Problem 1 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be an infinitely differentiable function. Show that the curl of ∇f is zero.

Solution. The curl of ∇f is

$$\left\langle \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}, \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \right\rangle = \langle 0, 0, 0\rangle$$

by Clairaut's theorem (the fact that you can take partial derivatives in any order and get the same answer).

Problem 2 Let F be the vector field on $\mathbb{R}^2$ defined by

$$F(x, y) = \langle x^2, xy\rangle.$$

(a) Determine if F is conservative.
This can be done in many ways, here are some vauge hints:

- *Compute some path integrals*
- *Compute some partial derivatives*
- *Compute a curl*

Solution. The curl of F is

$$\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2) = y \neq 0$$

so F is not conservative.

(b) Compute

$$\int_{\mathcal{C}} F \cdot d\vec{r}$$

where $\mathcal{C}$ is the unit circle around the origin, oriented counterclockwise.
Solution. Using the parametrization $\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, we get

$$\int_{\mathcal{C}} F \cdot d\vec{r} = \int_0^{2\pi} \langle \cos^2 t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt = \int_0^{2\pi} 0 \, dt = \boxed{0}.$$

(c) Compute $\operatorname{div} F$.
Solution.

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) = 2x + x = \boxed{3x}.$$

Problem 3 Let F be the vector field on $\mathbb{R}^2$ defined by

$$F(x, y) = \langle -ye^{-x}, e^{-x}\rangle.$$

(a) Show that F is conservative.
Solution. The curl of F is

$$\frac{\partial}{\partial x}(e^{-x}) - \frac{\partial}{\partial y}(-ye^{-x}) = -e^{-x} - (-e^{-x}) = 0,$$

and the domain of F is simply connected (i.e. has no holes), so F is conservative.

(b) Find two different potential functions for F.

Solution. If f is a potential function for F, then $\partial f/\partial y = e^{-x}$, so $f(x,y) = ye^{-x} + g(x)$ for some function g. We also need $\partial f/\partial x = -ye^{-x}$, which means we must have $g'(x) = 0$. Therefore, $f(x,y) = ye^{-x} + c$ for some constant c, and indeed we see that the gradient of any such function is F. We can produce two different potential functions by picking two different constants, e.g.

$$ye^{-x}$$

and

$$ye^{-x} + 1$$

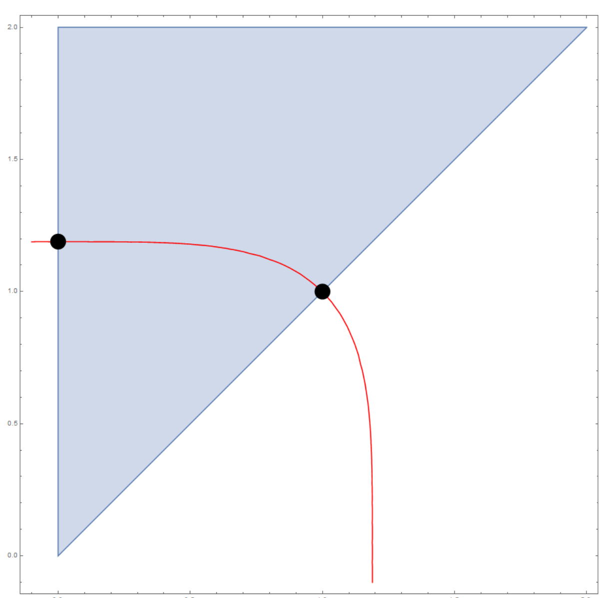
are two different potential functions for F.

(c) Compute

$$\int_{\mathcal{S}} F \cdot \, \mathrm{d}\vec{r}$$

where $\mathcal{S}$ is the portion of the curve $x^4 + y^4 = 2$, oriented counter-clockwise, in the region $0 \le x \le y$.

Solution. Here's a picture of what this path looks like:



The path starts at $(1,1)$ (the unique intersection point of the line $x = y$ and the curve $x^4 + y^4 = 2$ in the first quadrant) and ends at $(0, 2^{1/4})$ (the unique intersection point of the line $x = 0$ and the curve $x^4 + y^4 = 2$ in the first quadrant), so (using the potential function ye^{-x}) we have by the fundamental theorem of line integrals that

$$\int_{\mathcal{S}} F \cdot \, \mathrm{d}\vec{r} = \boxed{2^{1/4} - 1/e}.$$

Problem 4 One day, when Sandy is walking home with her bowling ball, she finds a beautiful hill whose shape is the graph of $f(x) = \frac{1}{1+x^2}$.

(a) Sandy decides to go to the top of the hill (at $(0, 1)$) and roll her bowling ball down. As she's climbing up, she passes her friend Ambrose, who says he'll catch the bowling ball when it rolls down. Ambrose is positioned at $(3, 0.1)$. Verify that Ambrose is on the hill.

Solution. Ambrose is on the hill because $1/(1 + 3^2) = 1/10 = 0.1$.

(b) Sandy tosses the ball down the hill and it follows some trajectory $r(t)$, $0 \le t \le 1$ such that $r(0) = (0, 1)$ and $r(1) = (3, 0.1)$. Assuming that the force of gravity on the ball as it rolls down the hill is always $\langle 0, -1\rangle$, find the work done by gravity on the ball during its trip down the hillside.

