Math 32B Practice Problems

[With Solutions]

Ben Spitz

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FAQ

1. *What's in this document?*

   20 practice problems for the Math 32B Midterm, mostly written by myself.

2. *Are these questions representative of the content/difficulty/format/etc. of the midterm?*

   I make no such guarantees and had no such intentions when making this worksheet. These questions are only designed to be instructive, i.e. so that solving them requires understanding some important concept from the course.

3. *What does* (($\star$)) *mean?*

   I've used the symbol (($\star$)) to mark problems which are particularly challenging or outside the scope of the course. I recommend at least thinking about how you could solve them, but don't stress if you find it difficult to tackle those problems.

4. *Should I do all of these problems in order?*

   Probably not; there's a variety of questions here, intended to cover most of the topics we've discussed so far. If you want to work on a particular concept, you should look for a question about that concept! Feel free to ask for recommendations.

5. *Would all of your solutions be good enough to get full credit if the question was asked on a midterm?*

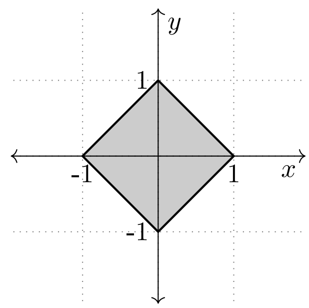
   Certainly not all; I'm not infallible and may have made mistakes (if so, email me!), and more importantly some of the solutions below are just sketches, or lack justification, etc. However, I've tried to make most of the solutions below of reasonably high quality.

Problems

1. Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$.

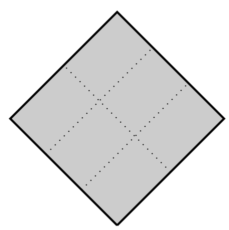
   (a) Draw the region $\mathcal{R}$.

   Solution: Your drawing should look something like this:

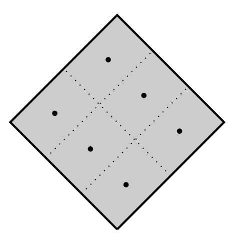
   (b) Divide $\mathcal{R}$ into 6 rectangles of equal area.

   Solution: Here's one way to do this:

   (c) Choose a sample point from each of the 6 rectangles you created in part b.

   Solution: One way to do this is to pick the center of each small rectangle as its sample point, like so:

   (d) Use your work in the previous parts to estimate $\iint_{\mathcal{R}} \frac{x^2}{1+y^2} \, dx dy$ via a Riemann sum.

> **Solution:** Using the points selected above, the estimate is
>
> $$1303718/3213969 \approx 0.405641$$

(e) Is the value of $\iint_{\mathcal{R}} \frac{x^2}{1+y^2}\,\mathrm{d}x\mathrm{d}y$ positive, negative, or zero?

> **Solution:** Positive, because $x^2/(1+y^2)$ is non-negative on all of $\mathcal{R}$ and not identically zero.

2. Compute

$$\int_1^2 \int_{-5}^{-4} \frac{x}{y}\,\mathrm{d}x\mathrm{d}y.$$

3. Let $\mathcal{R}$ be some rectangle in $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be some continuous function. If every Riemann sum approximation of $\iint_{\mathcal{R}} f\,\mathrm{d}A$ is positive, must it be the case that $\iint_{\mathcal{R}} f\,\mathrm{d}A$ itself is positive?

> **Solution:** Remember: we are allowed to pick any sample points we like from the rectangles in our Riemann sum. Indeed, we can perform a Riemann sum using only one rectangle (namely $\mathcal{R}$ itself), and using any point $p \in \mathcal{R}$ as the same point. The resulting sum is then (area of $\mathcal{R}$)$f(p)$. By assumption, this is positive for all p, so $f(p)$ is positive for all $p \in \mathcal{R}$. Now $\iint_{\mathcal{R}} f\,\mathrm{d}A$ is the integral of a positive function over a rectangle, which must be positive.

4. In the early 1600's, Bonaventura Cavlieri published what is now know as Cavalieri's Principle, which says that the volume of a region in $\mathbb{R}^3$ is unchanged if we slide its horizontal traces around horizontally. For example, the two stacks of coins in Figure 1 must have the same volume because we can turn one stack into the other by sliding the coins around horizontally (without lifting them at all). Justify Cavalieri's principle using calculus.

Figure 1: Two stacks of coins. Photo by Chiswick Chap, licensed under CC BY-SA 3.0.

