Chapter 1: Affine Varieties

1.1 Closed sets

Let $k$ be an algebraically closed field. Let $\mathbb{A}^n = \mathbb{A}^n_k = \{a = (a_1, \ldots, a_n) : a_i \in k\} = k^n$ be affine space. So $\mathbb{A}^0$ is a point, and $\mathbb{A}^1 = k$. Let $A_n = k[T_1, \ldots, T_n]$ be the $k$-algebra of polynomials. If $f \in A_n$, $a \in \mathbb{A}^n$, then $f(a) \in k$.

Suppose $S \subseteq A_n$ be a subset. Then $Z(S) = \{a \in \mathbb{A}^n : f(a) = 0, \forall f \in S\} \subseteq \mathbb{A}^n$. A subset $Z \subseteq \mathbb{A}^n$ is called closed if $Z = Z(S)$ for some $S \subseteq A_n$.

Example 1.1. If $n = 1$, $Z \subseteq \mathbb{A}^1$ is closed iff $Z$ is finite or $Z = \mathbb{A}^1$.

Some properties:

1. If $S \subseteq S' \subseteq A_n$, then $Z(S) \supseteq Z(S')$.

2. $S \subseteq A_n$, and $I$ an ideal in $A_n$ generated by $S$. Then $Z(S) = Z(I) = Z(\sqrt{I})$, where $\sqrt{I} = \{g \in A_n : \exists m, g^m \in I\}$, the radical of $I$. If $J = \sqrt{I}$, then $\sqrt{J} = J$, so $J$ is a radical ideal in $A_n$.

3. Hilbert Basis Theorem: every ideal in $A_n$ is finitely generated. Writing $J = \langle f_1, \ldots, f_k \rangle$, so $Z(S) = Z(J) = Z(f_1, \ldots, f_k)$. So every closed set can be given by finitely many polynomials.

4. If $S_n \subseteq A_n$, then $Z(\cup S_n) = \bigcap Z(S_n)$, so the intersection of closed sets is closed.

5. $Z(S \cdot T) = Z(S) \cup Z(T)$, so the union of finitely many closed sets is closed. Hence we get the Zariski topology on $\mathbb{A}^n$ with closed sets $Z(S)$. Going back to $\mathbb{A}^1$, the open sets are $\emptyset$, or the cofinite sets. So every two nonempty open sets intersect.

If $X \subseteq \mathbb{A}^n$ is a subset, then $I(X) = \{f \in A_n : f(x) = 0, \forall x \in X\}$. We get a correspondence between subsets of $A_n$ and subsets of $\mathbb{A}^n$ generated by $Z(-)$ and $I(-)$. In fact, $I(X)$ is a radical ideal in $A_n$.

Some properties of $I$:

1. If $X \subseteq X'$, then $I(X) \supseteq I(X')$.

2. $IZ(S) \supseteq S$, and $ZI(X) \supseteq X$.

3. $IZ = I$ and $ZIZ = Z$.

Proof. Note $IZI(X) = IZ(I(X)) \supseteq I(X)$. But $ZI(X) \supseteq X$, applying $I$, $IZI(X) \subseteq I(X)$. □

4. $ZI(X) = \overline{X}$

Proof. We have $ZI(X) \supseteq X$, but $ZI(X)$ is closed, so $ZI(X) \supseteq \overline{X}$. But $\overline{X} = Z(S)$ for some $S$, and $ZI(X) \subseteq ZI(\overline{X}) = ZIZ(S) = Z(S) = \overline{X}$. So $ZI(X) = \overline{X}$ iff $X$ is closed. □

5. If $S \subseteq A_n$, $IZ(S) = \sqrt{S}$. 
Proof. It’s clear that $IZ(S) \supset S$ is a radical ideal, so $IZ(S) \supset \sqrt{S}$. Recall Hilbert Nullstellensatz: If $S \subset A_n$, and $f \in A_n$ such that $f$ is zero on the set of all common zeroes of polynomials in $S$. So $f \in IZ(S)$, so $f^m \in (S)$ for some $m > 0$, and this gives the other containment.

So we get a correspondence of the radical ideals in $A_n$ and the closed sets in $\mathbb{A}^n$, with inverse bijections given by the restriction of $Z(\cdot)$ and $I(\cdot)$. We have $A_n \leftrightarrow \emptyset$, and $0 \leftrightarrow \mathbb{A}^n$, and maximal ideals correspond to points. (Warning: We need $k$ to be algebraically closed for this to work, as we use the Nullstellensatz.)

A subset $X \subset \mathbb{A}^n$ is quasi-affine if $X = Z \cap U$ where $Z$ is closed, and $U$ is open in $\mathbb{A}^n$.

Example 1.2. In $\mathbb{A}^1$, $X$ is quasi-affine iff $X$ is closed or open.

Some properties of quasi-affine sets:

1. Closed and open subsets are quasi-affine.
2. The intersection of two quasi-affine sets is also quasi-affine. Every closed (open) subset of a quasi-affine set is quasi-affine.
3. A subset $X$ is quasi-affine iff $X$ is open in $\overline{X}$.

Proof. If $X = U \cap \overline{X}$, for $U$ open in $\mathbb{A}^n$, then $X$ is quasi-affine. Conversely, if $X$ is quasi-affine, then since $Z \supset X$ implies $Z \supset \overline{X}$, we have $X = Z \cap U \supset \overline{X} \cap U \supset X$, which implies $X = \overline{X} \cap U$, so $X$ is open in $\overline{X}$.

1.2 Regular Functions

Let $X \subset \mathbb{A}^n$ be a closed subset. A function $f : X \rightarrow k$ is called regular if there exists $F \in A_n$ such that $f(x) = F(x)$ for all $x \in X$. Denote by $k[X]$, the $k$-algebra of all regular functions on $X$. We get a surjective map

$$A_n \rightarrow k[X] : F \mapsto F|_X$$

with kernel $I(X)$, since this is precisely the functions which are 0 on $X$. So $k[X] \simeq A_n/I(X)$ by the first isomorphism theorem, and is a reduced $k$-algebra, i.e., has no nonzero nilpotent elements.

Let $t_i = T_i|_X$ be the coordinate functions on $X$. So $k[X]$ is generated by the $t_i$, hence is finitely generated by the coordinate functions.

Proposition 1.3. Every commutative finitely generated reduced $k$-algebra is isomorphic to $k[X]$ for some closed set $X$ in $k^n$ for some $n$.

Proof. Take $A = k[a_1, \ldots, a_n]$. Consider the surjective homomorphism $A_n \rightarrow A$ by $T_i \mapsto a_i$, with kernel $J$, which is radical since the algebra is reduced, so $A \simeq A_n/J$. Let $X = Z(J)$, so $I(X) = IZ(J) = \sqrt{J} = J$. So $A_n/J = A_n/I(X) \simeq k[X]$.

Example 1.4.

1. $k[\mathbb{A}^1] = A_n$.

2. If $X = \{a\}$ a point, and $a \in \mathbb{A}^n$, so $k[X] = k$ since all functions on a point are constant.

3. Let $f \in A_n$ be irreducible polynomial, $X = Z(f) \subset \mathbb{A}^n$ a hypersurface. Then $k[X] = A_n/\sqrt{f} = A_n/(f)$, since $(f)$ is prime, since $f$ is irreducible, hence radical.

If $X \subset \mathbb{A}^n$ is closed, $k[\mathbb{A}^n] \rightarrow k[X]$ is surjective. If $S \subset k[X]$, define $Z_X(S) = \{x \in X : f(x) = 0, \forall f \in S\} \subset X$. If $T \subset k[\mathbb{A}^n]$ and maps onto $S \subset k[X]$, we have $Z_X(S) = Z_{\mathbb{A}^n}(T) \cap X$, closed in $X$.

If $Y \subset X$, define $I_X(Y) = \{f \in k[X] : f(y) = 0, \forall y \in Y\}$, a radical ideal in $k[X]$. This gives a correspondence between closed sets in $X$ and radical ideals in $k[X]$ given by $I_X(\cdot)$ and $Z_X(\cdot)$. In fact, observe that closed sets in $X$ are a subset of closed sets in $\mathbb{A}^n$. We have a correspondence between closed sets in $\mathbb{A}^n$ and radical ideals in $A_n$, given by $I$ and $Z$, and we have an inclusion of radical ideals in $k[X]$ into the radical ideals in $A_n$ by sending $J \mapsto \varphi^{-1}(J)$, where $\varphi : A_n \rightarrow k[X]$, is the surjection onto $k[X]$. It follows that $I_X$ and $Z_X$ are inverse bijections. We have the correspondence between $\emptyset \leftrightarrow k[X]$, $X \leftrightarrow 0$, and $\{x\} \leftrightarrow m$ maximal ideals. Also, if $S \subset k[X]$, then $Z_X(S) = \emptyset \iff \sqrt{S} = k[X] \iff \langle S \rangle = k[X]$. 

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Suppose $X \subset \mathbb{A}^n$ is closed. For $f \in k[X]$, define $D_X(f) = X \setminus Z_X(f) = \{x \in X : f(x) \neq 0\}$, called a principal open set in $X$. If $f, g \in k[X]$, then $D_X(fg) = D_X(f) \cap D_X(g)$. If $U \subset X$ is open, $X \setminus U = Z_X(S)$ is closed, then $U = X \setminus Z_X(S) = \bigcup_{f \in S} D_X(f)$. $S$ can be chosen to be finite, so principal open sets form a basis for the Zariski topology on $X$.

Recall that a quasi-affine set is of form $Z \cup U$ for $Z$ closed and $U$ open.

**Example 1.5.** Set $X = \mathbb{A}^1 \setminus \{0\}$, $T \in A_1 = k[\mathbb{A}^1]$, and $t = T|_X$. Suppose $t(x) \neq 0$ for all $x \in X$, so $\frac{1}{t} : X \to k$ should be regular.

If $X \subset \mathbb{A}^n$ is quasi-affine, $f : X \to k$ and $x \in X$, then $f$ is regular at $x$ if there exists a neighborhood $U$ of $x$ and $G, H \in A_n$ such that $H(y) \neq 0$ for all $y \in U$ and $f(y) = \frac{G(y)}{H(y)}$ for all $y$. First, there exist $G, H \in A_n$ such that $H(x) \neq 0$, $f = \frac{H}{G}$ in a neighborhood of $x$. If $X$ is closed, there exist $g, h \in k[X]$, $h(x) \neq 0$, $f = \frac{g}{h}$ in a neighborhood of $x$.

**Proposition 1.6.** Let $X \subset \mathbb{A}^n$ be closed, $f : X \to k$. Then, $f$ is regular iff $f$ is regular at every point $x \in X$.

**Proof.** Suppose $f$ is regular at every point. Then for all $x \in X$, there exists a neighborhood $U_x \ni x$ such that $f = \frac{g_x}{h_x}$ in $U_x$ for some $g_x, h_x \in k[X]$. So $g_x = f \cdot h_x$ on $U_x$. Then if $x \notin X \setminus U_x$ is closed, so there exists $\ell_x \in k[X]$ such that $\ell_x(x) \neq 0$, but $\ell_x|_{X \setminus U_x} \equiv 0$. Also, $g_x \ell_x = f h_x \ell_x$ holds in $U_x$ and $X \setminus U_x$. Since $(h_x \ell_x)(x) \neq 0$, we have $f = \frac{g_x \ell_x}{h_x \ell_x}$ in $U_x$. Replace $g_x$ with $g_x \ell_x$ and $h_x$ with $h_x \ell_x$. Then $g_x = f h_x$ in $k[X]$, and $f = \frac{g_x}{h_x}$ in $U_x$.

Put $S = \{h_x : x \in X\}$, $h_x(x) \neq 0$ implies $Z(S) = \emptyset$, so that $(S) = k[X]$. So we can write

$$1 = \sum_{x \in X} h_x m_x$$

for almost all $m_x \in k[X]$ equal to 0. Then

$$f = \sum_{g_x} f(h_x m_x) = \sum_{g_x} g_x m_x \in k[X]$$

so $f$ is regular. \qed

So if $X \subset \mathbb{A}^n$ is a quasi-affine subset, a function $f : X \to k$ is regular if $f$ is regular at every $x \in X$. Then $k[X]$ is the $k$-algebra of all regular functions on $X$, and is reduced, but not finitely generated in general.

**Example 1.7.** Set $X = \mathbb{A}^1 \setminus \{0\}$, and $t$ coordinate function on $X$. So $\frac{1}{t} \in k[X]$. So $k[t, t^{-1}] \subset k[X]$, but not generated by $t$. Also, consider the quasi-affine set $X = Z(T_1 T_2 - T_3 T_4) \setminus Z(T_2, T_4) \subset \mathbb{A}^4$. Let $f = \frac{T_3}{T_1} = \frac{T_4}{T_2}$, and for all $x \in X$, either $t_2(x) \neq 0$ or $t_4(x) \neq 0$. Then $X = U_2 \cup U_4$, where $U_2$ is the set $t_2 \neq 0$ and $U_4$ is the set $t_4 \neq 0$, so $f$ is regular.

Properties of regular functions

1. Let $Y$ be a quasi-affine subset of a quasi-affine set $X$, and $f \in k[X]$. Then $f|_Y \in k[Y]$. This follows since the restriction of polynomials to a neighborhood is still a polynomial.

2. Let $X$ be a quasi-affine set, and $f : X \to k$. Put $X = \bigcup_{\alpha} U_\alpha$ and open cover. Then $f \in k[X] \iff f|_{U_\alpha} \in k[U_\alpha]$. The direct statement follows from the above, and the converse follows from local statement of regularity.

3. if $f : X \to k$ is regular at $x \in X$, then $f$ is regular in a neighborhood of $x$.

4. Let $f \in k[X]$ be such that $f(x) \neq 0$ for all $x \in X$. Then $\frac{1}{f}$ is also regular.

**Proof.** Pick $x \in X$, in a neighborhood of $x$, $f(y) = \frac{G(y)}{H(y)}$ for polynomials $G$ and $H$. But $f(x) \neq 0 \implies G(x) \neq 0$, so $G$ is not zero in a nbhd of $x$. In a nbhd of $x$, $f = \frac{G}{H}$, and $G(y) \neq 0$ for all $y$. So $\frac{1}{f} = \frac{H}{G}$ is regular at $x$. \qed
5. First, we need a lemma.

**Lemma 1.8.** Let $X = \bigcup_{\alpha} U_{\alpha}$ be an open cover of a topological space $X$, and $Y \subset X$. Then $Y$ is closed in $X$ iff $Y \cap U_{\alpha}$ is closed in $U_{\alpha}$ for every $\alpha$.

**Proof.** For the forward direction, easy. Conversely, we prove that $X \setminus Y$ is open. Take $x \in X \setminus Y$, so $x \in U_{\alpha}$ for some $\alpha$. So there is a closed subset $Z_{\alpha} \subset X$ such that $Y \cap U_{\alpha} = Z_{\alpha} \cap U_{\alpha}$. Set $U := U_{\alpha} \cap (X \setminus Z_{\alpha})$, which is open in $X$. Since $x \notin Y$, $x \notin Z_{\alpha}$. So $x \in U_{\alpha} \cap (X \setminus Z_{\alpha})$. We claim $U \cap Y = \emptyset$. If $y \in U \cap Y$, then $y \in U_{\alpha} \cap Y = U_{\alpha} \cap Z_{\alpha} \subset Z_{\alpha}$, but $y \notin Z_{\alpha}$, a contradiction. So $U$ is a nbhd of $x$, and $U \subset X \setminus Y$.

Let $f \in k[X]$. Then the set $Z_X(f) = \{x \in X : f(x) = 0\}$ is closed in $X$.

**Proof.** Take $x \in X$, in a nbhd $U_x$ of $x$, $f = \frac{G_x}{H_x}$, where $G_x, H_x$ are polynomials. So

$$Z_X(f) \cap U_x = Z_{U_x}(f) = Z_{U_x}(G_x) = Z_X(G_x) \cap U_x$$

which is closed in $U_x$ since $Z_X(G_x)$ is closed in Zariski topology since $G_x$ is a polynomial. Then $X = \bigcup_{x \in X} U_x$, so by lemma, $Z_X(f)$ is closed in $X$.

6. Let $f \in k[X]$, then $f : X \to k = \mathbb{A}^1$ is continuous.

**Proof.** Take $Z \subset \mathbb{A}^1$ closed, need to show $f^{-1}(Z)$ is closed in $X$. If $Z = \mathbb{A}^1$, then $f^{-1}(Z) = X$ is closed in $X$. If $Z$ is finite, then

$$f^{-1}(Z) = \bigcup_{z \in Z} f^{-1}(z) = \bigcup_{z \in Z} Z_X(f - z)$$

which is a finite union of closed sets (by property 5), and the finite union of closed sets is closed.

Let $X$ be quasi-affine. We get maps between the closed subsets of $X$ and radical ideals in $k[X]$ given by $I$ and $Z$. Let $Z \subset X$ be closed, so $Z = Z_X(S)$. Then $(ZI)(Z) = ZIZ(S) = Z(S) = Z$. So $ZI$ is the identity. So $I$ is injective, and $Z$ is surjective. So we have "more" radical ideals in $k[X]$ than closed subsets.

**Example 1.9.** Consider $X = \mathbb{A}^2 \setminus \{(0,0)\}$. Then we will see $k[X] = k[\mathbb{A}^2] = A_2$. If $J_1 = \langle 1 \rangle$ is the unit ideal in $k[X]$, and $J_2 = \langle t_1, t_2 \rangle$. Then $Z(J_1) = \emptyset$, but $Z(J_2) = \emptyset$ since we’ve removed the only zero $(0,0)$, so the map $Z$ is not injective in general.

## 2 Regular Maps

Take $X \subset \mathbb{A}^n$ quasi-affine, and $f : X \to \mathbb{A}^m$ and write $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, where $f_i : X \to k$. We say $f$ is regular if $f_i \in k[X]$ for all $i$.

**Example 2.1.**

1. If $f : X \to k = \mathbb{A}^1$, so $f$ is regular as a function is equivalent to saying it’s regular as a map.

2. The inclusion $X \hookrightarrow \mathbb{A}^n$ is regular, since it’s given by coordinate functions which are always regular.

Suppose $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are quasi-affine sets. A map $f : X \to Y$ is regular if the composition $X \xrightarrow{f} Y \hookrightarrow \mathbb{A}^m$ is regular. So to give a regular map $f : X \to Y$ is to give $f_1, \ldots, f_m \in k[X]$ such that $(f_1(x), \ldots, f_m(x)) \in Y$ for all $x \in X$.

**Example 2.2.**

1. Inclusions are regular.

2. $f : X \to Y$ is regular, and $X' \subset X$ is a quasi-affine subset, then $f|_{X'} : X' \to Y$ is regular. Suppose $\text{im}(f) \subset Y'$ is a quasi-affine subset of $Y$, then $f' : X' \to Y'$ is regular, because we compose with inclusion. So we can also shrink the target provided we know the target is quasi-affine.
3. Consider \( f: \mathbb{A}^2 \to \mathbb{A}^2 \), and define \( f(x,y) = (xy,y) \), for \( x \) and \( y \) the coordinate functions. The image is everything except the \( x \)-axis, but we keep the origin. The image is not a quasi-affine subset. Call \( \text{Im}(f) = X \). But \( X = \mathbb{A}^2 \), and \( \mathbb{A}^2 \setminus X = \mathbb{A} \setminus \{0\} \). If \( X \) is quasi-affine, \( U \) is open in \( \mathbb{A}^2 \), which implies \( \mathbb{A} \setminus \{0\} \) is closed in \( \mathbb{A}^2 \). But \( \mathbb{A} \setminus \{0\} = 1 \).

**Proposition 2.3.** Regular maps are continuous.

*Proof.* Consider \( f: X \to Y \subset \mathbb{A}^n \) regular. Take \( Z \subset Y \) closed. There exists \( S \subset A_m \) such that \( Z = Z_Y(S) \). Then \( f^{-1}(Z) = \bigcap_{G \in S} f^{-1}(Z_Y(G)) \). If \( \bigcap_{G \in S} \mathbb{A} \) is regular. To show \( f^{-1}(Z_Y(G)) \) is closed in \( X \). But \( f^{-1}(Z_Y(G)) = \{ x \in X : G(f_1(x), \ldots, f_m(x)) = 0 \} \). So \( G(f_1, \ldots, f_m)(x) = 0 \), and \( G(f_1, \ldots, f_m) \) is regular, so this set is closed as the zero set of a regular function.

**Corollary 2.4.** If \( f: X \to Y \) is a regular map, \( Y' \subset Y \) quasi-affine, then \( f^{-1}(Y') \) is also a quasi-affine subset of \( X \).

*Proof.* Write \( Y' = Z \cap U \), for \( Z \) closed, \( U \) open in \( Y \), so \( f^{-1}(Y') = f^{-1}(Z) \cap f^{-1}(U) \), which is quasi-affine in \( X \) as an intersection of closed and open sets, since \( f \) is continuous.

Take \( f: X \to Y \) a map. If \( h: Y \to k \), take \( h \circ f: X \to k \).

**Proposition 2.5.** Let \( X \) and \( Y \) be quasi-affine sets, \( f: X \to Y \) a map. Then \( f \) is regular iff for every \( h \in k[Y] \), we have \( h \circ f \in k[X] \). So regular maps are those which take regular functions to regular functions after precomposing.

*Proof.* For the converse direction, let \( t_i \) be a coordinate function on \( Y \), and \( f = (f_1, \ldots, f_m) \). Then \( t_i \circ f = f_i \), and \( t_i \) is regular. By the condition, \( f_i \) is regular, so \( f \) is regular.

For the forward direction, suppose \( h \in k[Y] \). Is \( h \circ f \) regular? Take \( x \in X \), and \( y = f(x) \in Y \). In a neighborhood \( U \) of \( y \), \( h = \frac{G}{H} \) for \( G, H \) polynomials, \( H \neq 0 \). On \( f^{-1}(U) \), nbhd of \( x \), then

\[
\text{regular.}
\]

with both maps on the far RHS are regular on \( f^{-1}(U) \), so this is regular at \( x \), hence \( h \circ f \) is regular.

**Proposition 2.6.** If \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are regular maps, then \( g \circ f \) is also regular.

*Proof.* Take \( h \in k[Z] \). Then \( h \circ g \in k[Y] \) since \( g \) is regular, and thus \( h \circ g \circ f \in k[X] \) by the previous theorem. So \( g \circ f \) is regular.

For \( f: X \to Y \) regular, we have an induced map \( f^*: k[Y] \to k[X] \) by \( h \mapsto h \circ f \), which is a homomorphism of \( k \)-algebras. If \( X \xrightarrow{f} Y \xrightarrow{g} Z \), then \((g \circ f)^* = f^* \circ g^* \).

3 Categories and Functors

A category \( A \) is a class of objects \( \mathcal{O}(A) \), and for \( X, Y \in \mathcal{O}(A) \), we have a set of morphisms \( \text{Mor}_A(X,Y) \). We have an associative composition, and the identity morphism \( 1_X \in \text{Mor}_A(X,X) \).

For example, \( \text{QASets}(k) \). Here objects are quasi-affine sets, and the morphisms are regular maps. Another example is \( \text{CAlg}(k) \), where the objects are commutative \( k \)-algebras, and morphisms are \( k \)-algebra homomorphisms. Also, \( \text{CSet}(k) \) is the category of closed sets, with morphisms the regular maps.

A category \( B \) is a full subcategory of \( A \) if \( \mathcal{O}(B) \subset \mathcal{O}(A) \), and \( \text{Mor}_B(X,Y) = \text{Mor}_A(X,Y) \) for \( X, Y \in \mathcal{O}(B) \). We denote this \( B \subset A \). For example, \( \text{CSet}(k) \subset \text{QASets}(k) \).

If \( C \) is a category, the dual category \( C^{\text{op}} \), where \( \mathcal{O}(C) = \mathcal{O}(C^{\text{op}}) \), but \( \text{Mor}_{C^{\text{op}}}(X,Y) = \text{Mor}_C(Y,X) \).

Let \( C \) be a category, and \( f: X \to Y \). We say \( f \) is an isomorphism if there exists a morphism \( g: Y \to X \) such that \( g \circ f = 1_X \) and \( f \circ g = 1_Y \). We write \( g = f^{-1} \). For example, a map \( f: X \to Y \) of quasi-affine sets is an isomorphism in \( \text{QASets}(k) \) iff \( f \) is a bijection, and \( f^{-1} \) is regular.