Solution. The gravity vector field is converative, since $-y$ is a potential function for it. Thus, the work done by gravity on the ball is $-0.1-(-1) = \boxed{0.9}$.

(c) Explain how and why the specific shape of the hill and the specific trajectory r down the hill doesn't affect the answer in the previous part.

Solution. Because the vector field is conservative, the work done by gravity only depends on the start and end points of the trajectory, so the answer would have been the same for any trajectory r from $(0, 1)$ to $(3, 0.1)$ – the shape of the hill doesn't matter as long as it contains these two points.

In fact, since $-y$ is a potential function for the gravity vector field, the fundamental theorem of line integrals tells us that the work done by gravity along any trajectory is equal to the y-coordinate at the start of the path minus the y-coordinate at the end of the path. In other words, the answer would have been 0.9 for any trajectory along any hill that had a total decrease in height of 0.9.

Problem 5 An electron is placed in a newfangled particle decelerator. The electron starts at $(1, 0)$ with velocity $(0, 1)$, and experiences a force of $F = \langle -x, -y\rangle$ as it moves.

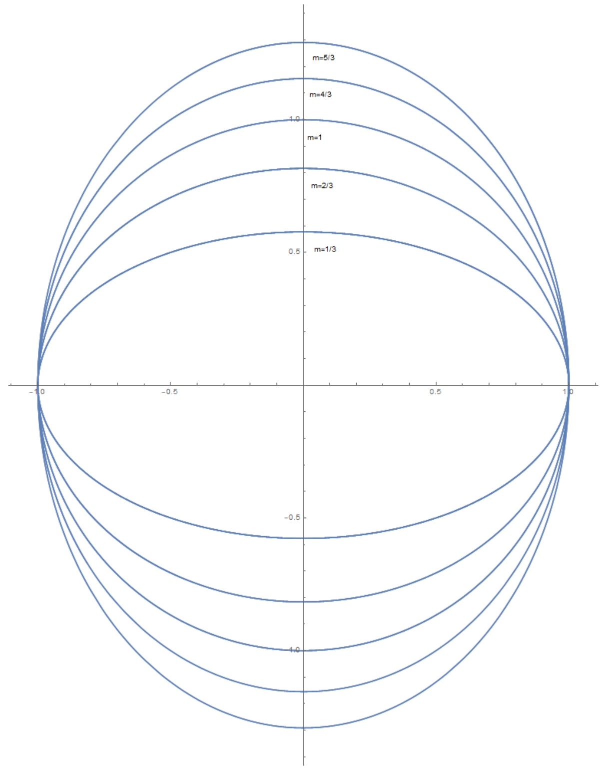
(a) Sketch the path the electron will take. This doesn't need to be precise, just give a rough idea.

Solution. The particle will orbit around the origin since the vector field is conservative and always points towards the origin. If you said the electron would spiral into the origin, you were on the right track, and this is a reasonable answer! It takes some knowledge of physics or differential equations to deduce that the electron will actually stay in a stable orbit.

Let m be the mass of the electron. By Newton's second law, the acceleration of the electron is $(1/m)\langle -x, -y\rangle$. By solving the relevant differential equations, we get that the path of the electron is parametrized by

$$\langle \cos(t/\sqrt{m}), \sqrt{m}\sin(t/\sqrt{m})\rangle.$$

Here are plots of this orbit for different values of m:



(b) Prove that F is conservative by showing its curl is 0.

Solution. The curl of F is

$$\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(-x) = 0,$$

and F is defined on all of $\mathbb{R}^2$, so F is conservative.

(c) Find a potential function for F.

Solution. If f is a potential function for F, then by definition we have $\nabla f = F$, i.e. $\frac{\partial f}{\partial x} = -x$ and $\frac{\partial f}{\partial y} = -y$. The first equation tells us that $f(x, y) = -x^2/2 + g(y)$ for some function g. The second equation then tells us that $g'(y) = -y$, so $g(y) = -y^2/2 + c$ for constant c. Choosing $c = 0$, we get the candidate potential function

$$f(x, y) = -(x^2 + y^2)/2,$$

which we can verify has the correct gradient.

(d) Determine the total amount of work done on the electron by F along its path. *Hint: it's not neccesary to parametrize the electron's path!*

Solution. This should have been worded more precisely: it would have better to ask how much work was done on the electron along one period of its path (from the time it starts to the next time it arives at $(1, 0)$), although this would have given away the fact that the path of the electron is periodic in part (a).

By the fundamental theorem of line integrals, the total work done on the electron over one period of its path is 0.

In general, using the parametrization given in the solution to part (a), the work done by the vector field on the electron between times $t = 0$ and $t = T$ is

$$\frac{m-1}{2}\sin^2\left(\frac{T}{\sqrt{m}}\right)$$

this doesn't approach a limiting value as $T \to \infty$ (unless $m = 1$), so it doesn't really make sense to ask how much work is done on the electron along its *entire* path.

Interestingly, if the mass of the electron is 1, then the path of the electron traces out a level curve of the potential function of the vector field, and consequently the total work done by the vector field on the electron is *always* 0 over any portion of the path it takes. In this case it would make sense to say that the total amoung of work done is 0.