①

Problem 1

a) False. The volume is missing an r.

ie $\int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr\,d\theta$ is a volume of a right cone.

b) False. we have

$x = u^2 - 2v \qquad y = 3v - 3uv$

$\frac{\partial x}{\partial u} = 2u \quad \frac{\partial x}{\partial v} = -2 \qquad \frac{\partial y}{\partial u} = -3v \quad \frac{\partial y}{\partial v} = 3-3u.$

Then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2 \\ -3v & 3-3u \end{vmatrix}$

$= 6u - 6u^2 - 6v.$

~~c) True. The cross partials agree, ie,~~

$\cancel{\frac{\partial}{\partial x}(xy^2) = y}$

c) False. The cross partials don't agree

$\frac{\partial}{\partial y}(x^2y) = x^2 \neq y^2 = \frac{\partial}{\partial x}(xy^2)$

②

d) False. The divergence gives a scalar field.

e) False. Counterexample: $\left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0 \right\rangle$.

we require $\nabla \times \vec{F} = 0$ and $\vec{F}$ defined on a simply connected domain to ensure conservative.

Problem 2

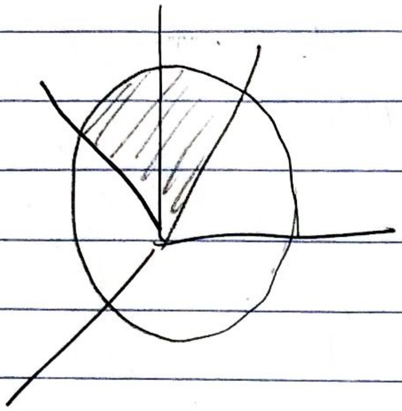
The bounds tell us we are after the volume of the half sphere. $(x \geq 0)$.

so $$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\pi} \int_{-\pi/2}^{\pi/2} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

$$= \frac{1}{3} \cdot 2 \cdot \pi = \frac{2}{3}\pi.$$

Problem 3:



The sphere in spherical coords is $\rho = 3$

The cone is
$\rho^2 \cos^2\phi = 3\sin^2\phi$

so $\tan^2\phi = \frac{1}{3} \Rightarrow \phi = \pi/6$

$$\iiint_B \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho\sin\phi \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left(\int_0^3 \rho^3 d\rho\right)\left(\int_0^{\pi/6} \sin^2\phi\, d\phi\right)$$

$$= 2\pi\left[\frac{\rho^4}{4}\Big|_0^3\right]\left(\int_0^{\pi/6} \frac{1-\cos 2\phi}{2}\, d\phi\right)$$

$$= \frac{81\pi}{2}\left(\frac{1}{2}\phi - \frac{\sin 2\phi}{4}\Big|_0^{\pi/6}\right)$$

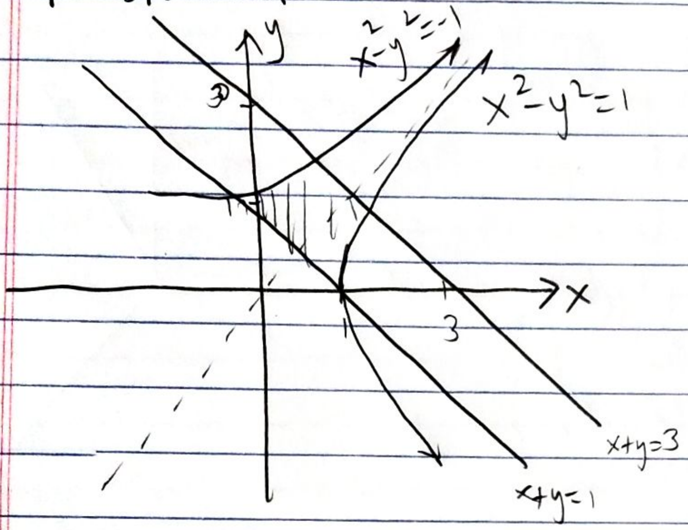
$$= \frac{81\pi}{2}\left(\frac{\pi}{12} - \frac{\sin \pi/3}{4}\right)$$