A functor \( F \) between two categories \( A \) and \( B \), \( F: A \to B \), takes an object \( X \in \mathcal{O}(A) \) to \( F(X) \in \mathcal{O}(B) \), and a morphism \( f: X \to Y \) to \( F(f): F(X) \to F(Y) \) in \( B \). Also, \( F(g \circ f) = F(g) \circ F(f) \), and \( F(1_X) = 1_{F(X)} \).
In particular, we have a functor \text{QASets}(f)^* \to \text{CAlg}(k)$ by the assignment $X \mapsto k[X]$, and $f: X \to Y$ to $f^*: k[Y] \to k[X]$. Alternatively, we can say this is a contravariant functor if we wish to avoid the dual category.

If $F: A \to B$ is a functor, and $f: X \to Y$ is an isomorphism in $A$, then $F(f)$ is an isomorphism in $B$. So functors map isomorphisms to isomorphisms. Furthermore, $F$ preserves isomorphisms where they exist. For example, $F$ preserves isomorphisms of objects.

\textbf{Example 3.1.} 1. Let $Y \subseteq \mathbb{A}^2$ give by $y^2 = x^3$. Let $X = \mathbb{A}^1$. Take $f: X \to Y$ to be $t \mapsto (t^2, t^3)$, i.e., $x = t^2$, and $y = t^3$. This is regular since the maps are given by polynomials. This is a bijection since $t = \frac{y}{x}$. We have

$$f^*: k[Y] \to k[X]$$

where the vertical maps are equality, and the bottom map is given by $x \mapsto t^2$ and $y \mapsto t^3$. But $t \notin \text{Im}(f^*)$ so $f^*$ is not an isomorphism and $f$ is not isomorphism.

We have

$$f^{-1} = \begin{cases} \frac{y}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

but is not regular.

2. Suppose $\text{char}(k) > 0$. Consider $\mathbb{A}^1 \to \mathbb{A}^1: t \mapsto t^p$, the Frobenius map. This is a bijection. Then $f^*: k[t] \to k[t]$ is given by $t \mapsto t^p$, but is not an isomorphism, since its image consists of polynomials in $t^p$. Note $f^{-1}(s) = s^{1/p}$, but is not regular since it’s not given by a polynomial.

Let $F: A \to B$ be a functor, $X, Y \in \mathcal{O}(A)$. Then $F$ yields a map of sets

$$\varphi_{X,Y}: \text{Mor}_A(X, Y) \to \text{Mor}_B(F(A), F(B)) : f \mapsto F(f)$$

We say $F$ is faithful if $\varphi_{X,Y}$ is injective for all $X, Y$. We say $F$ is full if $\varphi_{X,Y}$ is surjective for all $X, Y$.

\textbf{Example 3.2.} Let $A \subseteq B$ be a full subcategory. We have the embedding functor $F: \text{Mor}_A(X, Y) \to \text{Mor}_B(X, Y)$. If we have $f \in \text{Mor}_A(X, Y)$ and $g \in \text{Mor}_B(X, Y)$, then $F(f) \circ g = F(g) \circ F(f)$. So $F$ is fully faithful since these categories have the same morphisms.

\textbf{Lemma 3.3.} Let $F: A \to B$ be a fully faithful functor, $X, Y \in \mathcal{O}(A)$. Then $X \simeq Y$ iff $F(X) \simeq F(Y)$ in $B$. That is to say, fully faithful functors reflect isomorphism.

\textbf{Remark 3.4.} Consider the sets of isomorphism classes in $A$ and $B$. If we have functor $F: A \to B$, we get a map between them. If $F$ is fully faithful this map is injective.

Consider QASets($k$)$_{\text{op}} \to \text{CAlg}(k)$. Is it full or faithful? Take $f = (f_1, \ldots, f_m): X \to Y \subseteq \mathbb{A}^m$ to be a regular map of quasi-affine sets. We get a morphism $f^*: k[Y] \to k[X]$. We have the coordinate functions $t_i \in k[Y]$. Then $f^*(t_i) = f_i$.

So consider $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$. If $f^* = g^*$, then $f_i = f^*(t_i) = g^*(t_i) = g_i$, so $f = g$. So the functor is always faithful. It is not full however. So the algebras in the algebraic category coming from the sets in the geometric category have special properties.

\textbf{Proposition 3.5.} Let $X$ be a quasi-affine set and $Y$ a closed set. Then the map

$$\text{Mor}_{\text{QASets}(k)}(X, Y) \to \text{Mor}_{\text{CAlg}(k)}(k[Y], k[X])$$

is a bijection.

\textbf{Proof.} The map is given by $f \mapsto f^*$. We have seen that it is always injective. Take $\alpha: k[Y] \to k[X]$ a $k$-algebra homomorphism. We need to construct a regular map $f: X \to Y$ such that $\alpha = f^*$. We have $Y \subseteq \mathbb{A}^m$ is closed. Let $t_1, \ldots, t_m$ be the coordinate functions on $Y$, so $t_i \in k[Y]$. Set $f_i = \alpha(t_i) \in k[X]$. Put $f = (f_1, \ldots, f_m): X \to \mathbb{A}^m$ which is regular as the $f_i$ are regular. I claim $\text{Im}(f) \in Y$.

\textbf{Proof.} The map is given by $f \mapsto f^*$. We have seen that it is always injective. Take $\alpha: k[Y] \to k[X]$ a $k$-algebra homomorphism. We need to construct a regular map $f: X \to Y$ such that $\alpha = f^*$. We have $Y \subseteq \mathbb{A}^m$ is closed. Let $t_1, \ldots, t_m$ be the coordinate functions on $Y$, so $t_i \in k[Y]$. Set $f_i = \alpha(t_i) \in k[X]$. Put $f = (f_1, \ldots, f_m): X \to \mathbb{A}^m$ which is regular as the $f_i$ are regular. I claim $\text{Im}(f) \in Y$. 6
Take $F \in I(Y) \subset A_m$. By definition, $F|_Y = 0$. Then $F(T_1, \ldots, T_m)|_Y = 0$. Since $t_i = T_i|_Y$, we have $F(t_1, \ldots, t_m) = 0 \in k[Y]$. So

$$0 = \alpha(F(t_1, \ldots, t_m)) = F(\alpha(t_1), \ldots, \alpha(t_m)) = F(f_1, \ldots, f_m)$$

so $F(f_1(x), \ldots, f_m(x)) = 0$ for all $F \in I(Y)$. For any $x \in X$, $(f_1(x), \ldots, f_m(x)) \in ZI(Y)$. But $ZI(Y) = Y$ because $Y$ is closed. (This is why we must assume $Y$ closed.) So $\text{Im}(f) \subset Y$.

So consider $f: X \to Y$. Then $X \to Y \hookrightarrow A^m$ is regular, so $f$ is regular. Now $\alpha(t_i) = f_i = f^*(t_i)$, so $\alpha$ and $f^*$ agree on the generators $t_i$ of $k[Y]$. Thus $\alpha = f^*$. □

If $F: A \to B$ is a fully faithful functor, then $F$ induces a injection on the isomorphism classes of objects. A functor $F: A \to B$ is essentially surjective if every $Y \in \mathcal{O}(B)$ is isomorphic to $F(X)$ for some $X \in \mathcal{O}(A)$.

A functor $F: A \to B$ is an equivalence if $F$ is full, faithful, and essentially surjective. Such an $F$ gives an isomorphism on classes of objects and on morphisms: $\text{Mor}_A(X,Y) \cong \text{Mor}_B(F(X),F(Y))$.

**Remark 3.6.** 1. Suppose $A \subset B$ is a full subcategory such that every object of $B$ is isomorphic to an object of $A$. Then the inclusion functor is an equivalence.

In $B$, let $Y \cong Y'$ be isomorphic, and distinct. Set $A = B \setminus \{Y'\}$. Then $A$ and $B$ are equivalent.

For example, let $B$ be the category of finite dimensional vector spaces over a field $k$. Let $A$ be the category $\mathcal{O}(A) = \{k^n : n \geq 0\}$. So $A \hookrightarrow B$ is an equivalence. Moreover, let $A' = \{n : n \geq 0\}$, with $\text{Mor}(i,j) = M_{i \times j}(k)$.

2. If $H \subset G$ subgroup, and $H \to G$ an injection, then $H$ is isomorphic to its image in $G$.

If $A \subset B$ is full subcategory, then $A \to B$ is full and faithful. If $F: A \to B$ full and faithful, put $\mathcal{O}(B') = \{Y \in \mathcal{O}(B) : \exists F(X) \text{ for some } X \in \mathcal{O}\}$. Then $A$ is equivalent to $B' \subset B$.

**Theorem 3.7.** The functor $\text{CSets}(k)^{op} \to \text{CAlg}(k)$ given by $X \mapsto k[X]$ and $f \mapsto f^*$ is full and faithful. This functor gives an equivalence between $\text{CSets}(k)^{op}$ and the full subcategory of $\text{CAlg}(k)$ consisting of all reduced and finitely generated algebras.

4 Products

We identify $A^n \times A^m = A^{n+m}$. If $X = Z_{A^n}(S)$ is closed in $A^n$, and $Y = Z_{A^m}(T)$ is closed in $A^m$, then $X \times Y \subset A^{n+m}$ is equal to $Z_{A^{n+m}}(S \cup T)$, so the product of closed sets is closed.

Suppose $X \subset A^n$ and $Y \subset A^m$ are quasi-affine sets. Write $X = X_1 \setminus X_2$ for $X_1, X_2$ closed in $A^n$ and $Y = Y_1 \setminus Y_2$ for $Y_1, Y_2$ closed in $A^m$. Then

$$X \times Y = (X_1 \times X_2) \setminus (X_1 \times Y_2 \cup X_2 \times Y_1)$$

which are both closed, hence it is quasi-affine in $A^{n+m}$. Similarly, if $X, Y$ open, then $X \times Y$ is open.

Consider the maps $p: X \times Y \to X$ and $q: X \times Y \to Y$ which are regular as projections. They induce $p^*: k[X] \to k[X \times Y]$ and $q^*: k[Y] \to k[X \times Y]$. By the universal property of the tensor product, we get a map

$$\varphi: k[X] \otimes_k k[Y] \to k[X \times Y] : f \otimes g \mapsto ((x,y) \mapsto f(x)g(y))$$

Later, we will prove it is an isomorphism. For now,

**Proposition 4.1.** The map $\varphi$ is injective.

Proof. Let $\{f_i\}_{i \in I}$ be a basis for $k[X]$. Then every element in $k[X] \otimes_k k[Y]$ can be uniquely written in form $h = \sum_{i \in I} f_i \otimes g_i$ for $g_i \in k[Y]$ almost all zero. Then suppose

$$0 = \varphi(h)(x,y) = \sum_{i \in I} f_i(x)g_i(y) \quad \forall x, y$$

Fixing $y$, we get $\sum f_i(y)g_i(y) = 0$ in $k[X]$, but the $f_i$ are linearly independent, so $g_i(y) = 0$ for any $y \in Y$. So all $g_i = 0$, thus $h = 0$. □
Proof. This is because $k[X \times Y]$ is generated by the coordinate functions, so $\varphi$ is surjective.

Consider $f: X \to Y$ and $f': X' \to Y'$ to be regular maps of quasi-affine sets. Then $f \times f': X \times X' \to Y \times Y'$ defined by \((f \times f')(x, x') = (f(x), f'(x'))\). Write $f = (f_1, \ldots, f_m)$ and $f' = (f'_1, \ldots, f'_n)$. If $p, q$ are the projection maps, then $f \times f' = (f_1 \circ p, \ldots, f_m \circ p, f'_1 \circ q, \ldots, f'_n \circ q)$ which is also a regular map.

Let $f$ be a regular map. Take $f \times 1_Y : X \times Y \to Y \times Y \supset \Delta_Y$. Then \((f \times 1_Y)^{-1}(\Delta_Y) = \{(x, y) \in X \times Y : f(x) = y\}\), which is closed in $X \times Y$. This is the graph of $f$, denoted $\Gamma_f$. This is isomorphic to $X$ under the projection $p: \Gamma_f \to X$ and $X \to \Gamma_f : x \mapsto (x, f(x))$.

Proposition 4.4. (Universal Property of the Product) Let $X, Y, Z$ be quasi-affine sets and $f: Z \to X$ and $g: Z \to Y$ be regular maps. Then there exists a unique regular map $h: Z \to X \times Y$ such that $p \circ h = f$ and $q \circ h = g$.

Proof. Assuming $h$ exists, $p(h(z)) = f(z)$ and $q(h(z)) = g(z)$, so necessarily $h(z) = (f(z), g(z))$. This gives uniqueness.

On the other hand, if $h$ is defined as such, Then $h: Z \xrightarrow{\mathcal{D}} Z \times Z \xrightarrow{f \times g} X \times Y$ which is regular. Write $h = (f, g)$.

Proposition 4.5. Let $f, g: X \to Y$ be two regular maps. If $f$ and $g$ agree on a dense subset of $X$, then $f = g$.

Proof. Consider $h: X \xrightarrow{\mathcal{D}} X \times X \xrightarrow{f \times g} Y \times Y \supset \Delta_Y$ then $h^{-1}(\Delta_Y)$ is closed in $X$. But this set contains a dense subset of $X$. So $h^{-1}(\Delta_Y) = X$ since it is a closed set containing a dense set. So $f = g$.

5 Affine and Quasi-affine Varieties

Take $X = Z(xy - 1) \subset \mathbb{A}^2$ and $Y = \mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$ open. Then $X$ is closed in $\mathbb{A}^2$, and $Y$ is not closed in $\mathbb{A}^1$. But they are isomorphic under the maps $X \to Y: (x, y) \mapsto x$ and $Y \to X: t \mapsto (t, \frac{1}{t})$, which is regular since $\frac{1}{t}$ is regular since $t \neq 0$. So two sets can be isomorphic even though one is closed and the other is not.

Note $\mathbb{A}^1 \setminus \{0\} = D_A(t) = \{x \in \mathbb{A}^1 : t(x) \neq 0\}$.

Take $X$ a closed set, $f \in k[X]$, and $D_X(f) = \{x \in X : f(x) \neq 0\}$. Define $Z \subseteq X \times \mathbb{A}^1$ by $Z = (f \cdot t - 1)$, where $t$ is the coordinate function. I claim $D_X(f) \simeq Z$. The maps are $D_X(f) \to Z : x \mapsto (x, \frac{1}{f(x)})$ and $Z \to D_X(f) : (x, t) \mapsto x$.

Now $k[D_X(f)] \simeq k[Z] = k[X \times \mathbb{A}^1]/(f \cdot t - 1)$. But $k[X \times \mathbb{A}^1] \simeq k[X] \otimes k[\mathbb{A}^1] = k[X] \otimes k[t] = k[X][t]$. Thus $k[X \times \mathbb{A}^1]/(f \cdot t - 1) = k[X][t]/(f \cdot t - 1)$. Also $k[X][t]/(f \cdot t - 1) \simeq k[X]_{(f)}$

is the localization of $k[X]$ with respect to $\{f^n\}$. Then $k[X]$ is reduced so $k[X]_{(f)}$ is reduced. So $(f \cdot t - 1) = \sqrt{(f \cdot t - 1)}$ since it is radical as the quotient is reduced, so finally $k[D_X(f)] \simeq k[X]_{(f)}$. 


Example 5.1. 1. $k[A^1 \setminus \{0\}] = k[D_{A^1}(t)] = k[t]_{(t)} = k[t, t^{-1}]$

2. $k[A^2 \setminus \{0\}] = k[t_1, t_2]$. If $U \subset X$ is a dense subset, then $k[X] \rightarrow k[U]$, the restriction map, is injective. If $f : U \rightarrow X$ is the inclusion, the restriction map is $f^*$. It is injective because if they agree on $U$, they agree on $X$ since $U$ is dense.

Let $U_1 = D_{A^2}(T_1) \subset A^2$ and $U_2 = D_{A^2}(T_2) \subset A^2$. Then $U_1 \cap U_2 \subset X = A^2 \setminus \{0\} \subset A^2$. Then $k[A^2] \hookrightarrow k[X] \hookrightarrow k[U_1] \hookrightarrow k[U_1 \cap U_2]$ and likewise $k[X] \hookrightarrow k[U_2] \hookrightarrow k[U_1 \cap U_2]$.

But $k[U_1] = k[T_1, T_2, T_1^{-1}]$ and $k[U_2] = k[T_1, T_2, T_2^{-1}]$. And $k[U_1 \cap U_2] = k[D_{A^2}(T_1 T_2)] = k[T_1, T_2, T_1^{-1}, T_2^{-1}]$ so $k[T_1, T_2] \hookrightarrow k[X] \hookrightarrow k[U_1] \cap k[U_2] \subset k[U_1 \cap U_2] = k[T_1, T_2, T_1^{-1}, T_2^{-1}]$. But $k[T_1, T_2, T_1^{-1}] \cap k[T_1, T_2, T_2^{-1}] = k[T_1, T_2]$, so we get equality.

A quasi-affine variety is a quasi-affine set. An affine variety is a quasi-affine variety that is isomorphic to a closed set. Then $\text{QAffVar}(k) \supset \text{AffVar}(k)$ and we have a functor $\text{QAffVar}(k)^{\text{op}} \rightarrow \text{CAlg}(k)$ by $X \mapsto k[X]$ and $f \mapsto f^*$.

If $X$ is an affine variety, $f \in k[X]$, then $D_X(f)$ is also affine.

We have an embedding $\text{CSets}(k) \hookrightarrow \text{AffVar}(k)$ as a full subcategory, and this is an equivalence of categories. The functor $\text{AffVar}(k)^{\text{op}} \rightarrow \text{CAlg}(k)$ is full and faithful. The image lives in the set of reduced, finitely generated, commutative $k$-algebras.

If $X$ is an affine variety, and $Z \subset X$ is closed, then $Z$ is affine and $k[X] \rightarrow k[Z]$ is surjective.

Affine varieties $X$ and $Y$ are isomorphic iff $k[X] \simeq k[Y]$. This follows from the equivalence of categories. But $X, Y$ live in both $\text{AffVar}(k) \subset \text{QAffVar}(k)$, but it doesn’t matter which category we view them in.

Example 5.2. $X = A^2 \setminus \{0\}$ is not affine.

1. The map $Z$ from radical ideals in $k[X] = k[t_1, t_2]$ to closed subsets in $X$ is not injective, $Z(k[t_1, t_2]) = Z((t_1, t_2)) = \emptyset$, but if the punctured plane were affine, this map would be an isomorphism.

2. The embedding $f : X \hookrightarrow A^2$ is not an isomorphism. Applying $f^* : k[A^2] \rightarrow k[X]$ is an isomorphism as they are both polynomial rings in two variables. But the functor $\text{AffVar}(k)^{\text{op}} \rightarrow \text{CAlg}(k)$ is full and faithful.

3. View $A^1 \setminus \{0\}$ to be closed in $X$ under the map $g : t \mapsto (t, 0)$. But $g^* : k[X] = k[T_1, T_2] \rightarrow k[A^1 \setminus \{0\}] = k[t, t^{-1}]$ is given by $T_1 \mapsto t$ and $T_2 \mapsto 0$, so is not surjective.

Theorem 5.3. Let $X$ be a quasi-affine variety, $x \in X$. Then $x$ has an affine neighborhood.

Proof. We have $X \subset A^n$ is a quasi-affine subset, so $X = Z_1 \setminus Z_2$, both closed sets in $A^n$. So $x \notin Z_2$, thus there exists $F \in A_n$ such that $F|_{Z_2} = 0$, but $F(x) \neq 0$. Let $f = F|_{Z_1} \in k[Z_1]$. Consider the principal open $D_{Z_1}(f) = Z_1 \setminus Z(f)$ in $Z_1 \setminus Z_2 = X$ because $Z_2 \subset Z(F)$. And $x \in D_{Z_1}(f)$ since $f(x) \neq 0$, so $D_{Z_1}(f)$ is a neighborhood of $x$ in $X$, and $D_{Z_1}(f)$ is affine.

Corollary 5.4. Every quasi-affine variety admits an affine open cover.

6 Irreducible Varieties

Lemma 6.1. Let $X$ be a topological space. Then the following are equivalent:

1. Every sequence of closed sets $Z_1 \supset Z_2 \supset Z_3 \supset \cdots$ is stable. I.e., there exists $n$ such that $Z_m = Z_n$ for all $m \geq n$.

2. Every nonempty set $A$ of closed subsets of $X$ has a minimal element. That is, there exists $Z \in A$ such that $Z' \not\subseteq Z$ for all $Z' \in A$ distinct from $Z$.

We say $X$ is Noetherian if $X$ satisfies those conditions.

Example 6.2. $A^n$ is Noetherian since there is a correspondence between closed subsets and radical ideals, which reverses the inclusion, and $A_n$ is Noetherian as a ring. So every increasing sequence of ideals in $A_n$ stabilizes, so every decreasing sequence of closed sets in $A^n$ stabilizes.
Lemma 6.3. Every subspace of a Noetherian space is Noetherian.

Corollary 6.4. Every subset of $\mathbb{A}^n$ is Noetherian.

Lemma 6.5. Every Noetherian space is compact.

Proof. We use the definition of compactness that if $Z_n \subset X$ is a family of closed subsets, and $\bigcap \alpha Z_n = \emptyset$, then there exist $\alpha_1, \ldots, \alpha_n$ such that $Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n} = \emptyset$, i.e., the finite intersection property.

Let $\{Z_{\alpha}\}$ be as above. Let $A = \{Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n}\}$ be the set of all finite intersections of members of this family. So $\bigcap A \subset \bigcap_n Z_{\alpha} = \emptyset$. Also, $A$ is a set of closed sets. Let $Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n}$ be minimal in $A$. I claim this intersection is empty. If not, there exists $x \in Z_{\alpha_i}$ for $1 \leq i \leq n$. There exists $\alpha_{n+1}$ such that $x \notin Z_{\alpha_{n+1}}$ since $\bigcap A = \emptyset$. Then
\[ x \notin \bigcap_{i=1}^{n+1} Z_{\alpha_i} \subset \bigcap_{i=1}^{n} Z_{\alpha_i} \]
a contradiction to minimality. \qed

Proposition 6.6. Every quasi-affine variety admits a finite affine open cover.

Lemma 6.7. Let $X$ be a topological space. TFAE:

1. If $Z_1 \subset X$ and $Z_2 \subset X$ are closed, then $Z_1 \cup Z_2 \neq X$.
2. If $U_1 \subset X$ and $U_2 \subset X$ are nonempty open subsets, then $U_1 \cup U_2 \neq \emptyset$.

3. Every nonempty open subset of $X$ is dense in $X$.

Proof. (1) $\iff$ (2) just take complements. (2) $\Rightarrow$ (3). Take $U \subset X$ nonempty open, and $U \cap (X \setminus U) = \emptyset$. By (2), $X \setminus U = \emptyset$, so $X = U$, so $U$ is dense in $X$.

(3) $\Rightarrow$ (2) Take $U_1, U_2 \subset X$ to be nonempty, open. If $U_1 \cap U_2 = \emptyset$, then $U_1 \subset X \setminus U_2$, so $U_1 \subset X \setminus U_2 \neq X$, so $U_1$ is not dense in $X$. \qed

A topological space $X$ satisfying the above properties is called irreducible.

We have the following Irreducibility Test: Suppose $A \subset X$. Then $A$ is irreducible iff the following condition holds: if $A \subset Z_1 \cup Z_2$, for $Z_1, Z_2$ closed, then either $A \subset Z_1$ or $A \subset Z_2$.

Proposition 6.8. A quasi-affine variety $X$ is irreducible iff $k[X]$ is a domain.

Proof. Suppose $X$ is irreducible. Take $f, g \in k[X]$ such that $fg = 0$. But then $X = Z(fg) = Z(f) \cup Z(g)$. So $X$ is a union of two closed sets, so $X = Z(f)$ or $X = Z(g)$, whence $f = 0$ or $g = 0$.

Conversely, suppose $k[X]$ is a domain. Suppose $X = Z_1 \cup Z_2$, both closed. Towards a contradiction, suppose they are both proper subsets. So there exists $f_1 \in k[X]$ such that $f_1 \neq 0$ but $f_1|_{Z_1} = 0$, and $f_2 \in k[X]$ such that $f_2 \neq 0$ but $f_2|_{Z_2} = 0$. So $f_1f_2|_X = 0$, hence $k[X]$ has zero divisors, a contradiction. \qed

Example 6.9. 1. $\mathbb{A}^n$ is irreducible since coordinate ring is a polynomial ring, hence a domain.

2. $\mathbb{A}^1 \setminus \{0\}$ is irreducible, since $k[\mathbb{A}^1 \setminus \{0\}] = k[t, t^{-1}]$ which is a domain.

3. If $f \in \mathbb{A}_n$ is irreducible, then $Z(f) \subset \mathbb{A}^n$ is called a hypersurface, and is irreducible because $k[Z(f)] = A_n/(f)$ which is a domain since $(f)$ is prime.