5. A *regular tetrahedron* is a polyhedron made of four equilateral triangles of the same size, as shown in Figure 2.

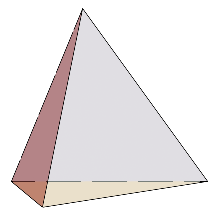


Figure 2: A tetrahedron.

Compute the volume of a regular tetrahedron with side length 1.

Solution:

$$\frac{1}{2\sqrt{6}}$$

6. (($\star$)) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that

$$\iint_{\mathcal{R}} f(x, y)\ \mathrm{d}A \geq \text{The area of } \mathcal{R}$$

for all rectangles $\mathcal{R}$. Show that $f(x, y) \geq 1$ for all points $(x, y) \in \mathbb{R}^2$.

Solution: If $f(x, y) < 1$ at some point (x, y), then we can find a small rectangle $\mathcal{R}$ around (x, y) on which the value of f is always less than $(1 + f(x, y))/2$ (since $f(x, y) < (1 + f(x, y))/2$). Then the integral of f over $\mathcal{R}$ cannot exceed $A(1+f(x, y))/2$, where A is the area of $\mathcal{R}$. Since $f(x, y) < 1$, $(1+f(x, y))/2 < 1$, which means that this integral is less than the area of $\mathcal{R}$. This contradicts the assumption about f.

7. Compute

$$\int_0^1 \int_0^1 \int_0^1 xyz\ \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

Solution: 1/8

8. (Source: Victoria Kala) Compute $\iiint_{\mathcal{E}} e^z\ \mathrm{d}V$ where $\mathcal{E}$ is the region enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

9. Let $\mathcal{R}$ be the region in $\mathbb{R}^2$ bounded by the y-axis, the line $x = 1$, and the curve $y = 1 + x^2$ (notably, $\mathcal{R}$ has infinite area). Sketch the region $\mathcal{R}$ and compute

$$\iint_{\mathcal{R}} \frac{\arctan(x)}{y^2}\, \mathrm{d}A.$$

> **Solution:** $\mathcal{R}$ is y-simple, so it makes sense to compute this as
>
> $$\int_0^1 \int_{1+x^2}^{\infty} \frac{\arctan(x)}{y^2}\, \mathrm{d}y\mathrm{d}x = \int_0^1 \frac{\arctan(x)}{1+x^2}\, \mathrm{d}x.$$
>
> Let $u = \arctan(x)$ so that $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$. Integrating with respect to x from 0 to 1 is the same as integrating with respect to u from 0 to $\arctan(1) = \pi/4$, so we have
>
> $$\int_0^{\pi/4} u\, \mathrm{d}u = \pi/8.$$

10. (Source: Victoria Kala) Compute $\iint_{\mathcal{R}} (x + y)\, \mathrm{d}A$ where $\mathcal{R}$ is the region to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

11. (($\star$)) In this problem, $\mathcal{R}$ will be the four-dimensional unit ball: that is, the region

$$\mathcal{R} = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 \le 1\}$$

in $\mathbb{R}^4$.

(a) Find the (four-dimensional) volume of $\mathcal{R}$.

> **Solution:** The cross-section of $\mathcal{R}$ for a fixed value of w is described by $x^2 + y^2 + z^2 \le 1 - w^2$, i.e. it is a ball in $\mathbb{R}^3$ of radius $\sqrt{1 - w^2}$. The volume of such a ball is $\frac{4}{3}\pi(1 - w^2)^{3/2}$ Thus, the volume is
>
> $$\int_{-1}^{1} \frac{4}{3}\pi(1 - w^2)^{3/2}\, \mathrm{d}w.$$
>
> We now perform a trigonometric substitution. Let $w = \sin(t)$ so that $\mathrm{d}w = \cos(t)\mathrm{d}t$. Integrating with respect to w from -1 to 1 is the same as integrating with respect to t from $-\pi/2$ to $\pi/2$, so we get
>
> $$\frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} (1 - \sin^2(t))^{3/2} \cos(t)\, \mathrm{d}t = \frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} (\cos^2(t))^{3/2} \cos(t)\, \mathrm{d}t$$
>
> Since $\cos(t)$ is non-negative when $-\pi/2 \le t \le \pi/2$, we can replace $(\cos^2(t))^{3/2}$ by $\cos^3(t)$. We now have
>
> $$\frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} \cos^3(t) \cos(t)\, \mathrm{d}t.$$

