Math 32B Practice Problems

Ben Spitz

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FAQ

1. *What's in this document?*

   20 practice problems for the Math 32B Midterm, mostly written by myself.

2. *Are these questions representative of the content/difficulty/format/etc. of the midterm?*

   I make no such guarantees and had no such intentions when making this worksheet. These questions are only designed to be instructive, i.e. so that solving them requires understanding some important concept from the course.

3. *What does* $((\star))$ *mean?*

   I've used the symbol $((\star))$ to mark problems which are particularly challenging or outside the scope of the course. I recommend at least thinking about how you could solve them, but don't stress if you find it difficult to tackle those problems.

4. *Should I do all of these problems in order?*

   Probably not; there's a variety of questions here, intended to cover most of the topics we've discussed so far. If you want to work on a particular concept, you should look for a question about that concept! Feel free to ask for recommendations.

Problems

1. Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$.
   (a) Draw the region $\mathcal{R}$.
   (b) Divide $\mathcal{R}$ into 6 rectangles of equal area.
   (c) Choose a sample point from each of the 6 rectangles you created in part b.
   (d) Use your work in the previous parts to estimate $\iint_{\mathcal{R}} \frac{x^2}{1+y^2}\, \mathrm{d}x\mathrm{d}y$ via a Riemann sum.
   (e) Is the value of $\iint_{\mathcal{R}} \frac{x^2}{1+y^2}\, \mathrm{d}x\mathrm{d}y$ positive, negative, or zero?

2. Compute
$$\int_1^2 \int_{-5}^{-4} \frac{x}{y}\, \mathrm{d}x\mathrm{d}y.$$

3. Let $\mathcal{R}$ be some rectangle in $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be some continuous function. If every Riemann sum approximation of $\iint_{\mathcal{R}} f \, \mathrm{d}A$ is positive, must it be the case that $\iint_{\mathcal{R}} f \, \mathrm{d}A$ itself is positive?

4. In the early 1600's, Bonaventura Cavlieri published what is now know as Cavalieri's Principle, which says that the volume of a region in $\mathbb{R}^3$ is unchanged if we slide its horizontal traces around horizontally. For example, the two stacks of coins in Figure 1 must have the same volume because we can turn one stack into the other by sliding the coins around horizontally (without lifting them at all). Justify Cavalieri's principle using calculus.

Figure 1: Two stacks of coins. Photo by Chiswick Chap, licensed under CC BY-SA 3.0.

5. A *regular tetrahedron* is a polyhedron made of four equilateral triangles of the same size, as shown in Figure 2.

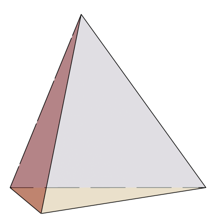


Figure 2: A tetrahedron.

Compute the volume of a regular tetrahedron with side length 1.

6. (($\star$)) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that

$$\iint_{\mathcal{R}} f(x, y) \, \mathrm{d}A \geq \text{The area of } \mathcal{R}$$

for all rectangles $\mathcal{R}$. Show that $f(x, y) \geq 1$ for all points $(x, y) \in \mathbb{R}^2$.

7. Compute
$$\int_0^1 \int_0^1 \int_0^1 xyz \, \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

8. (Source: Victoria Kala) Compute $\iiint_{\mathcal{E}} e^z \, \mathrm{d}V$ where $\mathcal{E}$ is the region enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

9. Let $\mathcal{R}$ be the region in $\mathbb{R}^2$ bounded by the y-axis, the line $x = 1$, and the curve $y = 1 + x^2$ (notably, $\mathcal{R}$ has infinite area). Sketch the region $\mathcal{R}$ and compute
$$\iint_{\mathcal{R}} \frac{\arctan(x)}{y^2} \, \mathrm{d}A.$$