$$= \frac{81\pi}{2}\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

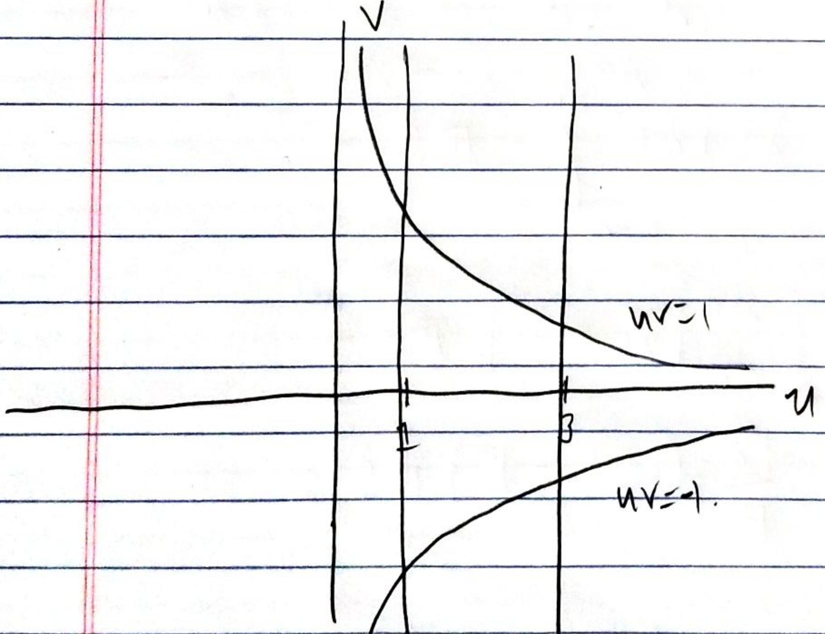
$$= \frac{81\pi}{8}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

(f)

Problem 4



let $u=x+y$ and $v=x-y$. Then the equations become, $u=1$, $u=3$, $uv=1$ $uv=-1$



The jacobian is given by (notice wrong way)

$$\left|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right|^{-1} = (-1-1)^{-1} = -\frac{1}{2}$$

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Hence, by change of variables, we have that

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \int_1^3 \int_{-1/u}^{1/u} v e^{uv} \left|-\frac{1}{2}\right| dv\,du$$

$$= \frac{1}{2}\int_1^3 \left(\frac{v}{u} e^{uv}\Big|_{-1/u}^{1/u} - \int_{-1/u}^{1/u} \frac{e^{uv}}{u}\,dv \right) du$$

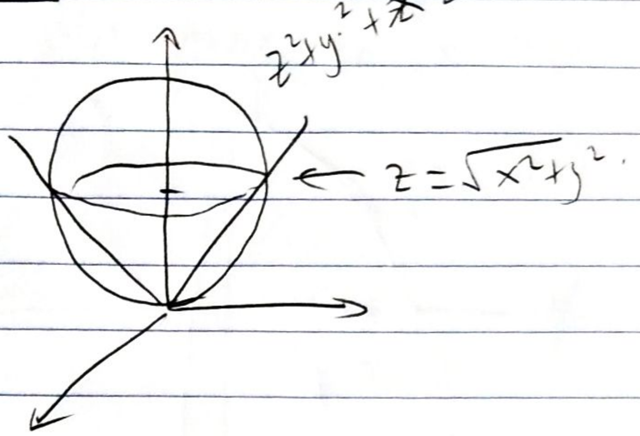
integration by part

$$= \frac{1}{2}\int_1^3 \frac{e}{u^2} + \frac{e^{-1}}{u^2} - \left(\frac{e}{u^2} - \frac{e^{-1}}{u^2}\right) du$$

$$= \int_1^3 \frac{e^{-1}}{u^2}\,du$$

$$= -\frac{e^{-1}}{u}\Big|_1^3 = \frac{-e^{-1}}{3} + \frac{e^{-1}}{1} = \frac{2}{3}e^{-1}$$