We now use integration by parts: let $u = \cos^3(t)$ and $\mathrm{d}v = \cos(t)\mathrm{d}t$. Then $\mathrm{d}u = -3\sin(t)\cos^2(t)\mathrm{d}t$ and $v = \sin(t)$, so we have

$$\frac{4\pi}{3}\left(\left[\sin(t)\cos^3(t)\right]_{-\pi/2}^{\pi/2} + 3\int_{-\pi/2}^{\pi/2} \sin^2(t)\cos^2(t)\ \mathrm{d}t\right).$$

Since sin is odd and cos is even, $\left[\sin(t)\cos^3(t)\right]_{-\pi/2}^{\pi/2} = 0$. Using the fact that $\sin(t)\cos(t) = \sin(2t)/2$, we are left with simply

$$\pi\int_{-\pi/2}^{\pi/2} \sin^2(2t)\ \mathrm{d}t.$$

Using the fact that $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we have

$$\frac{\pi}{2}\int_{-\pi/2}^{\pi/2}(1-\cos(4t))\ \mathrm{d}t = \frac{\pi^2}{2} - \frac{\pi}{2}\int_{-\pi/2}^{\pi/2}\cos(4t)\ \mathrm{d}t.$$

Letting $s = 4t$, we have

$$\int_{-\pi/2}^{\pi/2}\cos(4t)\ \mathrm{d}t = \frac{1}{4}\int_{-2\pi}^{2\pi}\cos(s)\ \mathrm{d}s = 0,$$

so the volume of the 4-dimensional unit ball is simply $\pi^2/2$.

(b) Let $\mathcal{S} = \{(w,x,y,z) \in \mathbb{R}^4 : w^2+x^2+y^2+z^2 = 1\}$ – this is the boundary of $\mathcal{R}$ in the same way that a sphere is the boundary of a normal three-dimensional ball. Use your work in part a to find the volume of this three-dimensional region.

Solution: We know that the volume of a 4-dimensional ball of radius r must be $\pi^2 r^4/2$. Taking the derivative with respect to r gives a "surface volume" of $2\pi^2 r^3$, and when $r = 1$ this is $2\pi^2$.

12. You've been challenged to an integration battle! The rules are as follows:

    1. Your opponent goes first and names a curve $\mathcal{C}$ of length 1 from a point on the x-axis to a point on the y-axis. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, and the curve $\mathcal{C}$. Your opponent recieves

    $$\iint_{\mathcal{R}} (x^2+y^2)\ \mathrm{d}A.$$

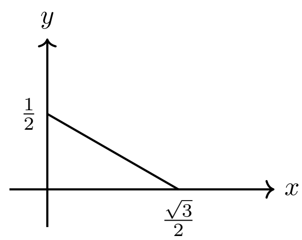
    points.

    2. You do the same thing: draw a curve of length 1 from the x-axis to the y-axis, and compute the integral of x^2+y^2 over the created region. You recieve points equal to the integral's value.

3. If your score is greater than your opponent's by at least 0.055, you win (and visa versa). Otherwise, play another round and add the points you recieve to your current score.

The good news is that your opponent doesn't know calculus, they've just realized that the function $x^2 + y^2$ increases with distance from the origin, so they want to make a region that encloses points which as "as far away as possible from the origin". Based on that observation, they're making educated guesses, but with your calculus knowledge, you should be able to beat them!

(a) Your opponent's first attempt was to make a triangle:



Make sure the path they drew has length 1, and figure out how many points they got.

Solution:

$$\frac{1}{16\sqrt{3}} \text{ points}$$

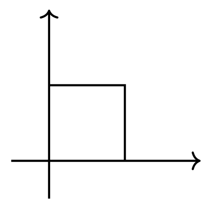
(b) It turns out your opponent didn't draw the best triangle. Figure out what the best triangle is and how many points you get for it!

Solution: The isosceles triangle gives the best possible value, which we can see as follows: if the triangle we draw has side length a on the x-axis and b on the y-axis, then the slope of the hypotenuse is $-b/a$. Thus, the line is given by $y = b - (b/a)x$, so the triangle is worth

$$\int_0^a \int_0^{b-(b/a)x} (x^2 + y^2)\, \mathrm{d}y\mathrm{d}x = \frac{1}{12}ab(a^2 + b^2) \text{ points}$$

We thus need to optimize the function $\frac{1}{12}ab(a^2 + b^2)$ with respect to the constraint $a^2 + b^2 = 1$. The function simplifies immediately to $\frac{1}{12}ab$, which has gradient $\frac{1}{12}\langle b, a\rangle$. By Lagrange multipliers, the optimal value is attained when this vector $\frac{1}{12}\langle b, a\rangle$ is parallel to $\nabla(a^2+b^2) = \langle 2a, 2b\rangle$. This only occurs if $a = b$, so the optimatal triangle is equilateral (with $a = b = 1/\sqrt{2}$). For this triangle you would recieve $1/24$ points.