4. Take $X = Z(T_1) \cup Z(T_2) \subset \mathbb{A}^2$. This is $Z(T_1T_2)$. Then $k[X] = k[T_1, T_2]/(T_1T_2)$, which is not a domain since $t_1t_2 \equiv 0$ on the axes. Note $Z(T_i) \simeq \mathbb{A}^1$, hence both axes are irreducible, so $X$ is a union of irreducible parts. Actually, these components are unique.

5. Take $X$ affine variety. We have a correspondence between the closed subsets of $X$ and the radical ideals of $k[X]$. So if $Y$ closed and $I$ the corresponding ideal, $k[Y] \simeq k[X]/I$ under this correspondence. This implies the irreducible closed subsets are in correspondence with the prime ideals in $k[X]$.

Theorem 6.10. Let $X$ be a Noetherian topological space. Then there are irreducible closed subsets $X_i \subset X$, $i = 1, \ldots, n$ such that $X = \bigcup_i X_i$ and $X_i \not\subset X_j$ for $i \neq j$. These subsets are unique.
Proof. Let \( Y \) be a minimal closed subset of \( X \) that has no decomposition \( Y = \bigcup Y_i \) of closed subsets. Then \( Y \) is not irreducible, therefore \( Y = Y' \cup Y'' \) for \( Y', Y'' \) closed and strictly smaller than \( Y \). But then \( Y' \) and \( Y'' \) have decompositions, hence so does \( Y' \cup Y'' = Y \), a contradiction.

For uniqueness, suppose \( X_1 \cup X_2 \cup \cdots \cup X_n = X_1' \cup X_2' \cup \cdots \cup X_m' \). Now \( X_i \) is contained in the RHS, and the irreducibility test says \( X_i \subset X_j' \) for some \( j \). Then find \( k \) such that \( X_j' \subset X_k \), so \( X_i \subset X_j' \subset X_k \), so \( i = k \). So \( X_i = X_j' \).

The \( X_i \)'s are called the irreducible components of \( X \).

**Example 6.11.** Take \( X = \mathbb{Z}(y^2 - xz, z^2 - y^2) \subset \mathbb{A}^3 \). Note
\[
z^3(z - x^3) = z^4 - x^3z^3 = z^4 - y^3 = y^4 - y^3 = 0
\]
So \( X = Z_X(z) \cup Z_X(z - x^3) \). Are they irreducible? Well \( Z_X(z) = Z_{\mathbb{A}^3}(z, y^2 - xz, z^2 - y^3) = Z_{\mathbb{A}^3}(y, z) \cong \mathbb{A}^1 \), hence irreducible.

On \( Z_X(z - x^3) \), note \( x^4 = xz = y^2 \), and \( z^2 = x^6 \), so
\[
x^4(y - x^2) = x^4y - x^6 = y^3 - x^6 = y^3 - z^2 = 0.
\]
So \( Z_X(z - x^3) = Z_X(z - x^3, x) \cup Z_X(z - x^3, y^2 - x^2) \). But \( Z_X(z - x^3, x) = Z_{\mathbb{A}^3}(x, y, z) = \{0\} \), which is in \( Z_X(z) \) already.

Now \( k[Z_X(z - x^3, y - x^2)] = k[x, y, z]/(z - x^3, y - x^2) \cong k[x] \). So \( Z_X(z - x^3, y - x^2) \cong \mathbb{A}^1 \), hence is irreducible. So \( X = Z_{\mathbb{A}^3}(y, z) \cup Z_{\mathbb{A}^3}(z - x^3, y - x^2) \). The first is embedded as the \( x \)-axis, and the second is a line embedded under the parametrization \( x = t, y = t^2 \), and \( z = t^3 \). The first set contains \( (1, 0, 0) \), and the second contains \( (1, 1, 1) \).

**Proposition 6.12.** Let \( A \) be a subset of a topological space \( X \). Then \( A \) is irreducible iff \( \overline{A} \) is irreducible.

Proof. For forward direction, write \( \overline{A} \subset Z_1 \cup Z_2 \) as a union of closed sets. Then \( A \subset Z_1 \cup Z_2 \), so by irreducibility test, \( A \subset Z_i \) for some \( i \), and thus \( \overline{A} \subset Z_i \), since \( \overline{A} \) is closed.

Conversely, suppose \( A \subset Z_1 \cup Z_2 \). Then \( \overline{A} \subset Z_1 \cup Z_2 \), so since \( \overline{A} \) is irreducible, \( \overline{A} \subset Z_i \) for some \( i \), whence \( A \subset Z_i \) for some \( i \).

**Corollary 6.13.** Let \( X \) be an irreducible topological space, \( U \subset X \) a nonempty open subset. Then \( U \) is also irreducible.

Proof. Since \( U \) is dense in \( X \), \( \overline{U} = X \) is irreducible, so \( U \) is irreducible by the proposition.

**Proposition 6.14.** If \( X \) and \( Y \) are irreducible quasi-affine varieties, then so is \( X \times Y \).

Proof. If \( y \in Y \), then \( X \cong X \times y \subset X \times Y \) is a closed subset isomorphic to \( X \). Suppose \( X \times Y = Z_1 \cup Z_2 \), for \( Z_i \) closed. Then \( X \times y \subset Z_1 \cup Z_2 \), so by irreducibility test, \( X \times y \subset Z_1 \) or \( X \times y \subset Z_2 \).

Now set \( Y_i = \{ y \in Y : X \times y \subset Z_i \} \), so \( Y = Y_1 \cup Y_2 \). Claim that \( Y_i \) is closed in \( Y \). For any \( x \in X \), consider \( (x \times Y) \cap Z_i = x \times Y_x \) where \( Y_x \subset Y \) is closed as the intersection of two closed sets. So
\[
\bigcap_{x \in X} Y_x = \{ y \in Y : (x, y) \in Z_i, \forall x \in X \} = Y_i
\]
so \( Y_i \) is closed as intersection of closed sets. Since \( Y \) is irreducible, we must have \( Y = Y_i \). Thus \( X \times Y = Z_i \).

**Corollary 6.15.** Let \( A \) and \( B \) be commutative \( k \)-algebras. If \( A \) and \( B \) are domains, then so is \( A \otimes_k B \). (Here \( k \) assumed to be algebraically closed.)

Proof. Assume \( A \) and \( B \) are finitely generated. Then \( A = k[X] \) and \( B = k[Y] \) for \( X \) and \( Y \) irreducible affine varieties. Then \( X \times Y \) is irreducible, then \( k[X \times Y] = k[X] \otimes_k k[Y] = A \otimes_k B \) is a domain.

Note this is not true if \( k \) is not algebraically closed, for instance, \( C \otimes_{\mathbb{R}} C \cong C \times C \).
7 Rational Functions and Maps

Let $X$ be an irreducible quasi-affine variety. Consider the set of pairs $(U, f)$ where $U \subset X$ is a nonempty, open subset of $X$, and $f$ is a regular function on $U$. We say $(U_1, f_1) \sim (U_2, f_2)$ if $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$. Note since $X$ is irreducible, $U_1 \cap U_2 \neq \emptyset$.

Suppose $(U_1, f_1) \sim (U_2, f_2) \sim (U_3, f_3)$. We have $f_1|_{U_1 \cap U_2 \cap U_3} = f_3|_{U_1 \cap U_2 \cap U_3}$. We need to show $f_1|_{U_1 \cap U_3} = f_3|_{U_1 \cap U_3}$. But $X \supset U_1 \cap U_3 \supset U_1 \cap U_2 \cap U_3 \neq \emptyset$. Where $X$ and $U_1 \cap U_3$ are irreducible, and $U_1 \cap U_2 \cap U_3$ is dense in $U_1 \cap U_3$, so the functions agree on $U_1 \cap U_3$, as they agree on a dense subset.

Denote by $k(X)$ the set of equivalence classes of pairs, denoted $[U, f]$. If $U' \subset U$ is open, then $(U, f) \sim (U', f')$ where $f' = f|_{U'}$. Thus $[U, f] = [U', f']$, so we can shrink open sets and replace functions by their restrictions.

These equivalence classes are called rational functions. We define operations

$$[U_1, f_1] + [U_2, f_2] = [U_1 \cap U_2, f_1|_{U_1 \cap U_2} + f_2|_{U_1 \cap U_2}]$$

and

$$[U_1, f_1] \cdot [U_2, f_2] = [U_1 \cap U_2, f_1|_{U_1 \cap U_2} \cdot f_2|_{U_1 \cap U_2}]$$

and

$$a \cdot [U, f] = [U, af]$$

This makes $k(X)$ a commutative $k$-algebra. Also $0 = [U, 0] = [X, 0]$. So if $[U, f]$ is nonzero in $k(X)$, then $f \neq 0$. Then note

$$[D_X(f), \frac{1}{f}] \cdot [U, f] = [D_X(f), 1]$$

which is the identity in $k(X)$. So in fact $k(X)$ is a field extension of $k$.

We can embed the domain $k[X] \to k(X) : f \mapsto [X, f]$.

We say $X$ is irreducible iff $k[X]$ is a domain. We say $X$ is irreducible if $X \neq \emptyset$ and $X$ is not the union of two proper closed sets. We know every $X$ is a unique union of irreducible varieties.

Let $X$ be irreducible in $\mathbb{A}^n$. We embed $k[X] \subset k(X)$ as $g \mapsto [X, g]$.

**Proposition 7.1.** $k(X)$ is a quotient field of $k[X]$.

**Proof.** Let $[U, f] \in k(X)$, so $U \subset X$ is open and nonempty, and $f \in k[U]$. Pick $x \in U$. In a neighborhood $U' \subset U$ of $x$, we can write $f = \frac{G}{H}$, where $G, H$ are polynomials, and $H \neq 0$ in $U'$. Let $h = G|_{U'}$ and $h = H|_{U'}$. On $U'$, we have $f|_{U'} = \frac{g}{h}$. So $[U, f] = [U', f|_{U'}] = [\frac{[X, g]}{[X, h]}]$.

We write $f$ for $[X, f]$ for $f \in k[X]$ and $[X, f] \in k(X)$. So every $f \in k(X)$ can be written as $f = \frac{g}{h} \in k[X]$, $h \neq 0$.

**Remark 7.2.** If $X \subset \mathbb{A}^n$, then $k(X)$ is generated by the coordinate functions $t_i$ as a field. In particular, $k(X)/k$ is finitely generated as a field.

Let $X$ be irreducible. Take $f \in k(X)$. So $f = [U, g]$, where $U \subset X$ is nonempty open, and $g \in k[U]$. Let $\hat{U} = \bigcup U$, where the union is taken over all $U \subset X$ open such that $f = [U, g]$ for some $g \in k[U]$. Then $\hat{U}$ is open in $X$. If $f = [U', g']$, then $g$ and $g'$ agree on $U \cap U'$. Then there exists $h \in k[U \cup U']$ such that $h|_U = g$ and $h|_{U'} = g'$.

So there exists a unique $\hat{g} \in k[\hat{U}]$ such that $f = [\hat{U}, \hat{g}]$. We call $\hat{U}$ the domain of definition of $f$. We say $f$ is regular at every point $x \in \hat{U}$ (or defined at $x \in U$), and $f(x) = \hat{g}(x)$.

Note $f$ is defined for all $x \in X$ iff $\hat{U} = X$ iff $f$ is regular.

**Example 7.3.** Take $X = Z(x^2 + y^2 - 1) \subset \mathbb{A}^2$. Then $k[X] = k[x, y]/(x^2 + y^2 - 1)$. Let $f = \frac{1-y}{x} \in k(X)$ is defined at $(x, y)$ with $x \neq 0$. But note that $(1-y)(1+y) = 1 - y^2 = x^2$, so $\frac{1-y}{x} = \frac{x}{1+y}$ in $k(X)$. This is defined at $(0, 1)$ if characteristic of $k$ is not zero.

But $\frac{1-y}{x} \notin k[X]$ so the domain of definition of $\hat{U} = X \setminus \{(0, -1)\}$.
Let $X$ and $Y$ be quasi-affine varieties, so $X$ is irreducible. Consider pairs $[U, f]$ and $U \subset X$ open, and $f: U \to Y$ be a regular map. We define $(U, f) \sim (U', f')$ if $f$ and $f'$ agree on $U \cap U'$. A rational map $X \to Y$ is an equivalence class $[U, f]$ of a pair $(U, f)$.

A rational function on $X$ is a rational map $X \to k^1$. Suppose $Y \subset k^m$, and $[U, f]: X \to Y$, then $f = (f_1, \ldots, f_n)$ for $f_i \in k[U]$, so we can view $f_i$ as rational functions on $X$. So to give a rational map $X \to Y$ is the same as to give rational functions $f_1, \ldots, f_n \in k(X)$ such that $(f_1(x), f_2(x), \ldots, f_n(x)) \in Y$ for all $x$ where all $f_i$'s are defined.

Every rational map $X \to Y$ has the domain of definition $\tilde{U} \subset X$. Let $f: X \to Y$ defined on $\tilde{U} \subset X$, so $f = \{\tilde{U}, \tilde{g}\}$ for $\tilde{g}: \tilde{U} \to Y$ regular. Define $\text{im}(f) = \text{im}(\tilde{g}) \subset Y$. We say $f$ is dominant if $\text{im}(f)$ is dense in $Y$.

Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

If $\text{im}(f) \cap \tilde{U} \neq \emptyset$, we cannot define $g \circ f$. But if $f$ is dominant, $\text{im}(f)$ is dense in $Y$, thus $\text{im}(f) \cap \tilde{U} \neq \emptyset$ and we can define $g \circ f$ as follows:

$$g \circ f = [f^{-1}(\tilde{U}) \cap \tilde{U}, h]$$

such that $h(x) = g(f(x))$.

Take $X \xrightarrow{f} Y \xrightarrow{g} \mathbb{A}^1$ where $g \in k(Y)$ and $X$ and $Y$ both irreducible. Assume that $f$ is dominant. Therefore, $g \circ f \in k(X)$. We get a map

$$f^*: k(Y) \to k(X) : g \mapsto g \circ f$$

which is a homomorphism of fields over $k$.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)^* = f^* \circ g^*$. So we get a functor from the opposite category of irreducible quasi-affine varieties and dominant rational maps to the category of field extensions of $k$ with morphisms homomorphisms of fields over $k$. It is given by $X \mapsto k(X)$ and $f \mapsto f^*$.

**Example 7.4.** Let $X$ be irreducible, and $U \subset X$ a nonempty open set, so $U$ is irreducible. We have the inclusion map $U \hookrightarrow X$ which is dominant because the image is $U$ is every nonempty open is dense in an irreducible variety. I claim this is an isomorphism in the above category. We define an inverse map $X \to U$ by $[U, \text{id}_U]$. So $U \approx X$ are “birationally isomorphic.” Therefore $k(X) \simeq k(U)$.

We have a functor from the opposite category of irreducible varieties over $k$ to the category of field extensions of $k$.

**Proposition 7.5.** This functor is full and faithful.

**Proof.** Take $X$ and $Y$ to be irreducible varieties. Let $\alpha: k(Y) \to k(X)$ be a homomorphism over $k$. We need to find $f: X \to Y$ such that $\alpha = f^*$. Replace $Y$ by a nonempty open affine subvariety. Such exists because $Y$ is covered by open affine subvarieties, and $Y$ is isomorphic to its open subvarieties since it is irreducible. So we may assume $Y$ is affine. So $Y \subset k^m$ closed, with $t_1, \ldots, t_m$ coordinate functions on $Y$. Now $t_i \in k[Y] \subset k(Y)$.

Let $g_i = \alpha(t_i) \in k(X)$. All the $g_i$ are defined on some nonempty open set $U \subset X$. So the restrictions are in $g_i \in k[U] \subset k(U) = k(X)$. We have

$$k(Y) \xrightarrow{\alpha} k(X)$$

$$t_i \in k[Y] \xrightarrow{\beta} k[U] \ni g_i$$

Since $Y$ is affine, there exists a regular map $h: U \to Y$ such that $h^* = \beta$. Also $h^*(t_i) = g_i$. So $h$ defines a rational map $f: X \to Y$ given by $[U, h]$. So $f^*: k(Y) \to k(X) = k(U)$ and $f^*(t_i) = g_i = \alpha(t_i)$. So $\alpha$ and $f^*$ agree on the $t_i$, which generate $F(Y)$ ($k(Y)$?) so $f^* = \alpha$.

We need to also show $f$ is dominant. Now

$$\text{im}(h) = h(Z(0)) = Z(f^{**}(0)) = \ker f^* = Z(\ker \beta) = Z(0) = Y.$$
since $\beta$ is injective as a field homomorphism. It follows that $h$ and $f$ are dominant.

Now we need to show that if $f: X \to Y$ and $g: X \to Y$ are such that $f^* = g^*$, then $f = g$. Assume $Y \subset A^m$. Then $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$, where $f_i, g_i \in k(X)$, and $f_i = f^*(t_i) = g^*(t_i) = g_i$, and hence $f = g$. \hfill \Box

Recall that the field $k(X)$ is always finitely generated.

**Corollary 7.6.** The functor from the opposite category of irreducible quasi-affine varieties with dominant morphisms over $k$ to the category of finitely generated field extensions of $k$ with homomorphism over $k$ is an equivalence.

**Proof.** Suppose $K/k$ is a finitely generated field extension. We need to find an irreducible quasi-affine variety $X$ such that $k(X) \cong K$. Suppose $K = k[f_1, \ldots, f_n]$. Then let $A = k[f_1, \ldots, f_n] \subset K$. Then $A$ is a finitely generated domain as a subring of a field. Therefore, there exists an irreducible affine $X$ such that $k[X] \cong A$ because of the equivalence between affine varieties and finitely generated reduced algebras. Then

$$K = \text{Quot}(A) = \text{Quot}(k[X]) = k(X).$$\hfill \Box

We will write $X \cong Y$ if they are birationally isomorphic. Note that $X \cong Y$ implies $X \cong Y$. However, if $U \subset X$ is open, then $U \cong X$, but it is not necessarily true that $U \cong X$ since $A^1 \setminus \{0\} \not\cong A^1$ since their coordinate rings are $k[t, t^{-1}]$ and $k[t]$, which are not isomorphic since any isomorphism must send constants to constants.

**Example 7.7.** Consider $X$ given by $x^3 = y^2$,

$$X \subset A^2 \xrightarrow{\pi} A^1$$

where the maps are $t \mapsto (t^2, t^3)$ and $(x, y) \mapsto \frac{y}{x}$. Then $X \cong A^1$ but $X \not\cong A^1$.

**Proposition 7.8.** Two irreducible varieties $X$ and $Y$ are birationally isomorphic if and only if there exist nonempty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$.

**Proof.** For the backward direction, we have $X \cong U \cong V \cong Y$. For the forward direction, suppose $X \cong Y$ by maps $f: U' \to Y$ and $g: V' \to X$ for $U'$ and $V'$ open in $X$ and $Y$, respectively. We get a map

$$f^{-1}(V') \xrightarrow{f} V' \xrightarrow{g} X$$

and

$$g^{-1}(U') \xrightarrow{g} U' \xrightarrow{f} Y$$

Both these maps are the identity, so these composites are inclusions, so $g(f(x)) = x$ for all $x \in f^{-1}(V')$ and $f(g(y)) = y$ for all $y \in g^{-1}(U')$.

But $X \supset U := f^{-1}(g^{-1}(U')) \subset f^{-1}(V')$ and $Y \supset V := g^{-1}(f^{-1}(V')) \subset g^{-1}(U')$. If $x \in U \subset f^{-1}(V')$, then $g(f(x)) = x$. Applying $f$ yields $f(g(f(x))) = f(x) \in V$. But $f(x) = f(g(f(x))) \in V'$ which implies the previous sentence.

Similarly, if $y \in V$, then $g(y) \in U$, and $f(g(y)) = y$, so we get inverse isomorphisms $f$ and $g$ between $U$ and $V$. \hfill \Box

**Proposition 7.9.** Every irreducible variety is birationally isomorphic to a hyperplane in some $A^n$. 

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Proof. Let $X$ be an irreducible variety. Take $K = k(X)/k$ which is a finitely generated field extension. We can find a sequence of extensions

$$K/k(t_1, \ldots, t_m)/k$$

where $K$ is finite and separable over $k(t_1, \ldots, t_m)$ (it is separable since $k$ is algebraically closed, hence perfect), and the second purely transcendental. Denote $E = k(t_1, \ldots, t_m)$. So $K = E(\alpha)$ for some $\alpha \in K$. Let $f \in E[t]$ be the minimal polynomial of $\alpha$. So we find $g \in k[t_1, \ldots, t_m, t] = R[t]$ where $R = k[t_1, \ldots, t_m]$ is a UFD where $g$ has relatively prime coefficients in $R$. So $f$ is primitive.

If $g$ is irreducible in $E[t]$, primitive in $R[t]$ and $R$ is a UFD, then this implies $g$ is irreducible in $F[t_1, \ldots, t, t]$. So $Z_{A=1}(g) \subset A^{m+1}$ is irreducible. Also,

$$k(Y) = \text{Quot}(k[Y]) = \text{Quot} F[t_1, \ldots, t_m, t]/(g) = \text{Quot} E[t]/(f) = \text{Quot} E(\alpha) = E(\alpha) = K = k(X)$$

and so $X \approx Y$. 

Example 7.10. Let $X = Z(x^2 + y^2 - 1) \subset \mathbb{A}^2$. Then $X \approx \mathbb{A}^1$ via the maps $t \mapsto \left(\frac{2t}{1 + t^2}, \frac{2 - 1}{1 + t^2}\right)$ and $(x, y) \mapsto \frac{1 + xy}{x}$.

8 Local Ring of a Subvariety

Let $X$ be a quasi-affine variety, $Y \subset X$ a closed irreducible subvariety. A neighborhood of $Y$ in $X$ is an open set $U \subset X$ such that $U \cap Y = \emptyset$. If $U_1$ and $U_2$ are two neighborhoods of $Y$, then so if $U_1 \cap U_2$, since $Y$ is irreducible.

Let $(U, f)$ denote $U$ a neighborhood of $Y$, and $f \in k[U]$. Define an equivalence relation such that $(U_1, f_1) \sim (U_2, f_2)$ if there exists a neighborhood $W$ of $Y$ in $X$ such that $W \subseteq U_1 \cap U_2$ and $f_1$ and $f_2$ agree on $W$. The set of classes $[U, f]$ forms a $k$-algebra denoted $O_{X,Y}$.

Example 8.1. If $Y$ is irreducible, then $O_{X,Y} = \{[U, f]\} = k(X)$.

Define a map $O_{X,Y} \xrightarrow{\pi} k(Y) : [U, f] \mapsto [U \cap Y, f|_{U \cap Y}]$. Define $\ker \pi = m_{X,Y}$, which is an ideal in $O_{X,Y}$.

Proposition 8.2. 1. The map $\pi$ is surjective.

2. The ring $O_{X,Y}$ is a local ring with maximal ideal $m_{X,Y}$ and residue field $k(Y)$.

Proof. Let $g \in k(Y)$, so $g = [W, h]$ where $W = U \cap Y$ and $U$ is open in $X$. Let $y \in W$. There exists an affine neighborhood $U'$ of $y \subset U' \subset U$. So $U' \cap Y \neq \emptyset$, and thus $U'$ is a neighborhood of $Y$ in $X$. Then $W' = U' \cap Y$ is closed in $U'$, so $W' \subset U'$ is affine and closed. The map $k[U'] \rightarrow k[W']$ is surjective so there exists $f \in k[U']$ such that $f|_{W'} = h|_{W'}$. Then

$$\pi[U', f] = [W', h|_{W'}] = g.$$  

For the second claim, since $k(Y)$ is a field, $m_{X,Y}$ is a maximal ideal in $O_{X,Y}$. Let $[U, f] \in O_{X,Y} \setminus m_{X,Y}$, so $f|_{U \cap Y}$ is not zero. Thus $D_U(f) \cap Y \neq \emptyset$, so $D_U(f)$ is a neighborhood of $Y$ in $X$. So

$$[D_U(f), f^{-1}] \cdot [D_U(f), f] = 1$$

but $[U, f] = [D_U(f), f]$, so $[U, f]$ is a unit. The claims follow. 