Question 5



The sphere in spherical cords:

$$x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\phi$$

$$\text{so } \rho=\cos\phi$$

The cone in spherical coords:

$$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi=\rho\sin\phi$$

$$\therefore \phi=\pi/4.$$

so in the $d\rho\, d\phi\, d\theta$ order, we get

$$0\le\rho\le\cos\phi$$

$$0\le\phi\le\pi/4$$

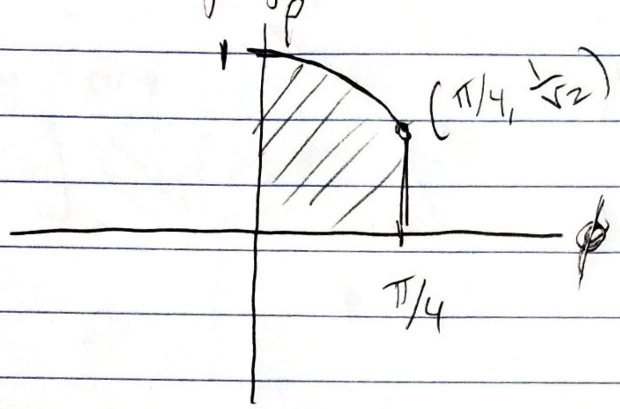
$$0\le\theta\le2\pi$$

and

$$V(E)=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

⑦

To get it into the $d\phi\, d\rho\, d\theta$ order, look at the projection onto the $\rho\phi$-plane:



so to swap the order, we need to break the inequalities up.

$0 \le \rho \le \cos\phi$ becomes $0 \le \rho \le 1/\sqrt{2}$ and $1/\sqrt{2} \le \rho \le 1$
$0 \le \phi \le \pi/4$ $\qquad$ $0 \le \phi \le \pi/4$ $\qquad$ $0 \le \phi \le \arccos\rho$

Hence

$$V(E) = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_0^{\pi/4} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta + \int_0^{2\pi}\int_{1/\sqrt{2}}^{1}\int_0^{\arccos\rho} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta.$$

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To calculate the volume, we use the first integral.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\phi \Big|_0^{\cos\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{\cos^4\phi}{12}\Big|_0^{\pi/4} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(-\frac{1}{4}+1\right) d\theta$$

$$= 2\pi \cdot \frac{1}{12} \cdot \frac{3}{4} = \frac{\pi}{8}$$

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Problem 6

(a) $\vec{F} = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \rangle$

$\vec{r}(t) = \langle t^2, t, t \rangle \quad 1 \le t \le e.$

Then

$\vec{F}(\vec{r}(t)) = \langle 2t^2 \ln t, t^3 + t^2, 2t^2 \rangle$

$\vec{r}'(t) = \langle 2t, 1, 1 \rangle.$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 4t^3 \ln t + t^3 + t^2 + 2t^2$$
$$= 4t^3 \ln t + t^3 + 3t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^e 4t^3 \ln t + t^3 + 3t^2 \, dt \quad (*)$$

split this up.

(1) $$\int_1^e 4t^3 \ln t = t^4 \ln t \Big|_1^e - \int_1^e t^3 dt$$ integration by parts

$$= e^4 - \left[\frac{t^4}{4} \right]_1^e$$

$$= \frac{3}{4} e^4 + \frac{1}{4}$$

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(2) $\int_1^e t^3\,dt = \frac{e^4}{4} - \frac{1}{4}$

(3) $\int_1^e 3t^2\,dt = e^3 - 1$

so (1),(2),(3) into $\bigstar$ gives

$$\int_C \vec{F}\cdot d\vec{r} = \frac{3}{4}e^4 + \frac{1}{4} + \frac{e^4}{4} - \frac{1}{4} + e^3 - 1$$
$$= e^4 + e^3 - 1$$

(b) we find a potential function for $\vec{F}$

integrating each component with respect to the corresponding variable gives the following:

$x^2\ln(y) + h(y,z)$
$x^2\ln(y) + z^2y + k(x,z)$
$yz^2 + l(x,y)$

Hence, we see $f(x,y,z) = x^2\ln(y) + z^2y$ each of these forms and so is a potential function.

(11)

The curve starts at $\vec{r}(1) = \langle 1,1,1 \rangle$
and ends at $\vec{r}(e) = \langle e^2, e, e \rangle$.

Hence the fundamental theorem tells us

$$\int_C \vec{F} \cdot d\vec{r} = f(e^2, e, e) - f(1,1,1)$$

$$= e^4 + e^3 - 1$$

which agrees with previous.

