \) for some \( \theta \) a character on a proper subgroup \( H \). Since \( H' \subseteq G' \), \( H' \) is abelian, so by the induction hypothesis, \( H \) is an \( M \)-group, and thus \( \theta = \eta^H \) for \( \eta \) a linear character on some subgroup of \( H \). Thus \( \chi = \eta^G \).

31
Now suppose $\chi$ is primitive. We may also assume $\chi$ is faithful, else we can pass to $G/\ker \chi$. Since $G'$ is abelian, $G' \leq Z(\chi) = Z(G)$ by a previous exercise. Then $G/Z(G)$ is abelian, hence nilpotent, and thus $G$ is nilpotent (of nilpotency class one greater). By the previous exercise, $G$ is an $M$-group.

**Exercise 8.38.** Let $\chi \in \text{Irr}(G)$ be monomial, and suppose that $A \leq G$ with $A$ abelian. Show that there exists a subgroup $H$ with $A \subseteq H \subseteq G$ and a linear character $\lambda$ of $H$ such that $\lambda^G = \chi$.

**Proof.** As $\chi$ is monomial, there exists a linear character $\mu$ on a subgroup $J$ such that $\chi = \mu^G$. Let $\theta \in \text{Irr}(A)$ be a constituent of $(\mu^A)^J$. Then

$$0 \neq (\theta, (\mu^A)^J) = (\theta, (\mu^A)\cap J) = (\theta, \mu^A \cap J) = 1.$$ 

The last equality follows since $\theta = \mu \cap J$ and $\mu^A \cap J$ are irreducible as $\mu$ and $\theta$ are both linear. It follows $\theta = \mu^A \cap J$. Let $T = I_{A, J}(\theta)$ be the inertia group. By the Clifford correspondence, we may find $\varphi \in \text{Irr}(T)$ such that $\varphi^A = \mu^A$ and $(\theta, \varphi^A) = (\theta, (\mu^A)^J)$. However,

$$(\theta, (\mu^A)^J) = (\theta, (\mu^A \cap J)^J) = (\theta, \mu^A \cap J) = 1.$$ 

By Clifford’s theorem, we then have $\varphi^A = (\theta, \varphi^A)\theta = \theta$, as there is only one $T$-conjugate of $\theta$. So $\varphi(1) = \theta(1) = 1$, so $\varphi$ is linear. Finally,

$$\varphi^G = (\varphi^A)^G = (\mu^A)^G = \mu^G = \chi,$$

and thus $\chi$ is monomial. 

\[\Box\]