Let $g : Y \subset X \rightarrow Y' \subset X'$ be a dominant regular map, with $Y \subset X$ closed and irreducible. Consider the map $g^* : O_{X', Y'} \rightarrow O_{X,Y}$. Let $[U', f']$, so $f' \in k[U']$, and thus $g \circ f' \in k[g^{-1}(U')]$. Since $g$ is dominant so that $g(Y)$ is dense in $Y'$, we have $g(Y) \cap U' \neq \emptyset$, so $g^{-1}(U') \cap Y \neq \emptyset$, so $g^{-1}(U')$ is a neighborhood of $Y$ in $X$. Then

$$g^*[U', f'] = [g^{-1}(U'), g \circ f'].$$

Also, $g^*(m_{X', Y'}) \subset m_{X,Y}$, $g^*$ is a local homomorphism of local rings.

Let $Y \subset X$ be a closed and irreducible subvariety. What is $O_{X,Y}$. We have a map

$$\alpha : k[X] \rightarrow O_{X,Y} : f \mapsto [X, f].$$
We have $\mathfrak{m}_{X,Y} \subset \mathcal{O}_{X,Y}$ is a maximal ideal, and $\alpha^{-1}(\mathfrak{m}_{X,Y}) = \{ f \in k[X] : f|_Y = 0 \} = I(Y)$ which is a prime ideal in $k[X]$.

If $g \in k[X] \setminus I(Y)$, then $\alpha(g) \in \mathcal{O}_{X,Y}^*$. By the universal property of localization, $\alpha$ extends to a unique ring homomorphism

$$\bar{\alpha} : k[X]_I(Y) \to \mathcal{O}_{X,Y} : \frac{f}{g} \mapsto \frac{\alpha(f)}{\alpha(g)}.$$

**Proposition 8.3.** $\bar{\alpha}$ is an isomorphism.

**Proof.** Suppose $\bar{\alpha}(\frac{f}{g}) = 0$. Then $0 = [X,f] \in \mathcal{O}_{X,Y}$. So $f|_U = 0$ for a neighborhood $U$ of $Y$ in $X$. We need to show $\frac{f}{g} = 0$ in $k[X]_I(Y)$. Pick $g \in Y \cap U$. Then $X \setminus U$ is closed in $X$, and does not contain $y$. So there exists $h \in k[X]$ such that $h|_{X \setminus U} = 0$ but $h(y) \neq 0$. So $h \notin I(Y)$. So $fh = 0$ in $k[X]$, and thus $\frac{f}{g} = 0$ in $k[X]_I(Y)$. This gives injectivity.

Now take $[U,f] \in \mathcal{O}_{X,Y}$, with $U$ a neighborhood of $Y$ in $X$, and $f$ a regular function on $U$. So pick $y \in Y \cap U$. In a neighborhood $W$ of $y$ in $U$, $f = \frac{h}{g}$ where $g,h \in k[X]$ $h(y) \neq 0$. But

$$[U,f] = [W,f|_W] = \frac{[W,g|_W]}{[W,h|_W]} = \frac{[X,g]}{[X,h]} = \frac{\alpha(g)}{\alpha(h)} = \bar{\alpha} \left( \frac{g}{h} \right) \in \text{Im}(\bar{\alpha}).$$

\[\square\]

**Example 8.4.**

1. Let $X = Z(xy) \subset \mathbb{A}^2$, the union of the two axes. So $X$ is reducible. Let $Y = \{0\}$. But $k[X] = \mathbb{k}[x,y]/(xy) \cong (x,y)/(xy) = I(Y)$. Then $\mathcal{O}_{X,Y} = k[X]_I(Y)$ is not a domain since $x \cdot \bar{y} = 0$ but $\bar{x} \neq 0$ and $\bar{y} \neq 0$. This reflects the fact that there are two irreducible components passing through $Y$.

2. Let $X = Z(y^2 - x^3 - x^2) \subset \mathbb{A}^2$. Put $Y = \{(0,0)\}$. Then $k[X] = k[x,y]/(y^2 - x^3 - x^2)$, but $y^2 - x^3 - x^2$ is irreducible, so $X$ is irreducible, and $I(Y) = (x,y)/(y^2 - x^3 - x^2)$. So $\mathcal{O}_{X,Y} = k[X]_I(Y)$ is a domain. This reflects the fact that their is one irreducible component passing through $Y$.

Let $Y \subset X$ be closed, with both $X$ and $Y$ irreducible. Then we have

$$k[X] \subset \mathcal{O}_{X,Y} \subset k(X) = \text{Quot} (\mathcal{O}_{X,Y})$$

so $\mathcal{O}_{X,Y} = \{ f \in k(X) : f \text{ is regular in a neighborhood of } Y \text{ in } X \}$. This is the same as $\{ f \in k(X) : f \text{ is regular at a point } y \in Y \}$. If $Y = \{y\}$, then $\mathcal{O}_{X,Y} = \mathcal{O}_{X,y} = \{ f \in k(X) : f \text{ is regular at } y \}$.

Now suppose $Y \subset Z \subset X$ with $Y$ and $Z$ both closed and irreducible. We have that

$$\ker (\mathcal{O}_{X,Y} \to \mathcal{O}_{X,Z} \to k(Z))$$

is a prime ideal in $\mathcal{O}_{X,Y}$. The map is just given by $[U,f] \mapsto [U,f]$. The ideal is $\{ [U,f] : f|_Z = 0 \} = I(Z)$.

Then the closed irreducible subsets $Z$ in $X$ containing $Y$ is in correspondence with the prime ideals in $\mathcal{O}_{X,Y}$ given by $Z \mapsto I(Z)$.

**Proposition 8.5.** This map is a bijection.

**Proof.** Idea: Let $U \subset X$ be a neighborhood of $Y$ in $X$. Consider the collection of closed irreducible subsets in $U$ containing $U \cap Y$. This is the same as the closed irreducible subsets $Z$ in $X$ containing $Y$. This corresponds to prime ideals in $\mathcal{O}_{U,U \cap Y}$, but $\mathcal{O}_{X,Y} = \mathcal{O}_{U,U \cap Y}$.

Now choose $U$ affine. We may assume that $X$ is affine. Then the collection of closed irreducible $Z \subset X$ are in correspondence with the prime ideals $p \subset k[X]$ with $p \subset I(Y)$. By commutative algebra, these prime ideals are in correspondence with the prime ideals in $k[X]_I(Y) = \mathcal{O}_{X,Y}$. The composition of these two maps is checked to be the same as the map above.

This map is inclusion reversing, so in particular, $Y$ corresponds to $\mathfrak{m}_{X,Y}$. The irreducible components of $X$ containing $Y$ correspond to minimal prime ideals in $\mathcal{O}_{X,Y}$. There is only one irreducible component of $X$ containing $Y$ if $\mathcal{O}_{X,Y}$ is a domain, because in that case (0) is the only minimal prime.

Suppose $X$ and $Y$ are quasi-affine varieties. Suppose $X, Y \subset \mathbb{A}^n$. Consider $X \cong X \times \{0\} \subset \mathbb{A}^{n+1}$ and $Y \cong Y \times \{1\} \subset \mathbb{A}^{n+1}$. Then write $X + Y$, the disjoint union of $X$ and $Y$, or sum of varieties, to be

$$X + Y = X \times \{0\} \cup Y \times \{1\}.$$

Finally, $k[X + Y] = k[X] \times k[Y]$. 

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9 Quasi-projective varieties

9.1 Quasi-projective Sets

Recall that if we have a map from a compact space to a Hausdorff space, then its image is always closed. However, the projection \( X = \mathbb{A}^1 \to \mathbb{A}^1 \) has image which is open but not closed. If \( X = X \cup \{ \infty \} \), then we can map \( \infty \to 0 \). These varieties are called projective. So projective varieties are “true compact.”

We will construct a full and faithful functor \( \text{QAVar} \to \text{QVar} \).

Consider \( \mathbb{A}^{n+1} \) with coordinates \((t_0, \ldots, t_n)\). A subset \( Y \subset \mathbb{A}^{n+1} \) is called a cone if the following holds: If \( y \in Y \) and \( c \in k \), then \( cy \in Y \). So note \( \emptyset \) and \( \{0\} \) are cones, and any nonempty cone must contain 0, so \( 0 \in Y \iff Y \neq \emptyset \).

Definition 9.1. A polynomial \( F \in k[t_0, \ldots, t_n] \) is called homogeneous of degree \( d \) if \( F(ct) = c^d F(t) \) for every \( c \in k \). Equivalently, every monomial in \( F \) has degree \( d \).

Definition 9.2. An ideal \( I \subset \mathbb{A}_{n+1} = k[t_0, \ldots, t_n] \) is called homogeneous if \( F \in I \) implies \( F_i \in I \) for all \( i \), where \( F_i \) is the degree \( i \) homogeneous component of \( F \).

Proposition 9.3.

1. An ideal \( I \subset \mathbb{A}_{n+1} \) is homogeneous iff \( I \) is generated by homogeneous polynomials.

2. If \( I \) is homogeneous, then so is \( \sqrt{I} \).

Lemma 9.4. A closed subset \( Y \subset \mathbb{A}_{n+1} \) is a cone iff \( I(Y) \subset \mathbb{A}_{n+1} \) is homogeneous.

Proof. For the forward direction, suppose \( Y \) is a cone, and \( F \in I(Y) \). Write \( F = \sum F_i \). We need to show \( F_i \in I(Y) \) for all \( i \). We have \( F(cy) = 0 \) for all \( c \in k \) and all \( y \in Y \), since \( cy \in Y \). But

\[
F(cy) = \sum_i c^i F_i(y)
\]

which implies \( F_i(y) = 0 \) for any \( i \) and any \( y \in Y \), so \( F_i \in I(Y) \). To see this, fix \( y \in Y \). Then \( F(Xy) = \sum_i F_i(y)X^i \equiv 0 \) is the zero polynomial in the polynomial ring \( k[X] \), so necessarily \( F_i(y) = 0 \) for all \( i \).

Conversely, suppose \( y \in Y \) and \( c \in k \). But \( I(Y) \) is generated by a set of homogeneous polynomials. If \( F \) is a homogeneous generator,

\[
F(cy) = c^d F(y) = 0
\]

which implies \( G(cy) = 0 \) for all \( G \in I(Y) \) which implies \( cy \in ZI(Y) = Y \). So \( Y \) is a cone.

We get a correspondence between the collection of closed cones in \( \mathbb{A}^{n+1} \) and the category of radical homogeneous ideals in \( \mathbb{A}_{n+1} \) given by the maps \( I \) and \( Z \). As usual, \( \emptyset \leftrightarrow \mathbb{A}_{n+1} \) and \( \{0\} \leftrightarrow (t_0, \ldots, t_n) \).

The unit group \( k^\times \) acts on \( \mathbb{A}_{n+1} \) by \( c \cdot (a_0, \ldots, a_n) = (ca_0, \ldots, ca_n) \). Then

\[
\mathbb{P}^n_k = \mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times
\]

is the projective space. The elements are classes \([a_0 : \cdots : a_n] \) with not all \( a_i \) are zero. So note \([a_0 : \cdots : a_n] = [b_0 : \cdots : b_n] \) iff there exists \( c \in k^\times \) such that \( b_i = ca_i \) for all \( i \). Equivalently, \( a_ib_j = a_jb_i \) for any \( i \) and \( j \).

We have a canonical map \( \pi: \mathbb{A}_{n+1} \setminus \{0\} \to \mathbb{P}^n \) defined by \((a_0, \ldots, a_n) \mapsto [a_0 : \cdots : a_n] \).

If \( X \subset \mathbb{P}^n \), then \( \pi^{-1}(X) \) is a cone in \( \mathbb{A}_{n+1} \).

We say \( X \subset \mathbb{P}^n \) is closed (resp. open) if \( \pi^{-1}(X) \subset \mathbb{A}_{n+1} \setminus \{0\} \) is closed (resp. open). Let \( S \subset \mathbb{A}_{n+1} \) be a set of homogeneous polynomials. Then \( Z(S) = \{ a \in \mathbb{P}^n : F(a) = 0 \} \), for all \( F \in S \). All closed sets take this form. If \( I \subset \mathbb{A}_{n+1} \) is a homogeneous ideal, then \( Z(I) = Z(S) \), where \( S \) is the set of all homogeneous polynomials in \( I \).

If \( F \) is a homogeneous polynomial, then \( D(F) = D_{\mathbb{P}^n}(F) = \{ a \in \mathbb{P}^n : F(a) \neq 0 \} \) are principal opens.

We get a correspondence of closed sets in \( \mathbb{P}^n \) and the radical homogeneous ideals in \( \mathbb{A}_{n+1} \) different from the ideal corresponding to \( \{0\} \), that is, \((t_0, \ldots, t_n) \). Then \( \emptyset \leftrightarrow \mathbb{A}_{n+1} \) and \( \mathbb{P}^n \leftrightarrow 0 \).

Theorem 9.5. (Projective Nullstellensatz) Let \( I \subset \mathbb{A}_{n+1} \) be a homogeneous ideal. Then \( Z(I) = \emptyset \) iff \( (t_0, \ldots, t_n)^m \subset I \) for some \( m > 0 \).
Proof. Note \( Z_{P^n}(I) = \emptyset \) iff \( Z_{A_n+1}(I) \subset \{0\} = Z_{A_n+1}(\langle t_0, \ldots, t_n \rangle) \). This holds iff \( \langle t_0, \ldots, t_n \rangle \subset \sqrt{I} \). \qed

We have \( P^n = \pi(\mathbb{A}^{n+1} \setminus \{0\}) \) is Noetherian and irreducible as the image of a Noetherian, irreducible space (the punctured space is an open subset of an irreducible space). So every subset of \( P^n \) is compact and admits a unique decomposition into a union of irreducible closed subsets.

A quasi-projective subset in \( P^n \) has the form \( U \cap Z \) for \( U \) open and \( Z \) closed. A subset \( X \subset P^n \) is quasi-projective iff \( X \) is open in \( \mathbb{X} \).

\[ \text{Example 9.6.} \quad \text{We have } k[\mathbb{P}^n] = k. \text{ We need to show every regular function is constant. Suppose } f \in k[\mathbb{P}^n] \text{ is not constant. So there exists } x \in \mathbb{P}^n \text{ such that } f = \frac{F}{G} \text{ in a neighborhood } U \text{ of } x \text{ and } F \text{ and } G \text{ are homogeneous nonconstant polynomials of the same degree. Assume } F \text{ and } G \text{ are coprime. Then } Z(G) \not\subseteq Z(F) \text{ because applying } I(-) \text{ implies } \sqrt{\langle G \rangle} \supset \sqrt{\langle F \rangle}, \text{ but } F \notin \sqrt{\langle G \rangle} \text{ since they are coprime. For if } F^m \in \langle G \rangle, \text{ then } G \mid F^m, \text{ contradiction. So there exists } x' \in \mathbb{P}^n \text{ such that } G(x') = 0 \text{ but } F(x') \neq 0. \text{ In a neighborhood } U' \text{ of } x', \text{ write } f = \frac{F'}{G'}, \text{ where } F' \text{ and } G' \text{ are homogeneous (these are not derivatives!).}

\text{So in the dense set } U \cup U' \text{ (since } \mathbb{P}^n \text{ is irreducible), we have } \frac{F}{G} = \frac{F'}{G'} \text{ so } FG' = F'G \text{ holds as equality of polynomials. So } F(x')G(x') = F'(x')G(x') = 0. \text{ But } F(x') \neq 0, \text{ and } G'(x') \neq 0, \text{ so we have a contradiction.}

\text{Some properties:}

1. If } f \in k[X] \text{ and } Y \subset X \text{ is a quasi-projective set, then } f|_Y \in k[Y].
2. If } X = \bigcup U_\alpha \text{ is an open cover, and } f : X \rightarrow k, \text{ then } f \in k[X] \iff f|_{U_\alpha} \in k[U_\alpha] \text{ for all } \alpha.
3. If } f : X \rightarrow k \text{ is regular at } x \in X, \text{ then } f \text{ is regular in a neighborhood of } x.
4. If } f \in k[X], \text{ then } \{ x \in X : f(x) = 0 \} \text{ is closed. So } f \text{ is continuous as a map } X \rightarrow \mathbb{A}^1.
5. If } f \in k[X] \text{ and } f(x) \neq 0 \text{ for all } x \in X, \text{ then } 1/f \in k[X].

\text{Suppose } f : X \rightarrow Y \subset \mathbb{P}^m. \text{ Take } x \in X. \text{ We say } f \text{ is regular at } x \text{ if there exists a neighborhood } U \text{ of } x \text{ and functions } f_0, f_1, \ldots, f_m \in k[U] \text{ such that } f_i \text{ is nowhere 0 on } U \text{ for some } i, \text{ with the property that } f(x') = [f_0(x') : f_1(x') : \cdots : f_m(x')] \text{ for all } x' \in U. \text{ We say } f \text{ is regular if } f \text{ is regular at every } x \in X.

\text{As before, } f \text{ is regular if and only if } X \xrightarrow{f} Y \rightarrow \mathbb{P}^m \text{ is regular.}

\text{We can write } f_i = \frac{G_i}{H} \text{ at a neighborhood of } x, \text{ where } G_i \text{ and } H \text{ are homogeneous polynomials of the same degree } d. \text{ We don’t subscript } H \text{ since we can find a common denominator. Then}

\[ f = \left[ G_0 : G_1 : \cdots : G_m \right] = \left[ G_0 : \cdots : G_1 : \cdots : G_m \right]. \]

\[ \text{Example 9.7.} \quad 1. \text{ If } f : X \rightarrow Y \text{ is regular, } X' \subset X \text{ and } Y' \subset Y \text{ such that } f(X') \subset Y', \text{ then } f|_{X'} : X' \rightarrow Y' \text{ is also regular.}
2. \text{ If } X \subset Y, \text{ then } X \hookrightarrow Y \text{ is regular.}

\[ \text{Example 9.8.} \quad 1. \text{ Let } L_1, L_2, \ldots, L_k \text{ be linearly independent linear forms on } \mathbb{P}^n. \text{ Then } Z := Z(L_1, L_2, \ldots, L_k) \subset \mathbb{P}^n, \text{ define}

\[ \mathbb{P}^n \setminus Z \rightarrow \mathbb{P}^{k-1} : x \mapsto [L_1(x) : \cdots : L_k(x)]. \]

This is called the projection with center } Z. \text{ When } k = n, \text{ then } Z \text{ is a point, so if we take } L_i = t_i \text{ the coordinate function, then } Z = \{[1 : 0 : \cdots : 0]\}. \text{ Thus we get a projection } \mathbb{P}^n \setminus \{*\} \rightarrow \mathbb{P}^{n-1}.\]
2. Fix $m,n$. Consider the monomials $t_0^\alpha t_1^n \cdots t_n^\alpha$ where $\sum \alpha_i = m$. There are $\binom{n+m}{m}$ such monomials. Write $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$. Introduce new variables $S_\alpha$. Define

$$v_{n,m} : \mathbb{P}^n \to \mathbb{P}^{d_{n,m}}$$

where $d_{n,m} = \binom{n+m}{m} - 1$, and $v_{n,m}([0 : t_1 : \cdots : t_n]) = [S_\alpha]$, where $S_\alpha = t^\alpha = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $\deg(\alpha) = \sum \alpha_i = m$. Note if $\alpha = (0, \ldots, m, \cdots 0)$ then $S_\alpha = t_1^m$. If $t_i \neq 0$, then $S_\alpha \neq 0$. This is the Veronese map.

Put $Z = Z(S_\alpha S_\beta - S_\gamma) : \alpha + \beta = \gamma + \delta \subset \mathbb{P}^{d_{n,m}}$. So $\text{Im}(v_{n,m}) \subset Z$. Claim: $v_{n,m}$ gives an isomorphism $\mathbb{P}^n \to Z$. We will construct $g : Z \to \mathbb{P}^n$.

Take $g([\alpha]) = [S(\beta_1) + \beta_2, \ldots, \beta_n] : S(\beta_0, \beta_1 + 1, \ldots, \beta_n) : \cdots : S(\beta_0, \ldots, \beta_n + 1)$ where $\deg(\beta) = m - 1$. We need to show that is independent of the choice of $\beta$, and for every $[\alpha] \in \mathbb{P}^{d_{n,m}}$, there exists $\beta$ of degree $m - 1$ such that one of the coordinates is not zero.

Consider the case where $n = 1$ and $m = 2$. The monomials are $t_0^2, t_0 t_1, t_1^2$. We get a map $v_{1,2} : \mathbb{P}^1 \to \mathbb{P}^2$

defined by $[t_0, t_1] \to t_0^2 : t_0 t_1 : t_1^2] =: [s_0 : s_1 : s_2]$. So $Z = \text{Im}(v_{1,2}) = Z(s_1^2 - s_0 s_2) \subset \mathbb{P}^2$. This is a conic in $\mathbb{P}^2$ given by $s_1^2 = s_0 s_2$, so $\mathbb{P}^2 \cong Z$. The map $g : Z \to \mathbb{P}^1$ sends $[s_0 : s_1 : s_2] \to [s_0 : s_1] = [s_1] : [s_2]$.

**Proposition 9.9.** A regular map $f : X \to Y \subset \mathbb{P}^n$ is continuous. It suffices to show that $f^{-1}(Z(G))$ is closed in $X$ where $G$ is a homogeneous polynomial since any closed subset is an intersection of closed sets given by one polynomials, and preimages behave well with intersections. Take $x \in f^{-1}(Z(G))$. In a neighborhood $U$ of $x$, $f = [f_0 : f_1 : \cdots : f_m]$, where $f_i \in k[U]$, where one is nowhere zero on $U$. So

$$f^{-1}(Z(G)) \cap U = \{ y \in U : G(f_0, \ldots, f_m)(y) = 0 \}$$

where $G(f_0, \ldots, f_m)$ is regular on $U$, hence this set is closed in $U$. So $f^{-1}(Z(G))$ is locally, therefore locally closed.

**Corollary 9.10.** If $f : X \to Y$ is regular, $Y' \subset Y$ is quasi-projective, then $f^{-1}(Y')$ is also quasi-projective.

**Lemma 9.11.** Let $f : X \to Y$ be a regular map, $g \in k[Y]$. Then $g \circ f \in k[X]$.

**Proof.** Take $x \in X$, $y = f(x)$. Since $g$ is regular, $g = \frac{F}{G}$ in a neighborhood $W$ of $y$ in $Y$, where $G$ is nowhere zero on $W$. In a neighborhood $U$ of $x$ in $X$, $f = [f_0 : \cdots : f_m]$, with $f_i \in k[U]$, one is nowhere zero on $U$. In $U \cap f^{-1}(W)$ is a neighborhood of $X$, and

$$g(f(y)) = \frac{F(f_0, \ldots, f_m)(y)}{G(f_0, \ldots, f_m)(y)} = \frac{F(f_0, \ldots, f_m)(y)}{G(f_0, \ldots, f_m)(y)} \cdot \frac{1}{G(f_0, \ldots, f_m)(y)}$$

which is regular on $U \cap f^{-1}(W)$. Hence this is regular in a neighborhood of $x$, hence is regular. 

So $f : X \to Y$ regular gives a map $f^* : k[Y] \to k[X]$ by $f^*(g) = g \circ f$, which is a $k$-algebra homomorphism.

**Proposition 9.12.** Let $f : X \to Y \subset \mathbb{P}^n$ be a map of quasi-projective sets. Then $f$ is regular if and only if for every open $V \subset Y$ and every $g \in k[V]$, the composition $g \circ f|_{f^{-1}(V)} \in k[f^{-1}(V)]$.

**Proof.** For the forward direction, $f^{-1}(V) : f^{-1}(V) \to V$ is regular, hence $g \circ f|_{f^{-1}(V)}$ is regular. Conversely, take $x \in X$ and $y = f(x) = [y_0 : \cdots : y_m]$ with $y_k \neq 0$. We may assume that $y_k = 1$. Let $s_0, \ldots, s_m$ be the coordinates of $\mathbb{P}^n$. Then $V = D_{\mathbb{A}^n}(s_k)$ is a neighborhood of $y$ in $Y$. Write $f = [f_0 : f_1 : \cdots : f_m]$ in a neighborhood $U$ of $x$ in $X$. Dividing all by $f_k$, we may assume that $f_k \equiv 1$ in that neighborhood $U$. We need to show that all $f_i$ are regular at $x$. Take $g = \frac{s_i}{s_k} \in k[V]$. Therefore, $g \circ f|_{f^{-1}(V)}$ is regular by assumption.

In $f^{-1}(V) \cap U$, $g \circ f = f_i$ so is regular at $x$.

**Corollary 9.13.** The composition of two regular maps is regular.

We will construct a fully faithful functor

$$\theta : \text{QAffVar}(k) \to \text{QProjSets}(k).$$
We have \( \mathbb{A}^n \) with coordinates \( T_1, \ldots, T_n \) and \( \mathbb{P}^n \) with coordinates \( S_0, \ldots, S_n \). We define \( U = D_{\mathbb{P}^n}(S_0) \subset \mathbb{P}^n \) open.