(12)

Question ~~8~~ 7

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

$$\int_C x^2 + y^2 + z^2 \, ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2) \sqrt{2} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} 1 + t^2 \, dt$$

$$= \sqrt{2} \left(t + \frac{t^3}{3} \Big|_0^{2\pi} \right)$$

$$= \sqrt{2} \left(2\pi + \frac{8\pi^3}{3} \right)$$

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Question. 8

we need to show that $z_{cm} = \frac{a+b}{2}$.

observe that $M = \iint_D \int_a^b \rho \, dz \, dA$

and $M_{xy} = \iint_D \int_a^b z \rho \, dz \, dA$.

since ρ doesn't depend on the z coord, we can bring it outside the inner integral.

ie $M = \iint_D \int_a^b \rho \, dz \, dA = \iint_D \rho \int_a^b dz \, dA$

$= \iint_D \rho (b-a) \, dA = (b-a) \iint_D \rho \, dA$

similarly $M_{xy} = \left(\frac{b^2 - a^2}{2}\right) \iint_D \rho \, dA$.

Hence, $z_{cm} = \frac{M_{xy}}{M} = \frac{a+b}{2}$.

(14)

Question 9

a) False.

if $r(t) = \langle t, t \rangle$, $r'(t) = \langle 1, 1 \rangle$, $\|r'(t)\| = \sqrt{2}$

so $\int_C xy\,ds = \int_0^1 t^2 \sqrt{2}\,dt$.

b) This is false. The standard example of this is the vortex field

$\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0 \rangle$.

c) This is true.

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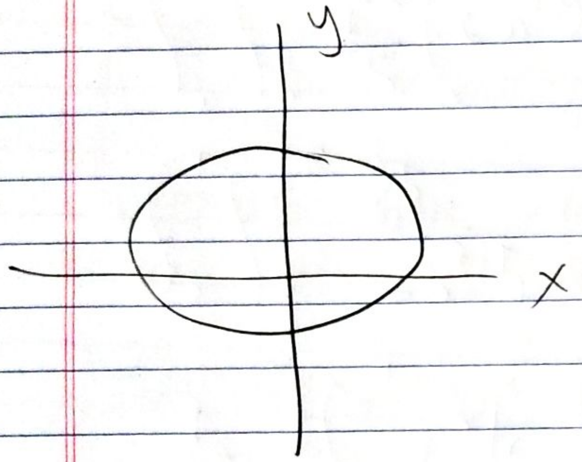
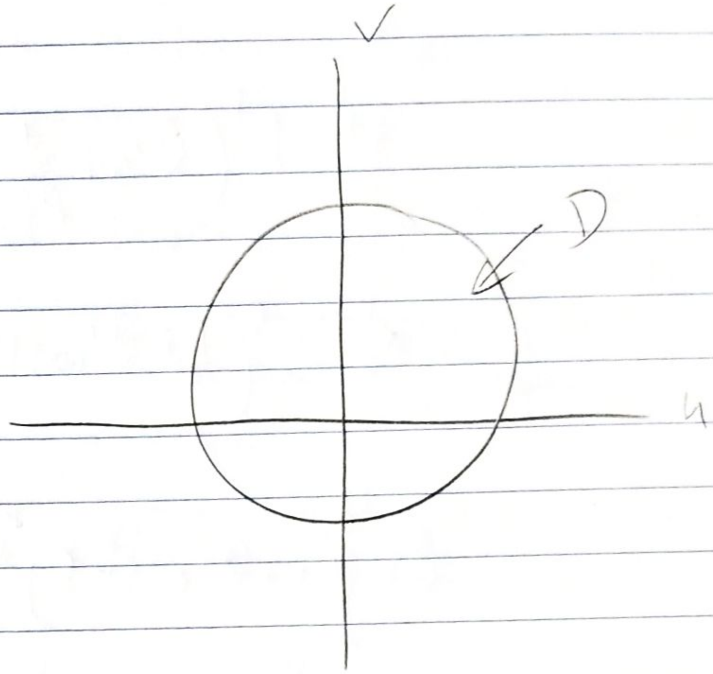
Question 10





