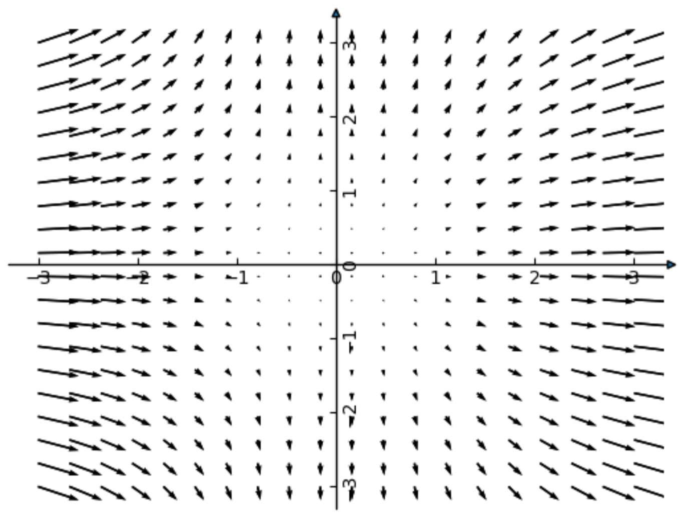
Problem 1 Sketch the following planar vector fields $\vec{F}$.

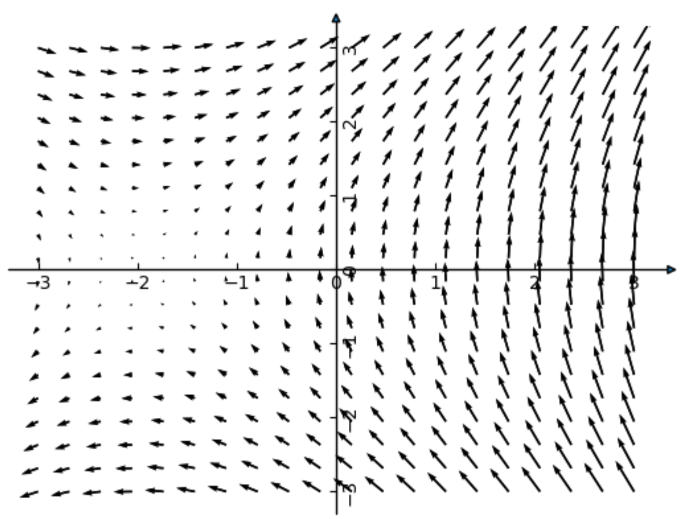
(a) $\vec{F} = \langle x^2, y \rangle$

Solution.



(b) $\vec{F} = \langle y, x+2 \rangle$

Solution.



Problem 2 Calculate $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$ of the following:

(a) $\vec{F} = \langle y, z, x \rangle$

Solution.

$$\nabla \times \vec{F} = (-1, -1, -1)$$
$$\nabla \cdot \vec{F} = 0.$$

(b) $\vec{F} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$

Solution.

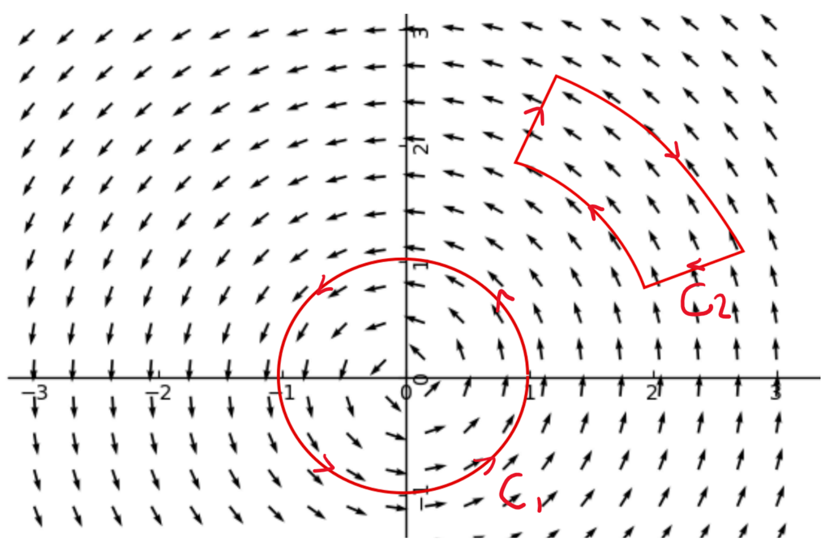
$$\nabla \times \vec{F} = \left(0, 0, \frac{1}{\sqrt{x^2+y^2}}\right)$$
$$\nabla \cdot \vec{F} = 0.$$

Problem 3 Show that $f(x, y, z) = r = \sqrt{x^2 + y^2 + z^2}$ is a potential function for the unit radial vector field $\vec{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$

Solution. We just check that $\nabla f = \vec{e}_r$.