We construct a map \( \mathbb{A}^n \overset{g}{\to} U \) by \( (T_1, \ldots, T_n) \mapsto [1 : T_1 : \cdots : T_n] \). We get an inverse map by \( [S_0 : S_1 : \cdots : S_n] \mapsto \left( \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0} \right) \). I claim that \( g \) is a homeomorphism.

Take \( F \in k[T_1, \ldots, T_n] \) and \( G = F \left( \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0} \right) : S_0^k \) for sufficiently large \( k \) so that \( G \) is a polynomial. Then \( g(Z_{\mathbb{A}^n}(F)) = Z_U(G) \).

Conversely, given \( G \) homogeneous in \( k[S_0, S_1, \ldots, S_n] \), define \( F = G(1, T_1, \ldots, T_n) \in k[T_1, \ldots, T_n] \). Then \( g^{-1}(Z_U(G)) = Z_{\mathbb{A}^n}(F) \). If \( V \subset \mathbb{A}^n \) is an open set and \( h: (V) \to k \) a function, then \( h \) is regular if and only if \( h \circ (g|_V): V \to k \) is regular. Locally, we may write \( h = \frac{K}{L} \) on \( g(V) \), where \( K \) and \( L \) are homogeneous of the same degree. Then

\[
\begin{align*}
 h \circ g &= \frac{K(1, T_1, \ldots, T_n)}{L(1, T_1, \ldots, T_n)} \\
\text{is regular. If } h \circ g &= \frac{F}{\overline{F}} \text{ locally, where } F, H \in k[T_1, \ldots, T_n], \text{ then} \\
 h &= \frac{F \left( \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0} \right) \cdot s_0^k}{H \left( \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0} \right) \cdot s_0^k} \\
\text{is regular.}
\end{align*}
\]

Then \( \theta \) is the assignment given by \( X \subset \mathbb{A}^n \mapsto g(X) \subset U \subset \mathbb{P}^n \). If \( f: X \to Y \), with \( X \subset \mathbb{A}^n \) and \( Y \subset \mathbb{A}^m \), we have \( g_n(X) \subset U_n \subset \mathbb{P}^n \), and \( g_m(Y) \subset U_m \subset \mathbb{P}^m \). We have a diagram

\[
\begin{array}{ccc}
\theta(f): g_n(X) & \xrightarrow{g^{-1}} & X \\
\downarrow & & \downarrow f \\
\mathbb{A}^n & \xrightarrow{g_m} & Y \\
\downarrow & & \downarrow g_m(Y) \\
\mathbb{A}^n & \xrightarrow{g_m} & \mathbb{A}^m \\
\downarrow & & \downarrow g_m(Y) \\
\mathbb{A}^n & \xrightarrow{g_m} & \mathbb{A}^m \\
\end{array}
\]

We claim \( \theta \) is full and faithful. We have

\[
f: X \xrightarrow{g_n} g_n(X) \xrightarrow{\theta(f)} g_m(Y) \xrightarrow{g^{-1}_m} Y
\]

So \( \text{QAffVar}(k) \) is a full subcategory of \( \text{QProjSets}(k) \).

**Definition 9.14.** A quasi-projective variety is a quasi-projective set. A quasi-affine variety is a quasi-projective variety isomorphic to a quasi-affine set. A projective variety is isomorphic to a closed set in \( \mathbb{P}^n \).

We have the inclusions

\[
\text{AffVar} \subset \text{QAffVar} \subset \text{QProjVar}
\]

Also, if \( \text{ProjVar} \) lies in \( \text{QProjVar} \) and \( \text{ProjVar} \cap \text{AffVar} \) consists of finite sets.

Some properties:

1. Every quasi-projective variety admits an open affine finite cover.

**Proof.** We have \( \mathbb{P}^n = \bigcup_{i=0}^n D_{\mathbb{P}^n}(S_i) \). But \( D_{\mathbb{P}^n}(S_i) \simeq \mathbb{A}^n \), so is affine. In general, if \( X \subset \mathbb{P}^n \) is quasi-projective, then

\[
X = \bigcup_{i=0}^n (D_{\mathbb{P}^n}(S_i) \cap X)
\]

and each member of this intersection is a quasi-affine variety, and every quasi-affine variety admits an open affine finite cover.

\[\square\]
2. A map \( f = (f_1, \ldots, f_n): X \to \mathbb{A}^n \) is regular if and only if all \( f_i \) are regular functions.

3. \( k[D_{\mathbb{P}^n}(S_i)] = k[\frac{s_0}{s_i}, \ldots, \frac{s_{i-1}}{s_i}, \frac{s_{i+1}}{s_i}, \ldots, \frac{s_n}{s_i}] \). To generalize, let \( L \) be a nonzero linear form on \( s_i \), so \( L = \sum a_is_i \) for \( a_i \in k \). Then

\[
k[D_{\mathbb{P}^n}(L)] = k[\frac{s_0}{L}, \frac{s_1}{L}, \ldots, \frac{s_n}{L}] \]

with \( \sum a_i \frac{s_i}{L} = 1 \).

Let \( F \in A_{n+1} \) be homogeneous, and nonconstant, so \( \deg(F) > 0 \). Consider

\[
D_{\mathbb{P}^n}(F) = \{ x \in \mathbb{P}^n : F(x) \neq 0 \} = \mathbb{P}^n \setminus Z(F).
\]

If \( F \) is linear, then \( D_{\mathbb{P}^n}(F) \cong \mathbb{A}^n \), which is affine.

**Proposition 9.15.** The variety \( D_{\mathbb{P}^n}(F) \) is always affine.

**Proof.** Consider the Veronese map \( v_{n,m}: \mathbb{P}^n \to \mathbb{P}^{d_{n,m}} \). Then \( Z = \text{Im}(v_{n,m}) \subset \mathbb{P}^{d_{n,m}} \) is closed, and \( \mathbb{P}^n \cong Z \).

Recall if \( \alpha = (\alpha_0, \ldots, \alpha_n) \), then \( |\alpha| = \sum \alpha_i \), and \( S^\alpha = S_0^{\alpha_0} \cdots S_n^{\alpha_n} \). Let

\[
F = \sum_{|\alpha|=m} a_{\alpha} S^\alpha
\]

for \( a_{\alpha} \in k \). Recall \( v_{n,m}([S_0 : \cdots : S_n]) = ([S^\alpha]) \). If \( T_\alpha \) are the coordinates of \( \mathbb{P}^{d_{n,m}} \), then this map is \( T_\alpha = S^\alpha \). Let \( L = \sum_{|\alpha|=m} a_{\alpha} T_\alpha \), which is linear on \( \mathbb{P}^{d_{n,m}} \). Then

\[
v_{n,m}(D_{\mathbb{P}^n}(F)) = D_Z(L)
\]

and moreover,

\[
D_{\mathbb{P}^n}(F) \cong D_Z(L) = Z \cap D_{\mathbb{P}^{d_{n,m}}}(L).
\]

Since \( Z \) is closed, this intersection is closed in \( D_{\mathbb{P}^{d_{n,m}}}(L) \cong \mathbb{A}^{d_{n,m}} \). Hence it is affine. \( \square \)

**Example 9.16.** Take \( X = Z(s_1 s_2 - s_0^2) \subset \mathbb{P}^2 \). We get a map

\[
X \cong \mathbb{P}^1: [s_0 : s_1 : s_2] \mapsto [s_0 : s_1] = [s_2 : s_0].
\]

Consider \( D_X(s_0) \). Let \( T_1 = \frac{s_0}{s_0} \) and \( T_2 = \frac{s_2}{s_0} \), so \( T_1 T_2 = 1 \). Then \( D_X(s_0) = Z_{\mathbb{A}^2}(T_1 T_2 - 1) \). Also,

\[
Z_X(s_0) = X \setminus D_X(s_0) = \{ [0 : 1 : 0], [0 : 0 : 1] \}.
\]

Call these points \( x_1 \) and \( x_2 \).

Let the coordinates of \( \mathbb{P}^1 \) be \( w_0, w_1 \). Then \( D_{\mathbb{P}^1}(w_0) = \mathbb{A}^1 \). The coordinate on the affine line is \( u = \frac{w_1}{w_0} \). We have the map

\[
[1 : T_0 : T_1] \mapsto [1 : T_0] \sim T_0 \in \mathbb{A}^1
\]

so the map \( X \to \mathbb{P}^1 \) restricts to the map \( D_X(s_0) \to \mathbb{A}^1 \) \( D_{\mathbb{P}^1}(w_0) \). This gives the projection map from the hyperbola to the affine line. But \( \mathbb{P}^1 \setminus D_{\mathbb{P}^1}(w_0) = \{ [0 : 1] \} \), call this point \( \infty \). So if \( f: X \to \mathbb{P}^1 \) is the map, we have \( f(x_1) = \infty \) and \( f(x_2) = 0 \in \mathbb{A}^1 \).

**10 Rational Maps and Functions**

Let \( X \) be an irreducible, quasi-projective variety. For a pair \( (U, f) \) with \( U \subset X \) nonempty and open, \( f \in k[U] \), we say \( (U, f) \sim (U', f') \) if \( f \) and \( f' \) agree on \( U \cap U' \). Denote by \( F(X) \) the field of rational functions on \( X \), which are equivalence classes of \( [U, f] \).

As before, we get a correspondence between the opposite category of irreducible quasi-projective varieties with morphisms dominant rational maps and the category of finitely generated extensions of \( k \), by the assignment \( X \mapsto F(X) \) and \( f \mapsto f^* \).

The category of irreducible quasi-affine varieties with dominant rational maps is an equivalence with both the above categories, so the functor in the previous paragraph is actually an equivalence.
11 Local Ring of a Subvariety

Let $X$ be a quasi-projective variety, and $Y \subset X$ closed and irreducible subvariety. Consider $(U, f)$ to be a neighborhood of $Y$ (i.e., $U \cap Y \neq \emptyset$), and $f \in k[U]$. Denote by $\mathcal{O}_{X,Y}$ the local ring of equivalence classes. If $U$ is a neighborhood of $Y$ in $X$, then $\mathcal{O}_{X,Y} \simeq \mathcal{O}_{U,Y \cap U}$, so we may reduce to the affine or quasi-affine case for the study of local objects.

12 Product of Quasi-projective Varieties

The main point is that $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}$, although $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$. Let $S_i \; 0 \leq i \leq n$ be the coordinates for $\mathbb{P}^n$ and $T_j \; 0 \leq j \leq m$ the coordinates for $\mathbb{P}^m$. Consider also $\mathbb{P}^k$ where $k = (n+1)(m+1) - 1$ with coordinates $w_{ij}$.

Define a map
$$
\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^k : ([S_i],[T_j]) \mapsto [w_{ij}]
$$
where $w_{ij} = S_iT_j$. This is well-defined because if we scale all $S_i$, we scale all $w_{ij}$, likewise if we scale the $T_j$.

Write $V = Z(w_{ij}w_{kl} - w_{il}w_{kj}) \subset \mathbb{P}^k$, which is closed. These equations can be rewritten as $S_iT_jS_kT_l - S_iT_jS_kT_l = 0$, so $\text{Im}(\varphi) \subset V$.

**Lemma 12.1.** The map $\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^k$ is a bijection.

**Proof.** We construct $\psi : V \to \mathbb{P}^n \times \mathbb{P}^m$. Take $[w_{ij}] \in V$. Send $[w_{ij}] \mapsto ([w_{il} : \cdots : w_{ni}], [w_{k0} : w_{k1} : \cdots : w_{km}])$. This is independent of $k$ and $l$. If $w_{kl} \neq 0$ and $w_{kl} \neq 0$ in both points above. Then $\psi = \varphi^{-1}$. ☐

Define the product $\mathbb{A}^n \times \mathbb{A}^m = V \subset \mathbb{P}^{(n+1)(m+1)-1}$. For example, $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. The map $\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^k$ is called the Segre embedding. We have canonical projections to $\mathbb{P}^n$ and $\mathbb{P}^m$, which are regular.

Let $F_r(S,T)$ be homogeneous in $S$ and in $T$, of possibly different degrees. Then $Z(F_r(S,T)) \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^k$ is closed. Since $w_{ij} = S_iT_j$, if $F_r$ is homogeneous in $S$ and $T$ of the same degree, then $F_r(S,T) = G_r(W)$ by replacing the $S_iT_j$ with $w_{ij}$. Suppose that $\deg_S(F_r) = a$ and $\deg_T F_r = b$ with $a < b$. Replace $F_r(S,T)$ with $S_i^{b-a}F_r(S,T)$. These polynomials defined the same closed set since at least one of the $S_i$ is nonzero.

Consider $\mathbb{P}^n \times \mathbb{P}^m$. Take $X = Z(S) \subset \mathbb{P}^n$ and $Y = Z(T) \subset \mathbb{P}^m$ closed. Then $X \times Y = Z(S \cup T) \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed.

If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are quasi-projective, say $X = Z_1 \setminus Z_1'$ and $Y = Z_2 \setminus Z_2'$, then
$$X \times Y = (Z_1 \setminus Z_2) \setminus (Z_1 \times Z_2') \subset \mathbb{P}^n \times \mathbb{P}^m$$
is quasi-projective. Similarly, if $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are open, then $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is open by taking $Z_1 = \mathbb{P}^n$ and $Z_2 = \mathbb{P}^m$.

In fact, $X \times Y$ is the categorical product in $\text{QProjVar}(k)$. If $X, Y$ are quasi-affine, then they are quasi-projective so in either category, $X \times Y$ is well-defined.

Some properties

1. If $X$ and $Y$ are affine, then so is $X \times Y$.
2. If $X$ and $Y$ are projective, then so is $X \times Y$.
3. If $X' \subset X$ and $Y' \subset Y$ is closed (resp. open), then $X' \times Y' \subset X \times Y$ is closed (resp. open).
4. If $f : X \to X'$ and $g : Y \to Y'$ are regular, then so is $f \times g : X \times Y \to X' \times Y'$. The components of $f \times g$ are $X \times Y \xrightarrow{\Delta} X \xrightarrow{f} X'$ and $X \times Y \xrightarrow{\Delta} Y \xrightarrow{g} Y'$. Now use the universal property.
5. If $X$ and $Y$ are irreducible, then so is $X \times Y$. (Same proof.)
6. The natural map $\varphi_{X,Y} : k[X] \otimes_k k[Y] \to k[X \times Y]$ is an isomorphism.

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Proof. We know that when $X$ and $Y$ are affine. The proof that $\varphi_{X,Y}$ is injective is the same. Choose basis $\{g_i\}$ a basis for $k[Y]$. Every element in $k[X] \otimes_k k[Y]$ can be uniquely written $\sum f_i \otimes g_i$, for $f_i \in k[X]$. Claim: Suppose $X = \bigcup U_\alpha$ is an open cover. If $\varphi_{U_\alpha,Y}$ is an isomorphism for all $\alpha$, so is $\varphi_{X,Y}$. To see this, Take $h \in k[X \times Y]$. Now $h|_{U_\alpha \times Y} \in \text{Im}(\varphi_{U_\alpha \times Y})$ implies there exists a unique $f_{\alpha,i} \in k[U_\alpha]$ such that

$$h|_{U_\alpha \times Y}(x,y) = \sum_i f_{\alpha,i}(x)g_i(y) = \varphi_{U_\alpha,Y}(\sum f_{\alpha,i} \otimes g_i)$$


for all $\alpha$. But

$$h|_{(U_\alpha \cap U_\beta) \times Y}(x,y) = \sum_i f_{\alpha,i}|_{U_\alpha \cap U_\beta}(x)g_i(y) = \sum_i f_{\beta,i}|_{U_\alpha \cap U_\beta}(x)g_i(y) = \sum_i f_{\alpha,i}|_{U_\alpha \cap U_\beta} \otimes g_i = \sum_i f_{\beta,i}|_{U_\alpha \cap U_\beta} \otimes g_i$$

By uniqueness, $f_{\alpha,i}|_{U_\alpha \cap U_\beta} = f_{\beta,i}|_{U_\alpha \cap U_\beta}$ so there exists unique $f_i \in k[X]$ such that $f_i|_{U_\alpha} = f_{\alpha,i}$. Then

$$\varphi_{X,Y}(\sum f_i \otimes g_i)|_{U_\alpha \times Y} = \varphi_{U_\alpha \times Y}(\sum f_i|_{U_\alpha} \otimes g_i) = \varphi_{U_\alpha \times Y}(\sum f_{\alpha,i} \otimes g_i) = h|_{U_\alpha \times Y}$$

Thus $\varphi_{X,Y}(\sum f_i \otimes g_i) = h$. \hfill \square

If $X$ is anything, and $Y$ is affine, then $\varphi_{X,Y}$ is an isomorphism. Let $X = \bigcup U_\alpha$ be an open affine cover. Then $\varphi_{U_\alpha,Y}$ is an isomorphism. By the claim, $\varphi_{X,Y}$ is an isomorphism.

If $X$ and $Y$ are arbitrary, take $Y = \bigcup W_\beta$ be an open affine cover. Then $\varphi_{X,W_\beta}$ is an isomorphism. By the claim, $\varphi_{X,Y}$ is an isomorphism.

7. Suppose $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. Then a function on $X \times Y$ is regular at $(x,y)$ is equal to $\frac{F(S,T)}{G(S,T)}$ is a neighborhood $U$ of $(x,y)$, where $F$ and $G$ are homogeneous in $S$ and $T$ of the same degree, with $G \neq 0$ everywhere in $U$.

8. The diagonal $\Delta_X \subset X \times X$ is closed. Note if $X \subset \mathbb{P}^n$, then $\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X)$. Now $\Delta_{\mathbb{P}^n} = Z(s_is'_j - s_j s'_i) \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. Here $S$ and $S'$ are the coordinates on the factors of $\mathbb{P}^n$ in the product.

9. Consider $\mathbb{P}^n$ and $\mathbb{A}^m$ with coordinates $s_i$ and $t_j$. Every closed set in $\mathbb{P}^n \times \mathbb{A}^m$ is given by the equations $F(s,t) = 0$ where $F$ is a polynomial, homogeneous in $S$. Embed $\mathbb{A}^m \subset \mathbb{P}^m$. A closed set in $\mathbb{P}^n \times \mathbb{A}^m = (\mathbb{P}^n \times \mathbb{A}^m) \cap Z$ where $Z$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$ which is given by $G(S,T) = 0$, homogeneous in $S$ and $T$.

Do the substitution $T_0 = 1$ and $T_i = t_i$ for $i > 0$.

10. If $f: X \to Y$ is regular $\Gamma(f) = \{(x,f(x)) \in X \times Y\}$ which is closed. The projection $p: \Gamma(f) \to X$ is an isomorphism.

**Proposition 12.2.** Let $X$ be a quasi-projective variety, and $Y,Z \subset X$ are subvarieties. Then if both $Y$ and $Z$ are affine, so is $Y \cap Z$.

**Proof.** Note

$$Y \cap Z \simeq (Y \times Z) \cap \Delta_X \subset X \times X$$

Observe $(y,z) \in \Delta_X \iff y = z \in Y \cap Z$. But $(Y \times Z) \cap \Delta_X$ is closed in $Y \times Z$, which is affine as the product of affine varieties, hence it is affine since any closed subset of an affine variety is affine. \hfill \square

13 Proper Maps

**Lemma 13.1.** The projection $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ takes closed sets to closed sets. (This fails if we replace the projective space by affine space, think the projection of the hyperbola.)
Proof. Let $S$ and $t$ be the coordinates of $\mathbb{P}^n$ and $\mathbb{A}^m$, respectively. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ so $Z = Z(G_i(S,t))$ with $G_i$ homogeneous in $S$. The image $p(Z) \subset \mathbb{A}^m$, we need to show it is closed.

Observe

$$p(Z) = \{ x \in \mathbb{A}^m : \text{ the system } G_i(S,x) = 0 \text{ has a solution } [S_0 : S_1 : \cdots : S_n] \in \mathbb{P}^n \}.$$ 

But

$$p(Z) = \{ x \in \mathbb{A}^m : \langle G_i(S,x) \rangle \not\supset \langle S_0, \ldots, S_n \rangle^k \text{ for all } k \}$$

Let $I_x = \langle G_i(S,x) \rangle \subset k[S_0, \ldots, S_n]$. For each $k > 0$, let

$$Y_k = \{ x \in \mathbb{A}^m : I_x \not\supset \langle S_0, \ldots, S_n \rangle^k \} \subset \mathbb{A}^m$$

Thus $p(Z) = \bigcap_k Y_k$. Is it closed? It suffices to show that every $Y_k$ is closed in $\mathbb{A}^m$. □

Let $G_i = (S,T)$, and $\deg_S(G_i) = d_i$, $x \in \mathbb{A}^m$, and $I_x = \langle G_i(S,T) \rangle \subset k[S_0, \ldots, S_n]$. Is $Y_k = \{ x \in \mathbb{A}^m : I_x \not\supset \langle S_0, \ldots, S_n \rangle \} \subset \mathbb{A}^m$? Note

$$I_x \supset \langle S_0, \ldots, S_n \rangle^k \iff I_x \text{ contains all monomials in } S \text{ of degree } k.$$ 

This holds iff every monomical of degree $k$ is equal to $\sum_i G_i(S,x) F_i(Z)$.

So $F_i(S)$ is a linear combination of monomials of degree $k - d_i$. The polynomials $G_i(S,x)$, where $N$ is a monomial of degree $k - d_i$, span the vector space $V_k$ of all monomials of degree $k$.

Let $A_x$ be the matrix of coefficients of all $G_i(S,x)N$, so $rk(A_x) = \dim(V_k)$. So $x \in Y_k$ iff $rk(A_x) < \dim(V_k)$ iff all $\dim(V_k) \times \dim(V_k)$ minors in $A_x$ are zero. But the minors are polynomials in $x$, hence in $k[t_1, \ldots, t_m]$. Therefore, $Y_k \subset \mathbb{A}^m$ is closed.

So the projection $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{P}^n$ is a closed map.

Corollary 13.2. Let $X$ be a projective variety and $Y$ a quasi-projective variety. Then the projection $p: X \times Y \to Y$ takes closed sets to closed sets.

Proof. Let $X = \mathbb{P}^n$ and $Y$ be affine. So we may assume $Y \subset \mathbb{A}^m$ is closed. So $Z \subset \mathbb{P}^n \times Y \subset \mathbb{P}^n \times \mathbb{A}^m$ is closed. By the lemma, $p(Z)$ is closed in $\mathbb{A}^m$, but $p(Z) \subset Y$, so $p(Z)$ is closed in $Y$.

Now let $X = \mathbb{P}^n$ and $Y$ arbitrary. Let $Y = \bigcup U_\alpha$, where $U_\alpha \subset Y$ are open and affine. Let $Z \subset X \times Y$ be closed. So $Z \cap (X \times U_\alpha)$ is closed in $X \times U_\alpha$. By the first case, $p(Z \cap (X \times U_\alpha))$ is closed in $U_\alpha$. This set is $p(Z) \cap U_\alpha$, so $p(Z)$ is locally closed, hence closed.

Now suppose $X$ is projective and $Y$ is arbitrary. So suppose $X \subset \mathbb{P}^n$ closed. Then $Z \subset X \times Y \subset \mathbb{P}^n \times Y$ is closed. So $p(Z)$ is closed in $Y$. □

A regular map $f: X \to Y$ is called proper if for every quasi-projective variety $T$, the map $f \times I_T: X \times T \to Y \times T$ takes closed sets to closed sets.

Example 13.3. The map $\mathbb{A}^1 \to \{ p \}$ is not proper. Consider $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{P}^2 \to \{ p \} \times \mathbb{A}^1 = \mathbb{A}^1$. If $X \subset \mathbb{A}^1 \times \mathbb{A}^1$ is the hyperbola, the image $p(X) = \mathbb{A}^1 \setminus \{ 0 \}$ is not closed.

Proposition 13.4. The map $X \to \{ p \}$ is proper if and only if $X$ is projective.

Proof. For the reverse direction, we have $X \times T \to \{ p \} \times T = T$, and by the corollary $p_2$ takes closed sets to closed sets.

Forward direction, let $X \subset \mathbb{P}^n$, and let $f: X \to \mathbb{P}^n$ be an embedding. Then $\Gamma f \subset X \times \mathbb{P}^n$ is closed. But $\Gamma f = \{ (x,x) : x \in X \}$. Take the projection $X \times \mathbb{P}^n \to \mathbb{P}^n = \{ p \} \times \mathbb{P}^n$. Then $\pi$ takes closed sets to closed sets. But $\pi(X) = X$ is closed in $\mathbb{P}^n$. Thus $X$ is projective. □

Some properties,

1. Closed embeddings are proper.
2. A composition of proper maps is proper.
3. If \( f: X \to Y \) is proper, then so is \( f \times 1_Z: X \times Z \to Y \times Z \).

