Week 8 Worksheet

Problem 1 – **Parametrization Practice.** We can describe a surface by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where u, v are two parameters. For example, the plane $x + y + z = 1$ could be described as the vector function

$$\mathbf{r}(x,y) = \langle x, y, 1 - x - y\rangle.$$

That same plane could also be described as

$$\mathbf{r}(y,z) = \langle 1 - y - z, y, z\rangle.$$

Find a parametric representation for each of the following surfaces:

(a) The plane $z = 1+2x+3y$ that lies above the rectangle $0 \le x \le 3, 0 \le y \le 2$
(b) The elliptic paraboloid $z = x^2 + 4y^2$
(c) The cylinder $x^2 + y^2 = 9$, $0 \le z \le 1$ (try cylindrical coordinates)
(d) The sphere $x^2 + y^2 + z^2 = a^2$ where a is a constant (try spherical coordinates)
(e) The "cone of shame" $z = \sqrt{x^2 + y^2}, 1 \le z \le 2$

a) let $u = x$, $v = y$, then $z = 1 + 2u + 3v$

Hence $\vec{r}(u,v) = \langle u, v, 1+2u+3v\rangle \quad 0 \le u \le 3, \; 0 \le v \le 2$

b) let $x = u$, $y = v$, then $z = u^2 + 4v^2$

so $\vec{r}(u,v) = \langle u, v, u^2 + 4v^2\rangle$

c) let $x = 3\cos t$, $y = 3\sin t$, $z = s$.

Then $\vec{r}(s,t) = \langle 3\cos t, 3\sin t, s\rangle$

d) $\vec{r}(\theta,\phi) = \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle$

e) $x = u\cos v$, $y = u\sin v$, $z = u$, $1 \le u \le 2$, $0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$

Problem 2 – **Tangent Planes to Surfaces.** The equation of a tangent plane can be written as

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

where (x_0,y_0,z_0) is a point on the plane and $\langle a,b,c\rangle$ is a normal vector to the plane. If we have a parametric representation $\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$ for a given surface, the normal to that surface is given by $\mathbf{r}_u\times\mathbf{r}_v$ where

$$\mathbf{r}_u=\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}$$
$$\mathbf{r}_v=\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}.$$

Find the equation of a tangent plane to the parametrized surface

$$\mathbf{r}(u,v)=uv\mathbf{i}+u\sin v\mathbf{j}+v\cos u\mathbf{k}$$

at $u=0, v=\pi$.

$$\vec{r}=\langle uv,\ u\sin v,\ v\cos u\rangle$$

$$r_u=\langle v,\ \sin v,\ -v\sin u\rangle$$

$$r_v=\langle u,\ u\cos v,\ \cos u\rangle$$

$$N=r_u\times r_v=\begin{vmatrix} i & j & k \\ v & \sin v & -v\sin u \\ u & u\cos v & \cos u \end{vmatrix}=\langle \sin v\cos u-uv\sin u\cos v,\ -uv\sin u-v\cos u,\ uv\cos v-u\sin v\rangle$$

at $u=0$, $v=\pi$, $N=\langle 0,-\pi,0\rangle$ (This is the $a\,b\,c$)

and $\vec{r}(0,\pi)=\langle 0,0,\pi\rangle$. (this is the x_0, y_0, z_0)

Hence the tangent plane is:

$$0(x-0)-\pi(y-0)+0(z-\pi)=0$$

$$\pi y=0$$
$$y=0.$$

Problem 3 – **Surface Area.** If S is a smooth parametric surface described by a parametrization $\mathbf{r}(u,v)$, $(u,v)\in D$, then the surface area of S is given by

$$A=\iint_D ||\mathbf{r}_u\times\mathbf{r}_v||\,dudv$$

Find the surface area of

(a) The sphere $x^2+y^2+z^2=a^2$ where a is a constant (use your parametrization found in 1d)

(b) The "cone of shame" $z=\sqrt{x^2+y^2}, 1\le z\le 2$ (use your parametrization found in 1e)

a) $\vec{r}(\theta,\phi) = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ a\sin\phi \rangle$ $\quad 0 \le \theta \le 2\pi$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

$\vec{r}_\theta = \langle -a\sin\theta\cos\phi,\ a\cos\theta\cos\phi,\ 0 \rangle$

$\vec{r}_\phi = \langle -a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ a\cos\phi \rangle$