(c) Next, your opponent tries a square:



How many points do they earn this round? How many do they have in total?

Solution: They get 1/24 points this round, bringing their total to

$$\frac{1}{16\sqrt{3}} + \frac{1}{24} \approx 0.0777511$$

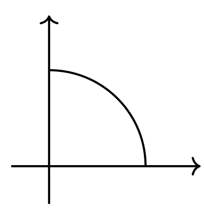
Your score is currently 1/24, so you need to score at least

$$\frac{1}{16\sqrt{3}} + 0.05 \approx 0.0860844$$

this round to win. It turns out this is impossible, so we have at least two rounds to go.

(d) Find a curve that beats the square your opponent tried and figure out how many points you get.

Solution: A reasonable choice would be the quarter-circle with radius $2/\pi$:



This gives you $2/\pi^3$ points (integrate with polar coordinates), bringing your total score to

$$\frac{1}{24} + \frac{2}{\pi^3} \approx 0.10617.$$

Of course, there are infinitely many correct answers to this question!

(e) (($\star$)) Looks like you haven't won yet. In good news, your opponent doesn't have any more original ideas, and will continue to use the same square every round from here on out. Try to win in as few rounds as possible!

Solution: Let

$$R = \left(\int_0^1 \sqrt{1 + \frac{t^6}{(1-t^4)^{3/2}}}\ \mathrm{d}t \right)^{-1}.$$

The best possible path is parametrized by

$$\rho(t) = R\langle t, (1-t^4)^{1/4} \rangle$$

from 0 to 1. It gives a score of roughly 0.0693975. If your curve in the previous part was at least as good as the quarter-circle, you can win this round by using this optimal path. If you use a quarter-circle repeatedly, it will take another another round after this one.

13. Show that

$$\int_0^1 \int_0^1 (\cos^4(x+y) + \sin^4(x-y))\ \mathrm{d}A \le 2$$

14. Let $\mathcal{B}$ be the unit disk in $\mathbb{R}^2$. Show that

$$\iint_{\mathcal{B}} (x^6 + y^6)\ \mathrm{d}A \le \frac{\pi}{2}.$$

15. Use a triple integral to find the volume of a circular cone with base radius R and height h.

Solution: Using cylindrical coordinates, we have

$$\int_0^h \int_0^{2\pi} \int_0^{R(1-z/h)} r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \frac{\pi}{3}R^2h.$$

16. Let $\mathcal{D}$ be the region in the first octant bounded by the surfaces $x^2+y^2+z^2=1$, $x^2+y^2+z^2=2$, $z^2=x^2+y^2$, and $z^2=2x^2+2y^2$. Compute

$$\iiint_{\mathcal{D}} \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\ \mathrm{d}V.$$

Solution: Use spherical coordinates!

17. Compute

$$\int_{-1}^1 \int_{-1}^1 \sin(xy)\ \mathrm{d}x\mathrm{d}y.$$

Solution: Since sin is odd, the integral over the right half of the square cancels with the integral over the left half of the square, and we get 0.

18. Compute

$$\int_1^2 \int_1^2 \frac{x}{x+y}\, \mathrm{d}x\mathrm{d}y.$$

Solution: Let $I = \int_1^2 \int_1^2 \frac{x}{x+y}\, \mathrm{d}x\mathrm{d}y$. By swapping x and y, we also get $I = \int_1^2 \int_1^2 \frac{y}{x+y}\, \mathrm{d}y\mathrm{d}x$. Thus,

$$2I = \int_1^2 \int_1^2 \left(\frac{x}{x+y} + \frac{y}{x+y} \right) \mathrm{d}x\mathrm{d}y = \int_1^2 \int_1^2 \mathrm{d}A = 1.$$

We conclude that $I = 1/2$.

19. Compute

$$\int_0^1 \int_0^1 (\cos(xy) - xy\sin(xy))\, \mathrm{d}x\mathrm{d}y$$

Solution: Noting that $\cos(xy) - xy\sin(xy) = \frac{\partial^2}{\partial x \partial y} \sin(xy)$, we get simply

$$\sin(0 \cdot 0) + \sin(1 \cdot 1) - \sin(0 \cdot 1) - \sin(1 \cdot 0) = \sin(1).$$

20. (Source: Victoria Kala) Compute

$$\int_0^1 \int_{3y}^3 e^{x^2}\, \mathrm{d}x\mathrm{d}y.$$