**Proposition 13.5.** Let \( f: X \to Y \) be a regular map. If \( X \) is projective, then \( f \) is proper.

**Proof.** We can factor the map as

\[
\begin{array}{c}
X \longrightarrow X \times Y \longrightarrow Y
\end{array}
\]

where the first map is the closed embedding \( x \mapsto (x, f(x)) \), and the second is the projection. The first is proper. The second map is proper by and Property 3. So by Property 2, the map is proper as the composition of proper maps.

**Corollary 13.6.**

1. The image of a regular map \( X \to Y \) with \( X \) projective is closed in \( Y \).

2. A subvariety \( X \subset \mathbb{P}^n \) is projective iff \( X \) is closed in \( \mathbb{P}^n \). We could define projective varieties as closed subsets of \( \mathbb{P}^n \).

**Proof.** For the first statement, if \( X \) is projective, then \( f: X \to Y \) is proper, and since \( X \subset X \) is closed, \( f(X) \) is closed in \( Y \).

For the second, if \( X \subset \mathbb{P}^n \) is closed, \( X \) is projective. If \( X \) is projective, consider the embedding \( f: X \to \mathbb{P}^n \). By 1, \( f(X) = X \) is closed in \( \mathbb{P}^n \).

Recall \( k[\mathbb{P}^n] = k \). This generalizes to

**Proposition 13.7.** If \( X \) is irreducible and projective, then \( k[X] = k \).

**Proof.** If \( f \in k[X] \), then \( f: X \to \mathbb{A}^1 \to \mathbb{P}^1 \). Then \( \text{Im}(f) \) is closed in \( \mathbb{P}^1 \), and \( \text{Im}(f) \not\ni \infty \). But closed sets in \( \mathbb{P}^1 \) are \( \mathbb{P}^1 \) and finite sets of points. So \( \text{Im}(f) \) is finite, and irreducible, hence is a one point set. So \( f \) is constant.

**Proposition 13.8.** If \( X \) is projective and quasi-affine, then \( X \) is finite.

**Proof.** Since \( X \) is a finite union of irreducible components, we may assume \( X \) is irreducible. Let \( f_i: X \subset \mathbb{A}^n \to \mathbb{A}^1 \) be the \( i \)th coordinate function. Since \( X \) is projective and irreducible, so \( f_i \) is constant. Thus \( X \) is a point.

### 14 Dimension

Let \( X \) be an irreducible quasi-projective variety. Define \( \dim(X) = \text{trdeg}(k(X)/k) \). If \( X \) is an arbitrary quasi-projective variety, then \( X = \bigcup X_i \), for \( X_i \) irreducible. Then \( \dim(X) = \max \dim(X_i) \). If \( X \approx Y \), then \( k(X)/X \simeq k(Y) \), so \( \dim(X) = \dim(Y) \). So if \( U \subset X \) is open in \( X \) irreducible, then \( \dim(U) = \dim(X) \).

**Example 14.1.**

1. \( k(\mathbb{A}^n) = k(t_1, \ldots, t_n) \), so \( \dim(\mathbb{A}^n) = n = \dim(\mathbb{P}^n) \) and \( \mathbb{A}^n \subset \mathbb{P}^n \) is open.

2. We have \( \dim(X \times Y) = \dim(X) + \dim(Y) \).

**Proof.** We may assume \( X \) and \( Y \) are irreducible and affine. We may find an intermediate field \( k(X)/E/k \) such that \( E/k \) is purely transcendental and \( k(X)/E \) is finite. Likewise we may find \( k(Y)/L/k \) such that \( L/k \) is purely transcendental, and \( k(Y)/L \) is finite. Then \( k[X \times Y] = k[X] \otimes_k k[Y] \).

So

\[
k(X \times Y) = \text{Quot}(k[X \times Y]) = \text{Quot}(k[X] \otimes_k k[Y]) = \text{Quot}(k(X) \otimes_k k(Y))
\]

We can write \( E = \text{Quot}(k[x_1, \ldots, x_n]) \) and \( L = \text{Quot}(k[y_1, \ldots, y_m]) \) so \( \dim(X) = n \) and \( \dim(Y) = m \). Then \( \text{Quot}(E) \otimes_k L = \text{Quot}(k[x_1, \ldots, x_n, y_1, \ldots, y_m]) = k[x_1, \ldots, x_n, y_1, \ldots, y_m] \). So we have the extension \( k(X \times Y) = \text{Quot}(k(X) \otimes_k k(X))/\text{Quot}(E \otimes L) \), which is finite.

So

\[
\dim(X \times Y) = \text{trdeg}(\text{Quot}(E \otimes L)) = n + m = \dim(X) + \dim(Y).
\]

\[\square\]
3. We have \( \dim(X) = 0 \) iff \( X \) is finite.

Proof. Suppose \( X \) is irreducible and affine. So if \( X \) is finite, \( X \) is a point, and \( k(X) = k \), so \( \text{trdeg}(k/k) = 0 \). If \( \dim(X) = 0 \), then \( k(X)/k \) is finite, but since \( k \) is algebraically closed, \( k(X) = k \). Then \( k \subseteq k[X] \subseteq k(X) \), so \( k[X] = k = k[\text{point}] \). So \( X \) is a point by the correspondence between affine varieties and coordinate rings. \( \square \)

4. Let \( F \in k[t_1, \ldots, t_n] \) be irreducible. Set \( X = Z(T) \subset \mathbb{A}^n \). Then \( \dim(X) = n - 1 \). Since \( F \) is not constant, assume \( t_1 \) is a variable in \( F \). Claim that \( t_2, \ldots, t_n \in k(X) = \text{Quot}(k[t_1, \ldots, t_n]/(F)) \) are algebraically independent over \( k \).

Suppose \( G(t_2, t_3, \ldots, t_n) = 0 \) on \( X \). If \( G \in I(X) \), then \( G^m \) is divisible by \( F \) by Nullstellensatz, but \( F \) depends on \( t_1 \), but \( G \) does not, a contradiction.

Then \( t_1 \) is algebraic over \( k(t_2, \ldots, t_n) \), so \( \dim(X) = \text{trdeg}(k(X)/k) = n - 1 \).

If \( R \) is a commutative ring, then the Krull dimension \( \dim(R) \) is the largest \( n \) such that there exists a chain of prime ideals \( P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \). It is a fact from commutative algebra that if \( R \) is a finitely generated domain over \( k \), then \( \dim(R) = \text{trdeg}(\text{Quot}(R)/k) \). So if \( X \) is affine, then \( \dim(X) = \dim(k[X]) \).

We say \( P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \) is maximal if we cannot properly insert other prime ideals in it.

It is a fact that if \( R \) is a localization of a finitely generated \( k \)-algebra, then all maximal chains of prime ideals in \( R \) have the same length. (= \( \dim(R) \)).

If \( X \) is affine, prime ideals in \( k[X] \) are in bijection with irreducible closed subvarieties of \( X \). If \( P \) is a prime ideal of \( k[X] \),

\[
\text{Spec}(k[X]/(P)) = \{ p' \in \text{Spec}(k[X]) : p' \supset p \}
\]

which can be interpreted as the irreducible closed subsets of \( Y \).

Also,

\[
\text{Spec}(k[X]_P) = \{ P'' \in \text{Spec}(k[X]) : P'' \supset P \} \leftrightarrow \{ \text{irreducible closed } \supset \text{ } Y \}
\]

So let \( P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m = P \supset P_{m+1} \supset \cdots \supset P_{m+k} \) be maximal, so \( m + k = \dim(k[X]) = \dim(X) \).

This gives a corresponding chain

\[
Y_0 \supset Y_1 \supset \cdots \supset Y_m = Y \supset Y_{m+1} \supset \cdots \supset Y_{m+k}
\]

So \( k = \dim(k[X]/(P)) \) and \( m = \dim(k[X]_P) \), hence

\[
\dim(k[X]) = \dim(k[X]/P) + \dim(k[X]_P)
\]

and

\[
\dim(Y) = \dim(k[X]/P) = k.
\]

Lastly, \( \dim(Y_i) = \dim(X) - i \), so we get subvarieties of every intermediate dimension.

If \( Y \subset X \) is closed, then \( \dim(Y) \leq \dim(X) \). If \( Y \subset X \) is closed, and \( X \) irreducible, then \( \dim(Y) < \dim(X) \).

To see this, we have \( \dim(X) = \dim(Y) + \dim(k[X]_P) \) where \( P \) is the prime corresponding to \( X \), so \( P \neq 0 \) since \( X \neq Y \).

Suppose \( Y \subset X \) is closed, and irreducible. Then \( \dim(X) = \dim(Y) + \dim(O_{X,Y}) \).

15 Smooth Varieties

Take \( F \in k[T_1, T_2, \ldots, T_n] \) nonzero. Set \( X = Z(F) \subset \mathbb{A}^n \). Take \( x \in X \).

Define the tangent space of \( X \) at \( x \) to be the space cut out by the linear equation

\[
\sum_{i=1}^n \frac{\partial F}{\partial T_i}(x)(T_i - x_i) = 0
\]

We view this as an (affine) subspace of \( \mathbb{A}^n \).
Example 15.1. Let \( F = T_1^3 - T_2^2 \). Then if \( x = 0 \),
\[
\frac{\partial F}{\partial T_1}(x)T_1 + \frac{\partial F}{\partial T_2}(x)T_2 = 0
\]
but both partial derivatives are 0, so the tangent space at \( x \) is \( \mathbb{A}^2 \).

Consider \( X \subset \mathbb{A}^n \) to be closed. For all \( F \in I(X) \subset A_n \) and \( x \in X \), then the equations
\[
\sum_{i=1}^{n} \frac{\partial F}{\partial T_i}(x) \cdot (T_i - x_i) = 0
\]
define the tangent space \( T_{X,x} \subset \mathbb{A}^n \). It suffices to consider only \( F \)'s from a generating set of \( I(X) \).

Take \( G \in A_n \). Define the differential of \( G \) at \( x \),
\[
D_x G = \sum_{i=1}^{n} \frac{\partial G}{\partial T_i}(T_i - x_i)
\]
to be a linear form on \( T_{X,x} \). (If \( G \in I(X) \), then \( D_x G = 0 \).

Consider the map
\[
k[T_1, \ldots, T_n] \to (T_{X,x})^* : G \mapsto D_x G
\]
where the latter space is the dual space. This is a \( k \)-linear map. This map vanishes on \( I(X) \). We have another map
\[
d_x : k[X] \to T^*_{X,x} \text{ : } f \mapsto d_x f = D_x F|_{T_{X,x}}
\]
for \( F \in A_n \) with \( F|_X = f \). This map is called the differential at \( x \).

We have \( d_x(f + g) = d_x f + d_x g \) and \( d_x(fg) = f(x)d_x g + g(x)d_x f \). Let
\[
I_x = \{ f \in k[X] : f(x) = 0 \} \subset k[X]
\]
be the maximal ideal. Then \( d_x|_{I_x} : I_x \to T^*_{X,x} \). Claim this map is still surjective. Every linear form on \( T_{X,x} \)
extends to a linear form \( \sum_{i=1}^{n} A_i(T_i - x_i) = f \in k[X] \). But \( f(x) = 0 \), so \( f \in I_x \) and \( d_x f = f \in T^*_{X,x} \).

Furthermore, \( \ker(d_x|_{I_x}) = I_x^2 \). If \( f, g \in I_x \), then \( d_x (fg) = f(x)d_x (g) + g(x)d_x (f) = 0 + 0 = 0 \). So \( I_x^2 \subset \ker(d_x|_{I_x}) \). For simplicity, assume \( x = (0, 0, \ldots, 0) \). If \( F \in k[T_1, \ldots, T_n] \), then \( D_x F|_{T_{X,x}} = 0 \) (\( f = F|_X \in \ker(d_x|_{I_x}) \)). Then
\[
D_x F = \sum_{i=1}^{m} a_i D_x F_i, \quad F_i \in I(X).
\]
Set \( F' = F - \sum a_i F_i \), then \( D_x F' = 0 \) on \( \mathbb{A}^n \). So \( F' \) has no constant and linear terms. Now \( I_x^2 = \langle T_1, T_2 | x \rangle \) since \( I_X = \langle T_1 | x \rangle \). Then \( f = F|_X = F'|_X \in I_x^2 \), so the other inclusion holds.

So \( I_x/I_x^2 \cong T^*_{X,x} \). Note \( k[X]/I_x = k \), and
\[
O_{X,x} = k[X]_{I_x} = \{ \frac{f}{g} : g(x) \neq 0 \}.
\]
But \( m_x = \{ \frac{f}{g} : f(x) = 0, g(x) \neq 0 \} \) is contained in \( O_{X,x} \). We get a map \( i : k[X] \to k[X]_{I_x} = O_{X,x} \) taking \( I_x^k \to m_x^k \) and \( I_x/I_x^k \to m_x/m_x^2 \). Claim that this is an isomorphism.

To see surjectivity, fix \( \frac{f}{g} \in m_x \), so \( f(x) = 0 \) but \( g(x) \neq 0 \). Let \( a = g(x) \neq 0 \), and consider \( \frac{f}{a} \in I_x \). Then \( i(f/a) = f/a \), and
\[
\frac{f}{a} - \frac{f}{g} = \frac{f(g - a)}{ag} \in m_x^2
\]
since \( ag(x) \neq 0 \), and \( f \in I_x \), and \( g - a \in I_x \). So \( i(f/a) \equiv f/g \) (mod \( m_x^2 \)).

For injectivity, take \( f \in I_x \), and suppose \( i(f) = f/1 = 0 \) in \( m_x/m_x^2 \). Hence there exists \( g \in k[X] \) such that \( g(x) \neq 0 \) and \( fg \in I_x^2 \). We need \( f \in I_x^2 \). Let \( a = g(x) \neq 0 \). Then
\[
af = \sum_{f \in I_x^2} + (a - g) f \in I_x^2
\]
so $a \neq 0$ implies $f \in I_x^2$. So $T_{X,x} \simeq m_x/m_x^2$. So $T_{X,x} \simeq (m_x/m_x^2)^*$. Note that $m_x/m_x^2$ is a module over $O_{X,x}/m_x = k$.

Now let $X$ be a quasi-projective variety. Pick $x \in X$. Then $O_{X,x} \supseteq m_x$ and $O_{X,x}/m_x = k$. Then $m_x/m_x^2$ is a finite dimensional vector space over $k$. So define the tangent now as $T_{X,x} = (m_x/m_x^2)^*$. If $U \subset X$ is open containing $x$, then $T_{U,x} = T_{X,x}$, so the definition is local.

If $f : X \to Y$ is a regular map, $x \in X$ and $y = f(x) \in Y$, then $f^* : O_{Y,y} \to O_{X,x}$ taking $m_y^k \to m_x^k$, so $m_y/m_y^2 \to m_x/m_x^2$. We get a linear map $df : T_{X,x} \to T_{Y,y}$ called the differential of $f$. If $f$ is a closed embedding, then $f^* : O_{Y,y} \to O_{X,x}$ is surjective, so $df$ is injective. Also, $d(gf) = df \circ df$.

Example 15.2. Let $X = Z(F) \subset \mathbb{A}^n$, for $F$ irreducible, then $\dim(X) = n - 1$. By Nullstellensatz, if $\partial F/\partial T_i \mid X = 0$, then $F \mid (\partial F/\partial T_i)^k$ implies $F \mid \partial F/\partial T_i$ so $\partial F/\partial T_i = 0$ since it’s of lesser degree. Claim that $\partial F/\partial T_i \mid X \neq 0$ for some $i$.

Suppose otherwise, that $\partial F/\partial T_i = 0$ for all $i$. If $\text{char}(k) > 0$, then $\partial F/\partial T_i = pT_i^{p-1} = 0$.

If $g = aT_1^{r_1}T_2^{r_2} \cdots T_n^{r_n}$, then $\partial F/\partial T_i = 0$ for all $i$ iff all $\alpha_i$ are divisible by $p$. So $F = G^p$, a contradiction since $F$ is irreducible.

Then $T_{X,x}$ is given by $\sum_{i=1}^n \partial F_i/T_i(x)(T_i - x_i) = 0$. Then $\partial F/\partial T_i \neq 0$ for some $i$. So this equation is not trivial for all $\{x : \partial F_i/T_i(x) \neq 0\} = D_X(\partial F/\partial T_i) \neq \emptyset$.

So $\dim(T_{X,x}) = n - 1 = \dim(X)$ for all $x$ in a nonempty open subset of $X$. For other $x$, then $T_{X,x} = \mathbb{A}^n$, so $\dim(T_{X,x}) = n > \dim(X)$.

We have some facts from commutative algebra: Let $O$ be a Noetherian local ring, $m$ a maximal ideal, and $k = O/m$. So $m$ is a finitely generated ideal, so $m/m^2$ is a $k$-vector space of finite dimension. Then $\dim_{Kull}(O) \leq \dim(m/m^2)$. We say $O$ is regular if $\dim(O) = \dim(m/m^2)$.

Elements $x_1, \ldots, x_d \in m$ for $d = \dim O$, then $\bar{x}_1, \ldots, \bar{x}_n \in m/m^2$ is a $k$-basis iff $x_1, \ldots, x_d$ generated $m$, and $(x_1, \ldots, x_n)$ are called local parameters.

Let $x \in X$, for $X$ quasi-projective. Then $\dim(O_{X,x}) = \max \dim(X_i) = \dim_x X$, where we take the maximum over all $X_i$ irreducible components containing $x$. If $X$ is irreducible, then $\dim_x X = \dim X).

We have $O_{X,x}$ is a Noetherian local ring, and $\dim_x X = \dim(O_{X,x}) \leq \dim(m_x/m_x^2) = \dim(T_{X,x})$.

We say is $x$ is a regular, aka nonsingular, point if $\dim_x X = \dim(T_{X,x})$ if $O_{X,x}$ is regular. If $x \in U \subset X$ is open, then $U_{U,x} = O_{x,x}$, and $T_{U,x} = T_{X,x}$ so $x$ is nonsingular on $X$ if $x$ is nonsingular on $U$. We say $X$ is smooth if all points are nonsingular. Note if $U \subset X$ is open and $X$ is smooth, then $U$ is smooth.

Proposition 15.3. Let $X$ be an irreducible and quasi-projective variety. Then there is a nonempty open subset $U \subset X$ such that every point of $U$ is nonsingular (on $U$ and $X$).

Proof. There exists an irreducible $F \in k[T_1, \ldots, T_n]$ such that $X \approx Z(F) \subset \mathbb{A}^n$. We have $U \subset X$ and $W \subset Z(F)$ nonempty opens which are isomorphic. Take $i$ such that $\partial F/\partial T_i \neq 0$, so from a previous example, for all $x \in D(\partial F/\partial T_i)$, then $\dim(T_{Z(F),x}) = n-1 = \dim(Z(F)) = \dim_x(Z(F))$. So $x$ is nonsingular on $Z(F)$.

All points in $W \cap D(\partial F/\partial T_i)$ are nonsingular, so there is $U' \subset U$ is isomorphic to $W \cap D(\partial F/\partial T_i)$. \hfill \Box

Suppose $x \in X \subset \mathbb{A}^n$ is closed, and $X = Z(S)$ for $S = \{F_1, \ldots, F_m\}$. Then $T_{X,x}$ is given by

$$\sum_{i=1}^n \frac{\partial F_j}{\partial T_i}(x) \cdot (T_i - x_i) = 0, \quad 1 \leq j \leq m$$

The matrix $(\partial F_j/\partial T_i(x))$ is an $n \times m$ matrix called the Jacobi matrix. Suppose it has rank $r$, so

$$\dim(T_{X,x}) = n - r \geq \dim_x X$$

and $x$ is nonsingular iff $n - r = \dim_x X$. 

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Example 15.4.  1. Both \( \mathbb{A}^n \) and \( \mathbb{P}^n \) are smooth.

2. Consider the curve \( y^2 = x^2 + x^3 \). Then \( 0 \) is singular since all partial derivatives vanish there. The Jacobi matrix is

\[
\begin{pmatrix}
-2x + 3x^2 & 2y
\end{pmatrix}
\]

which has rank 1 for all other points, so all other points are nonsingular.

We have the following fact: Every regular local ring is a domain. The minimal prime ideals of \( \mathcal{O}_{X,x} \) are in correspondence with the irreducible components containing \( x \).

There is exactly one irreducible component of \( X \) containing a nonsingular point \( x \). Equivalently, if \( X_i \) and \( X_j \) are distinct irreducible components of \( X \), then every point of \( X_i \cap X_j \) is singular on \( X \).

Suppose \( X \) is irreducible, and \( x_1, x_2 \in X \) are nonsingular. In general, \( \mathcal{O}_{X,x_1} \not\cong \mathcal{O}_{X,x_2} \). Recall \( k(X) = \text{Quot}(\mathcal{O}_{X,x_1}) \). If there exists an isomorphism \( \mathcal{O}_{X,x_1} \to \mathcal{O}_{X,x_2} \), then we get an automorphism of \( k(X)/k \), which gives a birational isomorphism \( f: X \to Y \) defined at \( x_2 \) and \( f(x_2) = x_1 \).

But there are many varieties \( \text{Aut}_{\text{birat}}(X) \) is finite.

Suppose \( \mathcal{O} \) is a local ring with maximal ideal \( m \). The completion is \( \hat{\mathcal{O}} = \lim_n \mathcal{O}/m^n \).

Example 15.5.  1. Let \( x \in X \) be a nonsingular point, so \( \mathcal{O}_{X,x} \) is a regular, with \( f_1, f_2, \ldots, f_m \in \mathfrak{m}_x \) a system of local parameters. We get a map

\[
k[[x_1, \ldots, x_n]] \to \hat{\mathcal{O}} : x_i \mapsto f_i
\]

If \( x_1, x_2 \in X \) are nonsingular, then it may be that \( \mathcal{O}_{X,x_1} \not\cong \mathcal{O}_{X,x_2} \) but \( \hat{\mathcal{O}}_{X,x_1} \cong \hat{\mathcal{O}}_{X,x_2} \) as they are both isomorphic to the ring of power series.

2. Let \( X = Z(y^3 - x^2 - x^3) \) and \( Y = Z(y^2 - x^2) \), with \( \text{char} = k = 0 \). Then \( \mathcal{O}_{X,0} \not\cong \mathcal{O}_{Y,0} \), but the completions are since \( y^2 = x^2 + x^3 = x^2(1 + x) = f(x)^2 \) for some \( f \) since \( (1 + x)^{1/2} = \sum_{i=0}^{\infty} a_i x^i \). So we have an isomorphism of completions with \( x \mapsto f(x) \) and \( y \mapsto y \).

Also from commutative algebra, every regular ring is a UFD.

Theorem 15.6. Let \( X \) be a irreducible quasi-projective variety, and \( Y \subset X \) a closed irreducible subvariety of codimension 1. Let \( x \in Y \) be a nonsingular point. Then there is an affine neighborhood \( U \) of \( x \) such that \( \mathcal{I}_U(Y \cap U) \) is principal in \( k[U] \).

Proof. Let \( I \subset \mathcal{O}_{X,x} \) be the ideal of \( Y \), prime since \( Y \) is irreducible. It is nonzero, so pick \( a \in I \) a nonzero unit. Since \( \mathcal{O}_{X,x} \) is a UFD, there exists \( f \in \mathcal{O}_{X,x} \) such that \( f \mid a \). Then \( Z(f) \) is irreducible in a neighborhood of \( x \). So \( Y \cap U \subset Z_U(f) \subset U \). Since \( Z(f) \subset X \), then \( \dim X - 1 = \dim Y \leq \dim Z(f) < \dim X \) so \( \dim Y = \dim Z(f) \), so \( Y = Z(f) \).

Then \( I_X(Y) = I_X(Z(f)) = (f) \) in a neighborhood of \( X \), so \( I_X(Y) \) is principal.

Example 15.7. Let \( X = Z(y^2 - x^3) \). So \( \mathcal{O}_{X,0} \supset \mathfrak{m}_x \) is not generated by one element. But \( Z_X(y) = \{0\} \), but \( y \) is not generated by one element. But \( Z_X(y) = \{0\} \). But \( \sqrt{y} \) is not generated by one element.

Theorem 15.8. Let \( f: X \to Y \) be a rational map of quasi-projective varieties. Let \( U \subset X \) be the domain of definition of \( f \). Note that \( (X \setminus U) \subset X \) is closed, so \( \text{codim}_X(X \setminus U) \geq 1 \). If \( X \) is smooth and \( Y \) is projective, then \( \text{codim}_X(X \setminus U) \geq 2 \).