Problem 4 Let $\vec{F} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$. This is sometimes known as the unit vortex field. We will investigate some properties of it.

(a) In the figure below, we have the vector field and two curves C_1 and C_2. Use a geometric argument to determine whether $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative or zero for the two curves.



Solution. The integral along C_1 is positive since the vector field is in the same direction as the tangent vector along the whole curve.

The integral along C_2 is negative. Along the two line segments, the tangent vector and the vector field are perpendicular and so the integral is zero along those two segments. Along the two curve segments, it is going in the same direction on the shorter segment but going in the oppositie direction for the longer segment. Hence, overall the line integral will be negative.

(b) The circle C_1 is the unit circle around the origin. Calculate $\int_{C_1} \vec{F} \cdot d\vec{r}$ explicitly by giving C_1 a parameterisation.

Solution. We can parameterise the circle by $g(\theta) = (\cos(\theta), \sin(\theta))$. We then get

$$\begin{aligned}\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(g(\theta)) \cdot \frac{dg}{d\theta} d\theta \\ &= \int_0^{2\pi} (-\sin(\theta), \cos(\theta)) \cdot (-\sin(\theta), \cos(\theta)) d\theta \\ &= \int_0^{2\pi} \cos^2(\theta) + \sin^2(\theta) d\theta \\ &= \int_0^{2\pi} 1 d\theta \\ &= 2\pi\end{aligned}$$

Problem 5 For this question, let $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. This is almost the same as the previous question except we've multiplied by a factor of $\frac{1}{r}$. This causes the length of each vector to get bigger as we approach the origin and get smaller the further from the origin we are. This is sometimes called the vortex field.

(a) Suppose that we had a curve γ parameterised by

$$\vec{r}(\theta) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta))$$

for $\theta \in [\theta_1, \theta_2]$ where f is a positive function. Show that $\vec{F} \cdot d\vec{r} = d\theta$ and conclude that

$$\int_\gamma \vec{F} \cdot d\vec{r} = \theta_2 - \theta_1.$$

Remember, given a parameterisation $\vec{r}(\theta)$, then $\vec{F} \cdot d\vec{r}$ means $\vec{F} \cdot \frac{d\vec{r}}{d\theta} d\theta$.

Solution. We have $F(\vec{r}(\theta)) = \frac{1}{f(\theta)}(-\sin(\theta), \cos(\theta))$ after simplification and

$$\frac{d\vec{r}}{d\theta} = f'(\theta)(\cos(\theta), \sin(\theta)) + f(\theta)(-\sin(\theta), \cos(\theta)).$$

Let $\vec{u} = (\cos(\theta), \sin(\theta))$ and $\vec{v} = (-\sin(\theta), \cos(\theta))$. Observe that we have $\vec{v} \cdot \vec{v} = 1$ and $\vec{u} \cdot \vec{v} = 0$ Hence we have that

$$\begin{aligned}\vec{F} \cdot \frac{d\vec{r}}{d\theta} &= \frac{1}{f(\theta)} \vec{v} \cdot (f'(\theta)\vec{u} + f(\theta)\vec{v}) \\ &= 1.\end{aligned}$$

Hence we have that

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1.$$

(b) Why is it true for a closed curve γ that wraps around origin n times (wrapping around in a positive sense is in the counterclockwise direction, and a negative sense in the clockwise direction) that we have

$$\int_\gamma \vec{F} \cdot d\vec{r} = 2\pi n.$$

Solution. Any curve that doesn't intersect a radial line more than once can be parameterised by the angle. This gives a parameterisation of the form in the previous question and the previous question tells us the integral is just the difference of the angle it started at and where it ends.

Any curve can be broken up into such curves and adding all these together gives us the result.