$\vec{r}_\theta \times \vec{r}_\phi = \langle a^2\cos\theta\cos^2\phi,\ a^2\sin\theta\cos^2\phi,\ a^2\cos\phi\sin\phi \rangle$

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = a^2\left(\cos^2\theta\cos^4\phi + \sin^2\theta\cos^4\phi + \cos^2\phi\sin^2\phi\right)^{1/2}$$

$$= a^2\left(\cos^2\phi\left(\cos^2\phi + \sin^2\phi\right)\right)^{1/2}$$

$$= a^2\cos\phi$$

$$A = \iint \|\vec{r}_\theta \times \vec{r}_\phi\|\, dA = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} a^2\cos\phi\, d\phi\, d\theta$$

$$= 2\pi a^2 \left[\sin\phi\right]_{-\pi/2}^{\pi/2}$$

$$= 4a^2\pi.$$

b) $\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u \rangle$ $\quad 0 \le v \le 2\pi$, $1 \le u \le 2$

$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$

$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -u\cos v,\ -u\sin v,\ u\cos^2 v + u\sin^2 v \rangle$

$= \langle -u\cos v,\ -u\sin v,\ u \rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \left(u^2\cos^2 v + u^2\sin^2 v + u^2\right)^{1/2}$

$$\|\vec{r}_u \times \vec{r}_v\| = \left(u^2\cos^2 v + u^2\sin^2 v + u^2\right)^{1/2}$$

$$= \sqrt{2}u$$

$$A = \int_1^2 \int_0^{2\pi} \sqrt{2}u \, dv \, du = 2\sqrt{2}\pi \left[\frac{u^2}{2}\right]_1^2 = 3\sqrt{2}\pi.$$

Problem 4 – **Scalar Surface Integrals.** If S is a smooth parametric surface described by a parametrization $\mathbf{r}(u,v)$, $(u,v) \in D$, then the surface integral of a scalar function f over S is given by

$$\iint_S f dS = \iint_D f(\mathbf{r}(u,v))||\mathbf{r}_u \times \mathbf{r}_v|| du dv$$

Evaluate the following surface integrals:

(a) $\iint_S x^2yz dS$ where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $0 \le x \le 3, 0 \le y \le 2$ (use your parametrization found in 1a)

(b) $\iint_S y dS$ where S is the cylinder $x^2 + y^2 = 9$, $0 \le z \le 1$ (use your parametrization found in 1c)

a) $r(u,v) = \langle u, v, 1+2u+3v \rangle \quad 0 \le u \le 3, \quad 0 \le v \le 2$

$r_u = \langle 1, 0, 2 \rangle$

$r_v = \langle 0, 1, 3 \rangle$

$r_u \times r_v = \langle -2, -3, 1 \rangle$

$\|r_u \times r_v\| = \sqrt{14}$

$$\iint_S x^2yz\,dS = \int_0^3 \int_0^2 u^2 v(1+2u+3v)\sqrt{14}\,dv\,du$$

$$= \sqrt{14} \int_0^3 \int_0^2 u^2v + 2u^3v + 3u^2v^2 \,dv\,du$$

$$= \sqrt{14} \int^3 \underline{u^2v^2} + u^3v^2 + u^2v^3 \Big|^2 du$$

$$= \sqrt{14}\int_0^3 \frac{u^2v^2}{2} + u^3v^2 + u^2v^3 \Big|_0^2 du$$

$$= \sqrt{14}\int_0^3 2u^2 + 4u^3 + 8u^2 du$$

$$= \sqrt{14}\int_0^3 4u^3 + 10u^2 du$$

$$= \sqrt{14}\left[u^4 + \frac{10}{3}u^3\right]_0^3$$

$$= \sqrt{14}(81 + 90)$$

$$= 171\sqrt{14}$$

b) $\iint y\,ds$ $\quad r(s,t) = \langle 3\cos t, 3\sin t, s\rangle$ $\quad 0 \le t \le 2\pi$, $0 \le s \le 1$

$r_s = \langle 0, 0, 1\rangle$

$r_t = \langle -3\sin t, 3\cos t, 0\rangle$

$r_s \times r_t = \langle -3\cos t, -3\sin t, 0\rangle$

$\|r_s \times r_t\| = 3$

so $\iint y\,dS = \int_0^{2\pi}\int_0^1 9\sin t\,ds\,dt$

$$= \int_0^{2\pi} 9\sin t\,dt$$

$$= -9\cos t \Big|_0^{2\pi} = 0$$