Proof. Write \( X \setminus U = \bigcup Y_i \) for \( Y_i \) irreducible. We need to show that \( \text{codim}_X(Y_i) \geq 2 \) for all \( i \). We claim that if \( Z \subset X \) is closed and irreducible of codimension 1, then \( Z \cap U \neq \emptyset \), i.e., \( f \) is defined at least one point in \( Z \). Pick \( x \in Z \subset X \) to be a nonsingular point. In a neighborhood of \( x \), the ideal \( I_X(Z) = (g) \) for some \( g \in \mathcal{O}_{X,x} \) is prime.

Assume \( Y = \mathbb{P}^n \). Then \( f = [f_0 : \cdots : f_n] \) where \( f_i \in k(X) = \text{Quot}(\mathcal{O}_{X,x}) \). So we may assume \( f_i \in \mathcal{O}_{X,x} \) are relatively prime by clearing denominators and dividing by the greatest common divisor. There exists \( i \) such that \( g \) does not divide \( f_i \) in \( \mathcal{O}_{X,x} \). This implies \( Z(g) \subset Z(f_i) \). For if \( Z = Z(g) \subset Z(f_i) \) in a neighborhood of \( x \), then \( f_i \) must be divisible by a power of \( g \), hence by \( g \). So there is a point \( z \in Z \) such that \( z \notin Z(f_i) \), so \( f_i(z) \neq 0 \). Thus \( f \) is defined at \( z \).

Now suppose \( Y \subset \mathbb{P}^n \) is closed. Let \( f': X \dashrightarrow Y \hookrightarrow \mathbb{P}^n \). Claim that the domain of definition of \( f \) and \( f' \) are the same. Write \( U' \) for the domain of definition of \( f' \). So \( U \subset U' \subset X \). Now \( U \subset U' \) is dense, so \( f(U) \) is dense in \( f'(U') \). So \( f(U) = f'(U') \). But \( f(U) \subset Y \) since \( Y \) is closed in \( \mathbb{P}^n \), so \( f'(U') \subset Y \). Thus \( U' = U \).
Corollary 15.9. Every rational map from an irreducible smooth curve to a projective variety is regular everywhere.

Corollary 15.10. Every birationally isomorphic irreducible smooth projective curves are isomorphic.

Some problems so far:
1. Previously, we had to assume $k$ is algebraically closed. We would like to replace $k$ with any commutative ring.
2. We had to assume varieties $X \subset \mathbb{P}^n$, but we would like to define them more abstractly as existing independently of some projective space.
3. Consider the projection $f : X \to \mathbb{A}^1 : (x, y) \mapsto y$ where $X = Z(y - x^2)$ is the parabola. Then $f^{-1}(a)$ is a pair of points. The map of coordinate rings $k[y] \to k[x, y]/(y - x^2)$, so $k[f^{-1}(a)] = k[x, y]/(y - x^2, y - a) \simeq k[x]/(x^2 - a) \simeq k \times k$ if $a \neq 0$. If $a = 0$, then $k[x]/(x^2)$. Thus $\bar{x}$ is nilpotent, hence $k[x]/(x^2)$ is not an algebra of regular functions. So we should allow nilpotence now.

16 Sheaves

Let $\mathcal{C}$ denote any one of the categories Rings, Ab or Sets. Let $\mathcal{A}$ be another category. A presheaf on $\mathcal{A}$ with values in $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{A}^{op} \to \mathcal{C}$. So $A \to A'$ is sent to $\mathcal{F}(A') \to \mathcal{F}(A)$.

Example 16.1. Let $X$ be a topological space, $\mathcal{A}_X$ be the category with objects open sets of $X$ and morphisms

$$\text{Mor}(U, U') = \begin{cases} \{\} & \text{if } U \subset U', \\ \emptyset & \text{if } U \not\subset U'. \end{cases}$$

A presheaf on $X$ is a presheaf on $\mathcal{A}_X$. For $\mathcal{F}(U) \in \mathcal{O}(\mathcal{C})$, and $U \subset U'$,

$$\mathcal{F}(U') \to \mathcal{F}(U) : f \mapsto f|_U$$

is the restriction.

Example 16.2. 1. Let $C \in \mathcal{C}$ and $A \in \mathcal{A}$. Let $\mathcal{F}(A) = C$ for all $A$ and $\mathcal{F}(\alpha) = 1_C$. This is the constant presheaf.

2. Let $X$ be quasi-projective over $k$, $U \subset X$ open is set to $k[U]$. And $U' \supset U$ is sent to $k[U']$ is the presheaf of regular functions on $X$.

Let $X$ be a topological space, and $\mathcal{F}$ a presheaf on $\mathcal{A}_X$ (on $X$) with values in $\mathcal{C}$. Suppose

1. $\mathcal{F}(\emptyset) = *$ a final object in $\mathcal{C}$.

2. If $U \subset X$ is open, $U = \bigcup U_\alpha$ is an open cover, such that $a, b \in \mathcal{F}(U)$: $a|_{U_\alpha} = b|_{U_\alpha}$ for all $\alpha$ implies $a = b$.

3. If $U \subset X$ is open, $U = \bigcup U_\alpha$ is an open cover, such that $a_\alpha \in \mathcal{F}(U_\alpha) : a_\alpha|_{U_\alpha \cap U_\beta} = a_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta$ then there exists a unique $a \in \mathcal{F}(U): a|_{U_\alpha} = a_\alpha$ for all $\alpha$.

Then $\mathcal{F}$ is a sheaf.

Recall a presheaf is a functor $\mathcal{F} : \mathcal{O}^{op} \to \mathcal{C}$. It assigns $U \subset X$ open to an object $\mathcal{F}(U)$ and restriction maps $r_{UV}$ from $\mathcal{F}(U) \to \mathcal{F}(V)$.
Definition 16.3. (Alternative) We have \( \mathcal{F} \) a presheaf is a sheaf if for all \( U \subset X \) open and for all open covers \( U = \bigcup U_\alpha \), the sequence

\[
\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \to \prod_{\alpha, \beta} \mathcal{F}(U_\alpha \cap U_\beta)
\]

is an equalizer, i.e., \( \mathcal{F}(U) = \{(s_\alpha)_\alpha \in \prod \mathcal{F}(U_\alpha) : r_1(s_\alpha)_\alpha = r_2(s_\alpha)_\alpha\} \), \( r_1, r_2 \) being the maps in the second mapping.

Some remarks, if \( \mathcal{C} = \text{Ab} \), then a sheaf is equivalent to \( r = \ker(r_1 - r_2) \).

Also \( \mathcal{F}(\emptyset) = \{\ast\} \) because \( \emptyset \) admits the empty cover, and then this follows since the empty product is the final object.

Let \( \mathcal{F} \) be a presheaf on \( X \).

Definition 16.4. The stalk of \( \mathcal{F} \) at \( x \) is \( \mathcal{F}_x = \lim_{x \to \bigcap U} \mathcal{F}(U) \) for \( U \) open neighborhoods of \( x \). Explicitly, consider the set of pairs \( (U, a) \) where \( U \ni x \) is an open neighborhood, and \( a \in \mathcal{F}(U) \). Define an equivalence relation on such pairs by \((U, a) \sim (U', a')\) if there exists a neighborhood \( W \ni x \) such that \( a|_W = a'|_W \). Then \( \mathcal{F}_x = \{(U, a)\} / \sim \)

Elements of \( \mathcal{F}_x \) are called germs of \( \mathcal{F} \) at \( x \).

Note if \( \mathcal{F} \) is a sheaf of rings or abelian groups, then \( \mathcal{F}_x \) is a ring or abelian group. IF \( U \ni x \) we have a natural morphism \( \mathcal{F}(U) \to \mathcal{F}_x \) by \( a \mapsto [(U, a)] = a_x \).

Example 16.5.  \begin{enumerate}
\item Let \( X \) be a quasi-projective variety. Then the stalk of \( \mathcal{O}_X \) at \( x \in X \) is the local ring \( \mathcal{O}_{X,x} \).
\item If \( \mathcal{F} \) is a constant presheaf with value \( A \in \mathcal{C} \), then for all \( x \in X \), \( \mathcal{F}_x = A \).
\item Let \( X \) be a domain in \( \mathbb{R}^n \), and \( \mathcal{F} \) be the sheaf of real analytic functions on \( X \), then a germ of \( \mathcal{F} \) at \( x \in X \) is determined by a sequence of real numbers.
\end{enumerate}

Proposition 16.6. Let \( \mathcal{F} \) be a sheaf on \( X \) and \( U \subset X \) open, and \( a, b \in \mathcal{F}(U) \). Then \( a = b \) iff for all \( x \in U \), \( a_x = b_x \).

Proof. Suppose that for all \( x \in U \), \( a_x = b_x \). Choose for each \( x \in U \) an open neighborhood \( U \subset W_x \ni x \) such that \( a|_{W_x} = b|_{W_x} \). This is possible by explicit description of the stalk. Then \( U = \bigcup_{x \in U} W_x \). Then we have the condition that \( a|_{W_x} = b|_{W_x} \) for all \( x \), so by the first part of the sheaf condition, (that \( \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}(W_x) \) is injective, so \( a = b \).

Definition 16.7. Let \( X \) be a topological space, and \( \mathcal{F} \) and \( \mathcal{G} \) (pre-)sheaves on \( X \). A morphism \( \mathcal{F} \to \mathcal{G} \) is a natural transformation of functors, i.e., for all \( U \subset X \) open, a morphism \( \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \) such that \( r_{UV} \circ \varphi_U = \varphi_V \circ r_{UV} \).

So clearly (pre-)sheaves on \( X \) with values in \( \mathcal{C} \) form a category. An isomorphism of (pre-)sheaves is one such that for all \( U \subset X \) is open, \( \varphi_U \) is an isomorphism in \( \mathcal{C} \).

Let \( f : \mathcal{F} \to \mathcal{G} \) be a morphism of presheaves \( x \in X \). Then we get an induced morphism in \( \mathcal{C} \), \( f_x : \mathcal{F}_x \to \mathcal{G}_x : [(U, a)] \mapsto [(U, f_U(a))] \).

The assignment \( \mathcal{F} \to \mathcal{F}_x \) for some fixed \( x \) is a functor from presheaves to \( \mathcal{C} \).

Proposition 16.8. Let \( \mathcal{F} \) and \( \mathcal{G} \) be sheaves on \( X \), and \( f : \mathcal{F} \to \mathcal{G} \) a morphism. Then \( f \) is an isomorphism iff for all \( x \in X \), \( f_x : \mathcal{F}_x \to \mathcal{G}_x \) is an isomorphism.

Proof. Suppose each \( f_x \) is an isomorphism. For injectivity, let \( a, b \in \mathcal{F}(U) \) for \( U \subset X \) open. Suppose \( f_U(a) = f_U(b) \). Then for all \( x \in U \), \( f_U(a)_x = f_U(b)_x \). Thus \( f_x(a_x) = f_x(b_x) \), so by hypothesis \( a_x = b_x \) for all \( x \in U \), so \( a = b \).

For surjectivity, let \( b \in \mathcal{G}(U) \). We need to produce \( \bar{b} \in \mathcal{F}(U) \) such that \( f_U(\bar{b}) = b \). Let \( x \in U \), then there is \( a^x \in \mathcal{F}_x \) such that \( f_x(a^x) = b_x \). Now each \( a^x \) is represented by a pair \( (U_x, a^x) \) such that \( b|_{U_x} = f_{U_x}(a^x) \), and \( U_x \ni x \), \( a^x \in \mathcal{F}(U_x) \). Note that \( U = \bigcup_{x \in U} U_x \). Take \( x, y \in U \).
We have \( f_U \circ f_U(a^x|U \cap U_y) = b|U \cap U_y = f_U \circ f_U(a^y|U \cap U_y) \) so since \( f_U \circ f_U \) is injective, so \( a^x|U \cap U_y = a^y|U \cap U_y \). Since \( \mathcal{F} \) is a sheaf, there exists a unique \( a \in \mathcal{F}(U) \) such that for all \( x \in U \), \( a|U_x = a^x \). Thus, \( f_U(a)|U_x = f_U(a^x) = b|U_x \) for all \( x \in U \). By sheaf condition for \( \mathcal{G} \), \( f_U(a) = b \).

**Example 16.9.** Let \( X \) be a space admitting a disconnected open subset. Let \( F \neq \{\ast\} \) to be a set. Take \( \mathcal{F} \) constant presheaf with value \( F \) and \( \mathcal{G} \) be the constant sheaf with value \( F \), so \( \mathcal{G}(U) = \text{Maps}(U,F) \) which is the locally constant \( F \)-valued functions. (Here \( F \) has the discrete topology.) We have an obvious morphism \( \mathcal{F} \to \mathcal{G} \) which is not an isomorphism, but is an isomorphism on all stalks, since \( X \) is locally connected.

Let \( \mathcal{F} \) be a presheaf on \( X \). Take \( U \subset X \) open. Define another presheaf \( \mathcal{F}|_U \) on \( U \) by \( \mathcal{F}|_U(W) = \mathcal{F}(W) \) for \( W \subset U \subset X \) all open. This is called the restriction of \( \mathcal{F} \). If \( \mathcal{F} \) is a sheaf, then \( \mathcal{F}|_U \) is also a sheaf.

Suppose \( f : X \to Y \) is continuous. Define a presheaf \( f_*\mathcal{F} \) on \( Y \) defined by \( f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \) for \( U \subset Y \) open. This is the pushforward of \( \mathcal{F} \). If \( \mathcal{F} \) is a sheaf, so if \( f_*\mathcal{F} \).

Consider \((X,\mathcal{F})\) for \( X \) a topological space, and \( \mathcal{F} \) is a presheaf on \( X \). Let these be the objects of a category. A morphism \((X,\mathcal{F}) \to (Y,\mathcal{G})\) is a pair \((f,g)\) where \( f : X \to Y \) is continuous, and \( g : \mathcal{G} \to f_*\mathcal{F} \) is a morphism of presheaves on \( Y \).

Note for \( U \subset Y \) open, we have maps \( \mathcal{G}(U) \to f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \). If \( x \in X \), we have a map of stalks 

\[
\mathcal{G}_{f(x)} \to (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x.
\]

We get a category of ringed spaces.

**Example 16.10.** Let \( X \) be a quasi-projective variety over \( k \). We have the structure ringed space \((X,\mathcal{O}_X)\) where \( \mathcal{O}_X(U) = k[U] \). If \( f : X \to Y \) is regular, we get a map \((X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)\) given by \( f \) and \( k[u] = \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)) \) defined by \( h \mapsto h \circ (f|_{f^{-1}(U)}) \).

If \( x \in X \), then \( (\mathcal{O}_X)_x = \mathcal{O}_{X,x} \).

## 17 Spectrum of a Ring

Let \( X \) be an affine variety, and \( k[X] \) its coordinate ring. For \( x \in X \), we have \( I_x = \{ f \in k[X] : f(x) = 0 \} \). So \( I_x = \ker(k[X] \to k : f \mapsto f(x)) \), so \( k[X]_{|I_x} \cong k \), so \( I_x \) is maximal.

There is a correspondence between maximal ideals in \( k[X] \) and (the points of) \( X \). So we recover \( X \) as a collection of maximal ideals.

Let \( R \) be a commutative ring. Note \( \text{Spec}(R) = \emptyset \) if \( R = 0 \). Let \( I \subset R \) be an ideal, and 

\[
V(I) = \{ P \in \text{Spec}(R) : I \subset P \} \subset \text{Spec}(R).
\]

Then \( V(0) = \text{Spec}(R) \), and \( V(I) = \emptyset \) iff \( I = R \). We have

1. If \( I \subset J \), then \( V(J) \subset V(I) \).
2. \( V(\sum I_\alpha) = \bigcap_\alpha V(I_\alpha) \)
3. \( V(I \cap J) = V(I) \cup V(J) = V(IJ) \). Now \( IJ \subset I \cap J \) so \( V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ) \). When \( P \in V(IJ) \), so \( IJ \subset P \), so \( I \subset P \) or \( J \subset P \) since \( P \) is prime, so either \( P \in V(I) \) or \( V(J) \). So \( V(IJ) \subset V(I) \cup V(J) \), and we conclude.

We have the Zariski topology on \( \text{Spec}(R) \), where the \( V(I) \) form the closed sets. Take \( T \subset \text{Spec}(R) \) a subset. If \( T = V(I) \), then \( T \subset V(I) \), so for all \( P \in T \), \( P \in V(I) \), so \( I \subset P \), and thus our desired \( I \) is \( I = \bigcap_{P \in T} P \). So if \( T = \{ P \} \), then \( \{ P \} = V(P) = \{ P' \in \text{Spec}(R) : P \subset P' \} \). Also, \( P \) is closed iff \( P \) is maximal. So not all points are closed.

For \( a \in R \), \( D(a) = \text{Spec}(R) \setminus V(a) \). These are the principal open sets. So 

\[
D(a) = \{ P \in \text{Spec}(R) : a \notin P \}.
\]

Some properties:
1. We have $V(I) = V(\sum_{a \in I} Ra) = \bigcap_{a \in I} V(a)$, implying $\text{Spec}(R) \setminus V(I) \cup \bigcup_{a \in I} D(a)$. So every open is a union of principal open sets.

2. So $D(0) = \emptyset$, and $D(1) = \text{Spec}(R)$.

3. $D(a) \cap D(b) = D(ab)$, so the $D(a)$ form a basis of the Zariski topology.

4. $D(a^n) = D(a)$.

5. $\bigcup D(a_\alpha) = \text{Spec}(R)$ if $\langle a_\alpha \rangle = R$. For this is equivalent to saying $\bigcap V(a_\alpha) = V(\sum a_\alpha R) = \emptyset$, which holds if $\sum a_\alpha R = R$.

6. $\text{Spec}(R)$ is compact. To see this, suppose $\text{Spec}(R) = \bigcup a_\alpha D(a_\alpha)$ if $\langle a_\alpha \rangle = R$, so $1 = \sum a_\alpha b_i$ for some $b_i \in R$. Thus $\langle a_\alpha \rangle = R$ iff $\text{Spec}(R) = \bigcup D(a_\alpha)$.

**Example 17.1.**

1. If $R$ is a field, $\text{Spec}(R) = \{0\}$.

2. If $R$ is a discrete valuation ring, then $\text{Spec}(R) = \{0, m\}$. The open sets are $\emptyset$, $\{0, m\}$, and $\{0\}$. And the closed sets are $\emptyset$, $\{0, m\}$ and $\{m\}$. This is because $D(a)$ is $\emptyset$ if $a \in R^\times$, is $\{0, m\}$ if $a = 0$, and $\{0\}$ otherwise. Also $\{0\} = \{0, m\}$.

3. If $R = \mathbb{Z}$, then $\text{Spec}(\mathbb{Z}) = \{0, p\mathbb{Z}\}$, and again $\overline{0} = \text{Spec}(\mathbb{Z})$.

Suppose $\alpha : R \to S$ is a ring homomorphism, with $\alpha^{-1}(P) = Q$ for $P$ a prime. Then we have an injection $R/Q \to S/P$, so $R/Q$ is a domain, so $Q$ is prime.

We have the pullback $\alpha^* : \text{Spec}(S) \to \text{Spec}(R) : P \mapsto \alpha^{-1}(P)$. We have

1. $I \subset R$ an ideal implies $(\alpha^*)^{-1}(V(I)) = V(\alpha(I)S)$. For

   $P \in (\alpha^*)^{-1}(V(I)) \iff \alpha^*(P) \in V(I) \iff \alpha^{-1}(P) \in V(I) \iff I \subset \alpha^{-1}(P) \iff \alpha(I) \subset P \iff \alpha(I)S \subset P \iff P \in V(\alpha(I)S)$.

2. $(\alpha^*)^{-1}(D(a)) = D(\alpha(a))$.

So we get a correspondence from the opposite category of commutative rings and topological spaces given by $R \mapsto \text{Spec}(R)$, and $\alpha \mapsto \alpha^*$.

Let $I \subset R$ be an ideal. Define $V(I) = \{p \in \text{Spec}(R) : I \subset P\} \subset \text{Spec}(R)$ to be closed. We have $\text{Spec}(R/I) = \{Q/I : Q \in \text{Spec}(R), Q \supset I\} \simeq V(I)$ are homeomorphic. Take $S \subset R$ to be a multiplicative subset. We have the canonical map $i : R \to S^{-1}R$.

We have $\text{Spec}(S^{-1}R) \simeq \{p \in \text{Spec}(R) : P \cap S = \emptyset\}$ under the maps $Q \mapsto i^{-1}(Q)$ and $P \mapsto S^{-1}P = S^{-1}R \cdot i(P)$.

**Example 17.2.**

1. Take $S = \{a^n\}, S^{-1}R = R_a$. Then $\{p \in \text{Spec}(R) : P \cap \{a^n\} = \emptyset\} = D(a)$

   so $\text{Spec}(R_a) \simeq D(a)$.

2. If $p \in \text{Spec}(R), S = R \setminus P$, then $S^{-1}R = R_p$, and

   $\text{Spec}(R_p) \simeq \{P' \in \text{Spec}(R) : P' \subset P\}$.

   Also,

   $\text{Spec}(R \setminus P) \simeq \{P' \in \text{Spec}(R) : P' \supset P\}$.
Recall \( \bigcap_{p \in \text{Spec}(R)} p = N(R) = \{a \in R : a^n = 0 \text{ for some } n > 0\} \).

Note \( D(a) = \emptyset \) iff \( a \in N(R) \) and \( V(I) = \text{Spec}(R) \) iff \( I \subset N(R) \). Take \( I \subset R \), so \( N(R/I) = \sqrt{I}/I \). So \( \bigcap_{p \supseteq I} p = N(I) \). Also, \( P \supset I \) iff \( P \supset \sqrt{I} \), so \( V(I) = V(\sqrt{I}) \).

**Proposition 17.3.** We have \( V(I) \subset V(J) \) iff \( J \subset \sqrt{I} \).

**Proof.** For the forward direction, every \( p \in \text{Spec}(R) \) and \( p \supset I \) implies \( P \supset J \). So \( J \subset \bigcap_{I \subset P} P = \sqrt{I} \).

Conversely, \( V(I) = V(\sqrt{I}) \supset V(J) \).

\( \square \)

**Corollary 17.4.** Note \( V(I) = V(J) \) iff \( \sqrt{I} = \sqrt{J} \).

**Proposition 17.5.** Let \( \alpha : R \to S \) be a ring homomorphism, and \( J \subset B \) an ideal, and \( I = \alpha^{-1}(J) \). Then \( \alpha^*(V(J)) = V(I) \)

If \( X \) is a topological space, a point \( x \in X \) is called generic if \( \overline{\{x\}} = X \). Note this implies \( X \) is necessarily irreducible, since \( \{x\} \) is irreducible, and hence its closure is as well.

**Proposition 17.6.** Let \( R \) be a commutative ring. The following are equivalent

1. \( \text{Spec}(R) \) is irreducible.
2. \( N(R) \) is a prime.
3. \( N(R) \) is prime and \( N(R) \) is a generic point in \( \text{Spec}(R) \).
4. There exists a unique generic point in \( \text{Spec}(R) \).

**Proof.** Suppose (1). Suppose \( ab \in N(R) \). Then \( \emptyset = D(ab) = D(a) \cap D(b) \). So either \( D(a) = \emptyset \) or \( D(b) = \emptyset \), so either \( a \in N(R) \) or \( b \in N(R) \). Hence (2).

Now suppose (2). So \( N(R) \subset P \) for all \( P \in \text{Spec}(R) \). So \( N(R) = V(N(R)) = \text{Spec}(R) \). Hence (3).

Now suppose (3). If \( P_1 \) and \( P_2 \) are generic points, then \( P_1 \in \{P_1\} = V(P_2) \), so \( P_2 \subset P_1 \). By symmetry, \( P_1 = P_2 \).

Now suppose (4). We have \( \text{Spec}(R) = \{\overline{x}\} \) is irreducible, where \( x \) is the generic point. \( \square \)

We have maps from the category of irreducible closed subsets of \( \text{Spec}(R) \) and \( \text{Spec}(R) \) sending \( Z = \text{Spec}(R/I) \) to a generic point of \( Z \) and \( P \) maps to \( \{P\} = V(P) \).

Recall an ideal of \( R \times S \) has form \( I \times J \) for ideals \( I \) and \( J \), and

\[ (R \times S)/(I \times J) \simeq (R/I) \times (S/J) \]

so \( I \times J \) is prime iff \( I \) is prime and \( J = S \) or \( I = R \) and \( J \) is prime. So

\[ \text{Spec}(R \times S) = \text{Spec}(R) \cup \text{Spec}(S) \].

