**Problem 1** Use geometric reasoning to find $\int_S \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F}$ and $\mathbf{S}$ are the following:

(a) $\mathbf{F} = \langle 1, 0, 0\rangle$ and $\mathbf{S}$ is the union of two squares $\mathbf{S}_1$ and $\mathbf{S}_2$ given by: $\mathbf{S}_1 : x = 0, 0 \le y \le 1, 0 \le z \le 1$ and $\mathbf{S}_2 : z = 0, 0 \le x \le 1, 0 \le y \le 1$ where $\mathbf{S}_1$ is oriented in the positive $x$ direction and $\mathbf{S}_2$ in the positive $y$ direction.

*Solution.* The vector field is perpendicular to the normal of $\mathbf{S}_2$ and so this doesn't add anything to the integral. The normal of $\mathbf{S}_1$ is in the same direction and constant so $\mathbf{F} \cdot \mathbf{n} = 1$ on this square. Hence the integral is 1.

(b) $\mathbf{F} = \langle 1, 1, 0\rangle$ and $\mathbf{S}$ is the square given by $\mathbf{S} : x = 0, 0 \le y \le 1, 0 \le z \le 1$.

*Solution.* $\mathbf{F} \cdot \mathbf{n} = 1$ in this case and so the integral is 1.

(c) $\mathbf{F} = \langle 1, 0, 0\rangle$ and $\mathbf{S}$ is the cylinder given by $x^2 + y^2 = 1$ from $z = 0$ to $z = 1$ oriented outwards.

*Solution.* Consider the value $(\mathbf{F} \cdot \mathbf{n})(P)$ for each point $P$ on the cylinder. Opposite points on the cylinder must be negatives of each other since the vectors form complementary angles to each other. Hence the integral is zero.

(d) $\mathbf{F} = \langle x, y, 0\rangle$ and $\mathbf{S}$ is the cylinder given by $x^2 + y^2 = 4$ and $1 \le z \le 3$ oriented outwards.

*Solution.* Observe that on the cylinder, the vector field is in the same direction as the normal at each point and always has the same magnitude of 2. Hence $\mathbf{F} \cdot \mathbf{n} = 2$ and the integral is then $16\pi$ (The surface area of the cylinder times 2).

**Problem 2** Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle xyz, xyz, xyz\rangle$ and $S$ is the five faces of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ missing $z = 0$ that is oriented outwards. *Hint: It is enough to just calculate one of the faces and multiply the result by 3. Why?*

*Solution.* Observe that the vector field is zero when any of the variables are zero. Hence we don't need to calculate 2 of the faces. The remaining faces and vector field are completely symmetric in all the variables so it's enough to calculate the integral over one face and multiply the answer by 3.

We parameterise the $z = 1$ face by $r(u, v) = (u, v, 1)$ and then the parameterised normal is $\mathbf{N} = \langle 0, 0, 1\rangle$ and $\mathbf{F}(r(u, v)) = \langle uv, uv, uv\rangle$ and so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 3\int_0^1 \int_0^1 \mathbf{F}(r(u, v)) \cdot \mathbf{N} dudv = 3\int_0^1 \int_0^1 uvdudv = \frac{3}{4}.$$

**Problem 3** Use green's theorem to calculate $\int_C x^2 ydx + (y - 3)dy$ where $C$ is the perimeter of the rectangle with vertices $(1, 1), (4, 1), (4, 5)$ and $(1, 5)$ oriented counterclockwise.

*Solution.* Let $R$ be the area enclosed by $C$. Using green's theorem we get

$$\begin{aligned}\int_C x^2 y dx + (y-3)dy &= \int_R 0 - x^2 dA \\ &= \int_1^4 \int_1^5 -x^2 dy dx \\ &= -84\end{aligned}$$

**Problem 4** Compute $\int_{\partial D} \left(\sin x - \frac{y^3}{3}\right) dx + \left(\sin y + \frac{x^3}{3}\right) dy$ where $D$ is the annulus given in polar coordinates by $0 \leq \theta \leq 2\pi, 1 \leq r \leq 2$.

*Solution.* We have

$$\begin{aligned}\int_{\partial D} \left(\sin x - \frac{y^3}{3}\right) dx + \left(\sin y + \frac{x^3}{3}\right) dy &= \iint_D x^2 + y^2 dA \\ &= \int_0^{2\pi} \int_1^2 r^3 dr d\theta \\ &= \frac{15\pi}{2}\end{aligned}$$

**Problem 5** Consider the vector field $\mathbf{F} = \langle y, 2x \rangle$. Suppose we have two paths $\gamma_1$ and $\gamma_2$ that both start and end at the same point. How do the two line integrals of $\mathbf{F}$ differ along the two paths?

*Solution.* Suppose we have it such that $\gamma_1 - \gamma_2$ is counterclockwise along the boundary of it's enclosed area $D$. Then green's theorem tells us that

$$\int_{\gamma_1 - \gamma_2} y dx + 2x dy = \iint_D 1 dA = \text{Area}(D).$$

Hence we get that

$$\int_{\gamma_1} y dx + 2x dy = \int_{\gamma_2} y dx + 2x dy + \text{Area}(D).$$

If $\gamma_1 - \gamma_2$ is clockwise instead, then we get

$$\int_{\gamma_1} y dx + 2x dy = \int_{\gamma_2} y dx + 2x dy - \text{Area}(D).$$