Problem 1 True or false?

(a) The integral $\int_0^{2\pi}\int_0^1\int_r^1 dzdrd\theta$ represents the volume of a right cone.
Solution. False.

(b) The jacobian of the transformation given by $x = u^2 - 2v, y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.
Solution. False.

(c) Is the vector field $\vec{F} = \langle x^2y, xy^2\rangle$ conservative?
Solution. False.

(d) The divergence of a vector field is a vector field.
Solution. False.

(e) If $\nabla \times \vec{F} = 0$ then $\vec{F}$ is conservative.
Solution. False.

Problem 2 Evaluate the integral

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx$$

Solution. $\frac{2\pi}{3}$

Problem 3 Calculate $\iiint_B \sqrt{x^2+y^2}dV$ where B is the region bounded above by the half sphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ and below by the cone $z^2 = 3(x^2 + y^2)$.

Solution. $\frac{81\pi}{8}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

Problem 4 Use a change of variables to evaluate the integral $\iint_R (x-y)e^{x^2-y^2}dA$ where R is the region bound by the lines $x+y = 1$, $x+y = 3$ and curves $x^2 - y^2 = -1$, $x^2 - y^2 = 1$.

Solution. $2/3e$

Problem 5 Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = z$. Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

(a) $d\rho d\phi d\theta$

(b) $d\phi d\rho d\theta$.

Solution. $\pi/8$

Problem 6 Let $\vec{F} = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz\rangle$ and let C be the curve parameterised by $\vec{r}(t) = \langle t^2, t, t\rangle$ for $1 \leq t \leq e$. Calculate $\int_C \vec{F}\cdot d\vec{r}$

(a) without using the Fundamental Theorem of Line Integrals and

(b) using the Fundamental Theorem of Line Integrals.

Solution. $e^4 + e^3 - 1$.

Problem 7 Calculate the line integral $\int_C (x^2 + y^2 + z^2)ds$ where C is the part of the helix parameterised by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 2\pi$.

Solution. $\sqrt{2}\left(2\pi + \frac{8\pi^3}{3}\right)$

Problem 8 A solid Q has the form $D \times I$ where D is some finite region in the xy-plane and $I = [a, b]$ is a finite interval. The density $\rho(x, y, z)$ of the solid Q doesn't depend on the variable z. Show that the center of mass of Q lies on the plane $z = \frac{a+b}{2}$.

Problem 9 True of False?

(a) If C is parameterised by $\vec{r}(t) = (t, t)$ for $0 \le t \le 1$, then

$$\int_C xyds = \int_0^1 t^2 dt.$$

Solution. False.

(b) If vector field $\vec{F}$ has zero curl on the open and connected region D, then line integrals of $\vec{F}$ are path independent on D.

Solution. False.

(c) If a vector field $\vec{F}$ is path independent on an open connected region D, then the vector field $\vec{F}$ is conservative on D.

Solution. True.

Problem 10 Use a change of variables to calculate $\iint_R (x^2 + 25y^2)^2 dA$ where R is the ellipse with boundary $x^2 + 25y^2 = 1$.

Solution. $\pi/15$

Problem 11 Find a potential function for $\vec{F} = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z \rangle$.

Solution. $f(x, y, z) = 4x^3 + \sin y \cos z + z$

Problem 12 Compute $\int_C \cos x \cos y dx - \sin x \sin y dy$ where C is parameterised by $\vec{r}(t) = (t, t^2)$ for $0 \le t \le 1$.

Solution. $\sin(1)\cos(1)$

Problem 13 Prove that if $\vec{F}$ is a conservative vector field in $\mathbb{R}^3$ with continuously differentiable component functions, then $\nabla \times \vec{F} = 0$.

Problem 14 Let $\vec{F} = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right\rangle$, the vortex field.

(a) Consider the function $g(x, y) = \arctan(x/y)$. Show that $\vec{F} = \nabla g$.

(b) Is $\vec{F}$ conservative?

(c) Consider the path P that is the two line segments from $(1, 1)$ to $(1, 7)$ and then to $(2, 2)$. Evaluate $\int_P \vec{F} \cdot d\vec{r}$.