We construct a sheaf of rings on \( \text{Spec}(R) \). Take \( U \subset \text{Spec}(R) \) is open. Recall if \( p \in \text{Spec}(R) \), then \( R_p = \{ \overline{f} : f \notin P \} \) and \( \overline{f} \) makes sense in \( R_p \) for all \( P \in D(f) \). Denote \( O_R \) to be the structure sheaf of rings on \( \text{Spec}(R) \). Define

\[ O_R(U) = \{ \varphi : U \to \bigcup_{P \in U} R_p : \varphi(P) \in R_P \text{ and in a nbhd } W \text{ of every } P \in U, \varphi(P) = \frac{a}{f}, f \notin Q, \forall Q \in W \} \].

The restriction maps are given by restriction of functions. So \( O_R \) is a sheaf of rings. Take \( a \in R \). Then \( \text{Spec}(R_a) = D(a) \subset \text{Spec}(R) \). Then \( O_R|_{D(a)} = O_{R_a} \).

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Proposition 17.7. The map

\[ R \to \mathcal{O}_R(\text{Spec}(R) : a \mapsto (P \xrightarrow{\varphi|_{\text{Spec}(R)}} \frac{a}{1} \in R_p) \]

is a ring isomorphism.

Proof. For injectivity, suppose \( a \in R \) such that \( \varphi(a) = 0 \) for all \( P \). Then there exists \( p \in R \setminus P \) such that \( ab_p = 0 \). Then Ann(\( A \)) \( \not\subset P \) for all \( P \), so Ann(\( A \)) = \( R \), so \( a = 0 \).

For surjectivity, let \( \varphi : \text{Spec}(R) \to \bigsqcup R_p \) be as above. There is an open cover \( \text{Spec}(R) = \bigsqcup U_i \) such that \( \varphi(P) = \frac{a}{f_i} \) for all \( P \in U_i \), \( f_i \notin P' \) for all \( P' \in U_i \). We may assume \( U_i = D(g_i) \). Then \( D(f_i) \supset D(g_i) \). Then \( V(f, R) \subset V(g, R) \) implies \( g_i = \sqrt{f_i R} \). Then \( g_i = f_i b_i \). So

\[ \varphi(P) = \frac{a_i}{f_i} = \frac{a_i b_i}{f_i b_i} = \frac{a_i b_i}{g_i^{n_i}} \]

but \( \text{Spec}(R) = \bigsqcup D(g_i) = \bigsqcup D(g_i^{n_i}) \). Replacing \( g_i \) by \( g_i^n \) and \( a_i \) by \( a_i b_i \), we may assume in \( U_i = D(g_i) \) and \( \varphi(P) = \frac{a_i}{g_i} \). We may assume the cover is finite by compactness.

For \( i, j \) we have

\[ D(g_i) \cap D(g_j) = D(g_i g_j) = \text{Spec}(R_{g_i g_j}) \]

We get

\[ R_{g_i g_j} \hookrightarrow \mathcal{O}_{R_{g_i g_j}}(\text{Spec}(R_{g_i g_j})) = \mathcal{O}_R(D(g_i g_j)) \]

We have the functions \( \alpha : D(g_i g_j) \to \bigsqcup R_p \) for \( \alpha(P) = \frac{a_i}{f_i} \) and \( \beta : D(f_i f_j) \to \bigsqcup R_p \) with \( \beta(P) = \frac{a_j}{g_j} \).

Now \( a_i g_i = a_j g_j \) in \( R_{g_i g_j} \) and \( (a_i g_j - a_j g_i)(g_i g_j)^n = 0 \) for \( a_i g_i^n g_j^{n+1} = a_j g_i^{n+1} g_j \). So

\[ \frac{a_i}{g_i} = \frac{a_i g_i^n}{g_i^{n+1}} \]

and

\[ \frac{a_j}{g_j} = \frac{a_j g_j^n}{g_j^{n+1}} \]

so by replacing \( a_i \) by \( a_i g_i^n \) and \( a_j \) by \( a_j g_j^n \) with \( g_i \) by \( g_i^n \) and \( g_j \) by \( g_j^n \), we may assume that \( a_i g_j = a_j g_i \). So \( \langle g_j \rangle \subseteq R \) so there exists \( h_i \in R \) such that \( \sum g_i h_i = 1 \). Set \( a = \sum_k a_k h_k \in R \), so \( \frac{a}{g_i} = \varphi(P) \) in \( R_{g_i} \).

Then

\[ a \cdot g_i = \sum_k a_k g_i h_k = \sum_k a_i g_i h_k = a_i \cdot 1 = a_i \]

and thus \( \frac{a_i}{g_i} = \frac{a_i}{g_i} \).

Corollary 17.8. We have \( \mathcal{O}_R(D(a)) = R_a \).

Proof. Note

\[ \mathcal{O}_R(D(a)) = \mathcal{O}_{R_a}(D(a)) = R_a \]

and apply the statement to \( R_a \).

Proposition 17.9. For every \( P \in \text{Spec}(R) \), \( (\mathcal{O}_R)_P \simeq R_P \).

Proof. We have a map \( (\mathcal{O}_R)_P \to R_P \) by \( (U, \varphi) \mapsto \varphi(P) \).

For injectivity, in a neighborhood \( D(f) \) of \( P \) in \( U \), \( \varphi = \frac{a}{f} \), with \( \frac{a}{f} = 0 \) in \( R_P \), so \( a = 0 \) for \( s \in R \setminus P \) then \( \frac{a}{f} = 0 \) in \( R_{a f} \) implies \( \varphi|_{D(a f)} = 0 \). Then

\[ (U, \varphi) \sim (D(s f), \varphi|_{D(s f)}) = 0. \]

For surjectivity, take \( \frac{a}{f} \in R_P \). Take \( f \in R \setminus P \). Then \( D(f, P) \) with \( \varphi = \frac{a}{f} \) maps to \( \frac{a}{f} \in R_P \).

Take \( U \subset \text{Spec}(R) \) to be open. Let \( \varphi \in \mathcal{O}_R(U) \). If \( P \in U \), we have \( \varphi_P \in R_P \to R_P / P R_P = \text{Quot}(R/P) \), so \( \varphi(P) \) is in a field.

If \( U = \text{Spec}(R) \) and \( \varphi \in \mathcal{O}_R(U) \), then \( \varphi(P) = 0 \) for all \( P \in \text{Spec}(R) \) iff \( \varphi \in P \) for all \( P \), iff \( \varphi \in N(R) \). So \( \varphi \in N(P) \) iff \( \varphi(P) = 0 \) for every \( P \), but \( \varphi_P = 0 \) in \( R_P \) for all \( P \) iff \( \varphi = 0 \).
Let $\alpha: R \rightarrow S$ be a ring homomorphism. It induces a continuous map $\alpha^*: \text{Spec}(S) \rightarrow \text{Spec}(R)$. We want to make a map from $\mathcal{O}_R(U)$ to $\mathcal{O}_S((\alpha^*)^{-1}(U))$ by the diagram

$$
\begin{array}{ccc}
(\alpha^*)^{-1}(U) & \longrightarrow & \bigsqcup_{Q \in (\alpha^*)^{-1}(U)} S_Q \\
\downarrow & & \downarrow \\
U & \stackrel{\varphi}{\longrightarrow} & \bigsqcup_{P \in U} R_P
\end{array}
$$

The morphisms are then $(\text{Spec}(S), \mathcal{O}_S) \rightarrow (\text{Spec}(R), \mathcal{O}_R)$. So we get a functor of ringed spaces from the opposite category of commutative rings to the category of ringed spaces sending $R$ to $(\text{Spec}(R), \mathcal{O}_R)$.

This functor is faithful. The morphism $R \rightarrow S$ goes to $(\text{Spec}(S), \mathcal{O}_S) \rightarrow (\text{Spec}(R), \mathcal{O}_R)$. Note $R = \mathcal{O}_R(\text{Spec}(R)) \rightarrow \mathcal{O}_S(\text{Spec}(S)) = S$, so different morphisms give rise to different morphisms.

Observe $(\text{Spec}(R), \mathcal{O}_R)$ is a local ringed space since all stalks are local rings.

A ringed space $(X, \mathcal{F})$ is called local if $\mathcal{F}_x$ is a local ring for all $x \in X$. A morphism $f: (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ of local ringed spaces is local if for all $x \in X$, $\mathcal{G}_f(x) \rightarrow \mathcal{F}_x$ is a local ring homomorphism. So $(\text{Spec}(R), \mathcal{O}_R)$ is a local ringed space and if $\alpha: R \rightarrow S$ is a local ring homomorphism, then $\alpha^*: (\text{Spec}(S), \mathcal{O}_S) \rightarrow (\text{Spec}(R), \mathcal{O}_R)$ is local.

**Example 17.10.** Let $R$ be a DVR, so $\text{Spec}(R) = \{0, m\}$, so $S = \text{Quot}(R)$ and $\text{Spec}(S) = \{0\}$. We make a map $f: \text{Spec}(S) \rightarrow \text{Spec}(R)$ by sending $0$ to $m$. The nonempty open sets in $\text{Spec}(R)$ are $\{0\}$ and $\text{Spec}(R)$. We also need to make a map $\mathcal{O}_R(U) \rightarrow \mathcal{O}_S(f^{-1}(U))$. But $R \rightarrow S$ is not a local homomorphism, so the map of ringed spaces is not local, so $f \neq \alpha^*$ for any $\alpha: R \rightarrow S$.

So we adjust the later category to just be local ringed spaces. This functor is full.

**Theorem 17.11.** This functor is full and faithful.

*Proof.* Let $f: (\text{Spec}(S), \mathcal{O}_S) \rightarrow (\text{Spec}(R), \mathcal{O}_R)$ is local. Take $Q \in \text{Spec}(S)$ and $P = f(Q) \in \text{Spec}(R)$. We have

$$
\begin{array}{ccc}
R = \mathcal{O}_R(\text{Spec}(R)) & \longrightarrow & \mathcal{O}_S(\text{Spec}(S)) = S \\
\downarrow & & \downarrow \\
R_P = (\mathcal{O}_R)_P & \longrightarrow & (\mathcal{O}_S)_Q = S_Q
\end{array}
$$

So $f_P^{-1}(Q)S_Q = PR_P$. Then

$$
\alpha^*(Q) = \alpha^{-1}(Q) = P = f(Q)
$$

so $\alpha^* = f$ as maps of topological spaces.

Now take $a \in R$ and $b = \alpha(a) \in S$, then $f^{-1}(D(a)) = (\alpha^*)^{-1}(D(a)) = D(b)$. Then

$$
\begin{array}{ccc}
R = \mathcal{O}_R(\text{Spec}(R)) & \longrightarrow & \mathcal{O}_S(\text{Spec}(S)) = S \\
\downarrow & & \downarrow \\
\mathcal{O}_R(D(a)) & \longrightarrow & \mathcal{O}_S(D(b))
\end{array}
$$

We need $f(D(a))(\frac{x}{a^n}) = \frac{\alpha(x)}{b^n}$. But $f_D(a)(a) = \alpha(a) = b \in S_b$ and $f_D(a)(x) = \alpha(x)$ so

$$
\frac{f(D(a))(x)}{f(D(a))(a^n)} = \frac{\alpha(x)}{b^n}
$$

so $f = \alpha^*$ on the structure sheaves. \hfill \square
18 Definition of a Scheme

We have the opposite category of commutative rings is a full subcategory of local ringed spaces. If \((X, \mathcal{F})\) is a local ringed space, and \(U \subset X\) is open, then \((U, \mathcal{F}|_U)\) is a local ringed space. A local ringed space \((X, \mathcal{F})\) is a scheme if there is an open cover \(X = \bigcup U_\alpha\) such that \((U_\alpha, \mathcal{F}|_{U_\alpha}) \simeq (\text{Spec } R_\alpha, \mathcal{O}_{R_\alpha})\) for some commutative ring \(R_\alpha\) for every \(\alpha\). So the category of schemes sits between the opposite category of commutative rings and local ringed spaces.

A scheme \((X, \mathcal{O}_X)\) is affine if \(\mathcal{O}_X = (X, \mathcal{F}) \simeq (\text{Spec } R, \mathcal{O}_R)\) for some \(R\). In short, we write \(X \simeq \text{Spec } R\) and \(\mathcal{O}_X \simeq \mathcal{O}_R\).

Suppose \(X\) is a scheme, and \(U \subset X\) is open, then \(U\) is a scheme. That is \((U, \mathcal{O}_X|_U)\) is a scheme. Need to check it’s locally affine. Pick \(x \in X\). There exists an open neighborhood \(W\) of \(x\) such that \(W \simeq \text{Spec } R\). Then \(U \cap W \subset W\) is open and corresponds to an open subset of \(\text{Spec } R\). There is some \(D(a) \subset \text{Spec } R\) corresponding to \(U \cap W\).

Then \(\mathcal{O}_X|_{D(a)} \simeq \mathcal{O}_R|_{D(a)} = \mathcal{O}_{R_\alpha}\). Then \(U|_{D(a)} = X|_{D(a)} \simeq \text{Spec } R_\alpha\) is the desired affine neighborhood of \(x\).

Suppose \(U \subset X\) is open, and \(f: U \rightarrow X\). A map \((U, \mathcal{O}_U) \rightarrow (X, \mathcal{O}_X)\) is given by the inclusion \(f: U \rightarrow X\) and \(\mathcal{O}_X \rightarrow f_* (\mathcal{O}_X|_U)\) by

\[
\mathcal{O}_X(W) \rightarrow f_* (\mathcal{O}_X|_U)(W) = \mathcal{O}_X|_U(f^{-1}(W)) = \mathcal{O}_X|_U(U \cap W) = \mathcal{O}_X(U \cap W).
\]

If \(x \in X\) for a scheme, then \((\mathcal{O}_X)_x = \mathcal{O}_{X,x}\) is the local ring at \(x\). If \(f: X \rightarrow Y\) is a morphism of schemes, and for \(x \in X\), \(y = f(x)\), we get a local homomorphisms \(\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}\). This descends to a map of the residue fields. If \(x \in X\), and \(x \in U \subset X\) is open, and \(y \in \mathcal{O}_X(U)\) is a “function” on \(U\). Then \(g_x \in \mathcal{O}_{X,x}\) and then \(g_x \in \mathcal{O}_{X,x} \rightarrow g(x) \in k(x)\) is the value of \(y\) at \(x\).

Let \(X\) and \(Y\) be schemes. Define a presheaf \(\mathcal{F}_Y\) of sets on \(X\): If \(U \subset X\) is open, then \(\mathcal{F}_Y(U) = \text{Mor}(U, Y) = Y(U)\), called the \(U\)-values of \(Y\) or set of points of \(Y\) with values in \(U\). If \(U = \bigcup U_\alpha\) is an open cover, then

\[
\text{Mor}(U, Y) \rightarrow \prod_\alpha \text{Mor}(U_\alpha, Y) \rightarrow \prod_{\alpha, \beta} \text{Mor}(U_\alpha \cap U_\beta, Y)
\]

exact. Then \(\mathcal{F}_Y\) is a sheaf of sets. So every scheme gives us a sheaf of sets.

Let \(X\) be a scheme, and \(R\) a commutative ring. If \(f: X \rightarrow \text{Spec } (R)\), we get \(R = \mathcal{O}_R(\text{Spec } (R)) \rightarrow \mathcal{O}_X(X)\).

We get a map

\[
\text{Mor}(X, \text{Spec } (R)) \rightarrow \text{Hom}_{\text{Rings}}(R, \mathcal{O}_X(X)).
\]

**Theorem 18.1.** This map is a bijection.

**Proof.** If \(X\) is affine, \(X = \text{Spec } (S)\), then \(\text{Mor}(\text{Spec } S, \text{Spec } R) \rightarrow \text{Hom}_{\text{Rings}}(R, S)\) But recall that opposite category of commutative rings is a full subcategory of ringed spaces, so this is a bijection.

In general, take \(X = \bigcup U_\alpha\) to be an affine open cover. We get an exact sequence of sets

\[
\text{Mor}(X, \text{Spec } R) \rightarrow \prod_\alpha \text{Mor}(U_\alpha, \text{Spec } R) \rightarrow \prod_{\alpha, \beta} \text{Mor}(U_\alpha \cap U_\beta, \text{Spec } R) \rightarrow \text{Hom}_R(\mathcal{O}_X(X)) \rightarrow \prod_{\alpha, \beta} \text{Hom}(R, \mathcal{O}_X(U_\alpha \cap U_\beta))
\]

A standard diagram chase shows bijectivity of the leftmost vertical map.

Then we have a correspondence between opposite category of commutative rings and the category of schemes given by \(R \mapsto \text{Spec } R\) and \(X \mapsto \mathcal{O}_X(X)\). So \(R \mapsto \text{Spec } (R)\) is right adjoint, and the other functor is the left adjoint.

The homomorphism \(R \rightarrow \mathcal{O}_X(X)\) makes \(\mathcal{O}_X(X)\) an \(R\)-algebra. This induces an \(R\)-algebra structure on every \(\mathcal{O}_X(U)\) is an \(R\)-algebra. So to give a morphism \(X \rightarrow \text{Spec } R\) is equivalent to giving \(R\)-algebra structure on all \(\mathcal{O}_X(U)\).

When \(R = \mathbb{Z}\), there exists a unique morphism \(X \rightarrow \text{Spec } \mathbb{Z}\), so \(\text{Spec } \mathbb{Z}\) is the final object in the category of schemes. We say \(X\) is a scheme of a commutative ring \(R\) if a morphism \(X \rightarrow \text{Spec } R\) is given. Every scheme is a scheme over \(\mathbb{Z}\).
Define the category of schemes over $R$ to have objects $X \to \text{Spec } R$, and the morphisms to be a map $X \to Y$ making the following commute:

\[
\begin{array}{ccc}
X & \longrightarrow & \text{Spec } R \\
\downarrow & & \downarrow \\
Y & \longrightarrow & ~
\end{array}
\]

So the category of schemes is the same of schemes over $\mathbb{Z}$.

Note for Spec $S$ to be a scheme over $R$, we need $S$ to be an $R$-algebra.

Let $k$ be an algebraically closed field, and $X$ a quasi-projective variety over $k$. Let $\tilde{X} = \{ Z \subset X : Z$ closed and irreducible $\}$. So we may view $S \subset \tilde{X}$ be the identification $x \mapsto \{ x \}$. We want $Z \subset X$ closed to imply $\tilde{Z} \subset X$ be closed for the topology on $\tilde{X}$. We have $X \cap \tilde{Z} = Z$ and the Zariski topology is induced on $\tilde{X}$ so that $X \to \tilde{X}$ is continuous.

For $U \subset X$ open, if $Z = X \setminus U$, so that $\tilde{Z} \subset \tilde{X}$, we define

\[
\tilde{U} := \tilde{X} \setminus \tilde{Z} = \{ W \subset U : \text{W is irreducible } , \ W \cap U \neq \emptyset \}.
\]

These are the open sets in $\tilde{X}$.

If $Z \in \tilde{X}$, then $\{ Z \} = \tilde{Z}$, so $\{ Z \}$ is closed in $\tilde{X}$ iff $Z = \{ z \}$ is a singleton. So $X \subset \tilde{X}$ is a set of closed points.

Take $\tilde{U} \subset \tilde{X}$ for some $U \subset X$ open. Note we can recover $U$ by $U = \tilde{U} \cap X$. We define

\[
\mathcal{O}_X(\tilde{U}) = k[U].
\]

So $(\tilde{X}, \mathcal{O}_X)$ is a ringed space. Suppose $X$ is affine, and $R = k[X]$. We have Spec$(R) \simeq X$ are homeomorphic. We want to construct a map $\mathcal{O}_X \to \mathcal{O}_R$ by

\[
k[U] = \mathcal{O}_X(\tilde{U}) \to \mathcal{O}_R(\tilde{U}) = \{ \varphi : \tilde{U} \to \prod \mathcal{O}_{X,Z} \}
\]

Given $g \in k[U]$, take $P \in \mathcal{O}_R(\tilde{U})$ corresponding to $z$ which maps to image of $g$ under $k[U] \to \mathcal{O}_{U,Z} = \mathcal{O}_{U,Z \cap U}$.

This gives map $\text{Spec } R, \mathcal{O}_R) \to (\tilde{X}, \mathcal{O}_X)$. It’s sufficient to check $k[X]_g = \mathcal{O}_X/(D(g)) \simeq \mathcal{O}_R(D(g)) = R_g$ where $g \in R = k[X]$.

Let $X$ be arbitrary quasi-projective, and $X = \bigcup X_\alpha$ for $U_\alpha$ open affines. Then $\tilde{X} = \bigcup \tilde{U}_\alpha$, and $\mathcal{O}_X|_{\tilde{U}_\alpha} = \mathcal{O}_{\tilde{U}_\alpha}$. So $\{ U_\alpha, \mathcal{O}_X|_{\tilde{U}_\alpha} \} = \{ U_\alpha, \mathcal{O}_{\tilde{U}_\alpha} \}$ is an affine scheme and $(X, \mathcal{O}_X)$ is a scheme.

Suppose $f : X \to Y$ is a regular map, and $Z \subset X$ is a closed irreducible set. Then $f(Z)$ is also closed and irreducible. We get a map

\[
\tilde{f} : \tilde{X} \to \tilde{Y} : Z \mapsto f(Z)
\]

So if $\tilde{W} \subset \tilde{Y}$ is open and $W \subset Y$ is open. We want to construct a map $f^*$ from $\mathcal{O}_Y(\tilde{W}) = k[W]$ to $\mathcal{O}_X(\tilde{f}^{-1}(\tilde{W})) = k[f^{-1}(W)]$ since $\tilde{f}^{-1}(\tilde{W}) = f^{-1}(W)$.

So $\tilde{f} : (\tilde{X}, \mathcal{O}_X) \to (\tilde{Y}, \mathcal{O}_Y)$ and we say $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism of schemes.

We’ve constructed a functor $\theta : \text{QProjVar}(k) \to \text{Schemes}$ by the assignment $X \mapsto \tilde{X}$ and $f \mapsto \tilde{f}$.

**Theorem 18.2.** The functor $\theta$ is fully faithful.

**Lemma 18.3.** If $Z \in \tilde{X}$, then $(\mathcal{O}_X)_Z \simeq \mathcal{O}_{X,Z}$, and $\text{Quot}((\mathcal{O}_X)_Z) \simeq k(Z)$.

**Proof.** (Of Theorem) if $f : X \to Y$ and $\tilde{f} : \tilde{X} \to \tilde{Y}$, then $f$ the “restriction” of $\tilde{f}$ implies $\theta$ is faithful.

Take $g : \tilde{X} \to \tilde{Y}$ be a morphism of schemes over $k$. We need to find a regular map $f : X \to Y$ such that $g = \tilde{f}$. We need $g(X) \subset Y$.

Take $x \in X$, so $\{ x \} \in \tilde{X}$. Then $k(x) = k$ since $x$ corresponds to a maximal ideal. Take $Z \in \tilde{Y}$ such that $g(\{ x \}) = Z$. This induces a map $k(Z)/k(k(x)) = k$ over $k$. This implies dim(Z) = 0, so $Z$ is a point. So $g(X) \subset Y$, so set $f = g|_X : X \to Y$.

We need to check that $f$ is regular, and $g = \tilde{f}$ on top space and on structure sheaves. The proof is omitted.
and the bottom inclusion is that of reduced finitely generated $k$-algebras into commutative $k$-algebras.

Let $k[\mathbb{A}^n] = k[T_1, T_2, \ldots, T_n]$ and $\mathbb{A}^n_k = \text{Spec}(k[T_1, \ldots, T_n])$. So $\mathbb{A}^1_k = \text{Spec}(k[T])$ is in bijection with the set of monic irreducible polynomials and 0, which are the closed points and the generic point. So if $P = (h)$ and $\mathcal{O}_{X,P} = k[T]|_P$ and $k(P) = k[T]/(h)/k$ is of degree $\text{deg}(h)$.

Let $V$ be an $n$-dimensional $k$-vector space. Then $V \simeq k^n$ and view $V$ as $\mathbb{A}^n_k$. It turns out $V = \text{Spec}(\text{Sym}(V^*))$. For suppose $x_1, x_2, \ldots, x_n$ are the coordinates, so $x_i \in V^*$. Then $k[x_1, \ldots, x_n]$ is the symmetric algebra on $x_1, \ldots, x_n$. 

\[
\begin{array}{c}
\text{QPorjVar}(k) \hookrightarrow \text{Schemes}/k \\
\downarrow \\
\text{AffVar}(k) \hookrightarrow \text{AffSchemes}/k
\end{array}
\]