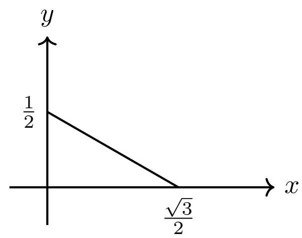
10. (Source: Victoria Kala) Compute $\iint_{\mathcal{R}} (x + y) \, \mathrm{d}A$ where $\mathcal{R}$ is the region to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

11. (($\star$)) In this problem, $\mathcal{R}$ will be the four-dimensional unit ball: that is, the region
$$\mathcal{R} = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 \leq 1\}$$
in $\mathbb{R}^4$.
    (a) Find the (four-dimensional) volume of $\mathcal{R}$.
    (b) Let $\mathcal{S} = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1\}$ – this is the boundary of $\mathcal{R}$ in the same way that a sphere is the boundary of a normal three-dimensional ball. Use your work in part a to find the volume of this three-dimensional region.

12. You've been challenged to an integration battle! The rules are as follows:
    1. Your opponent goes first and names a curve $\mathcal{C}$ of length 1 from a point on the x-axis to a point on the y-axis. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, and the curve $\mathcal{C}$. Your opponent recieves
    $$\iint_{\mathcal{R}} (x^2 + y^2) \, \mathrm{d}A.$$
    points.
    2. You do the same thing: draw a curve of length 1 from the x-axis to the y-axis, and compute the integral of $x^2 + y^2$ over the created region. You recieve points equal to the integral's value.
    3. If your score is greater than your opponent's by at least 0.055, you win (and visa versa). Otherwise, play another round and add the points you recieve to your current score.

    The good news is that your opponent doesn't know calculus, they've just realized that the function $x^2 + y^2$ increases with distance from the origin, so they want to make a region that encloses points which as "as far away as possible from the origin". Based on that observation, they're making educated guesses, but with your calculus knowledge, you should be able to beat them!

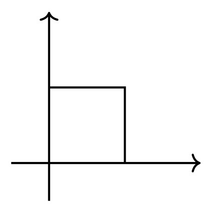
(a) Your opponent's first attempt was to make a triangle:



Make sure the path they drew has length 1, and figure out how many points they got.

(b) It turns out your opponent didn't draw the best triangle. Figure out what the best triangle is and how many points you get for it!

(c) Next, your opponent tries a square:



How many points do they earn this round? How many do they have in total?

(d) Find a curve that beats the square your opponent tried and figure out how many points you get.

(e) (($\star$)) Looks like you haven't won yet. In good news, your opponent doesn't have any more original ideas, and will continue to use the same square every round from here on out. Try to win in as few rounds as possible!

13. Show that

$$\int_0^1 \int_0^1 (\cos^4(x+y) + \sin^4(x-y))\, \mathrm{d}A \leq 2$$

14. Let $\mathcal{B}$ be the unit disk in $\mathbb{R}^2$. Show that

$$\iint_{\mathcal{B}} (x^6 + y^6)\, \mathrm{d}A \leq \frac{\pi}{2}.$$

15. Use a triple integral to find the volume of a circular cone with base radius R and height h.

16. Let $\mathcal{D}$ be the region in the first octant bounded by the surfaces $x^2+y^2+z^2=1$, $x^2+y^2+z^2=2$, $z^2=x^2+y^2$, and $z^2=2x^2+2y^2$. Compute

$$\iiint_{\mathcal{D}} \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\, \mathrm{d}V.$$

17. Compute
$$\int_{-1}^{1}\int_{-1}^{1} \sin(xy)\ \mathrm{d}x\mathrm{d}y.$$

18. Compute
$$\int_{1}^{2}\int_{1}^{2} \frac{x}{x+y}\ \mathrm{d}x\mathrm{d}y.$$

19. Compute
$$\int_{0}^{1}\int_{0}^{1} (\cos(xy) - xy\sin(xy))\ \mathrm{d}x\mathrm{d}y$$

20. (Source: Victoria Kala) Compute
$$\int_{0}^{1}\int_{3y}^{3} e^{x^2}\ \mathrm{d}x\mathrm{d}y.$$