we use $x = u$ $y = \frac{1}{5}v$ to change the to a circle

$x^2 + 25y^2 = 1 \Rightarrow u^2 + 25\left(\frac{1}{5}v\right)^2 = 1$

$u^2 + v^2 = 1.$

$g(u,v) = (u, \frac{1}{5}v)$, $dg = \begin{pmatrix} 1 & 0 \\ 0 & 1/5 \end{pmatrix}$

so $|dg| = 1/5.$

Then

$$\iint_R (x^2+25y^2)^2 \, dA = \iint_D (u^2+v^2)^2 \cdot \frac{1}{5} dA$$

$$= \frac{1}{5}\int_0^1 \int_0^{2\pi} r^5 \, d\theta \, dr$$

$$= \frac{2\pi}{5} \cdot \frac{1}{6} = \pi/15$$

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Question 11

$\vec{F} = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z \rangle$.

Suppose there exists a function $f(x,y,z)$ such that $\nabla f = \vec{F}$.

Then $\frac{\partial f}{\partial x} = 12x^2$, $f = 4x^3 + C(y,z)$

then $\frac{\partial f}{\partial y} = \cos y \cos z$ and $f_y = C_y(y,z)$

so $C_y(y,z) = \cos y \cos z$

$\Rightarrow \quad C(y,z) = +\sin y \cos z + d(z)$

and so $f = 4x^3 + \sin y \cos z + d(z)$.

$\frac{\partial f}{\partial z} = 1 - \sin y \sin z$ but also $\frac{\partial f}{\partial z} = -\sin y \sin z + d_z$

and $\frac{\partial f}{\partial z} = 1 - \sin y \sin z$. Hence $d_z = 1$

and so $d = z$.

Therefore, $f = 4x^3 + \sin y \cos z + z$ is a potential.

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Question 12

Note, this is the same thing as

$$\int_C \langle \cos x \cos y, -\sin x \sin y \rangle \cdot d\vec{r}$$ if you

are used to that notation. I'll do this

question using the alternate notation.

if $r(t) = (t, t^2)$ then $x = t$, $y = t^2$
so $dx = dt$, $dy = 2t\,dt$ and then

$$\int_C \cos x \cos y\,dx - \sin x \sin y\,dy = \int_0^1 \cos t \cos t^2\,dt - \sin t \sin t^2 \cdot 2t\,dt$$

$$= \int_0^1 \cos t \cdot \cos(t^2) - \sin t \cdot \sin(t^2) \cdot 2t\,dt.$$

This seems impossible to integrate. Instead we will look for a potential function and find one:

$$f(x,y) = \sin(x)\cos(y)$$

The curve starts at $r(0) = (0,0)$ and ends at $r(1) = (1,1)$. Hence,

$$\int_C \cos x \cos y\,dx - \sin x \sin y\,dy = \sin(1)\cos(1)$$

(18)

Question 13.

if $\vec{F} = \nabla f$ then $\vec{F} = \langle f_x, f_y, f_z \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

since partials are continuously differentiable, we can swap order of differentiation.

Hence $\nabla \times \vec{F} = 0$.

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Question 14

(a) $\frac{\partial g}{\partial x} = \frac{1}{(x/y)^2+1} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$

$\frac{\partial g}{\partial y} = \frac{1}{(\frac{x}{y})^2+1} \cdot \frac{-x}{y^2} = \frac{-x}{x^2+y^2}$

(b) No. Atleast not over $\mathbb{R}^2 \setminus (0,0)$.

The problem is that $g(x,y)$ is not defined every where.

Technically, conservativity of a vector field is with respect to a domain it's defined over, if we restrict $\vec{F}$ to the upper half plane $\{(x,y) \in \mathbb{R}^2 \mid y>0\} = D$ then $g(x,y)$ is defined here and $\vec{F}$ is conservative on this domain.

(c) The path $(1,1) \to (1,7) \to (2,2)$ is contained in the upper half plane which $\vec{F}$ conservative on. Hence fundamental theorem..

$\int_P \vec{F} \cdot dr = \arctan\left(\frac{2}{2}\right) - \arctan\left(1/1\right) = 0$